2024 January

数学月刊

Mathematics Monthly

Author :
Guanzhong Yang(student)
Yan Wang(teacher)

Mathematics changes our lives

PREFACE

This month, we are going to talk about the following questions.

Q1

A pair of distinct two-digit integers (n, d) is misleading if the first digit of d equals the second digit of n, and if you write the numbers as a fraction $\frac{n}{d}$ *and "cancel" (i.e., cross out) the common digits, you get an equal fraction. For example, (16, 64) is a misleading pair since*

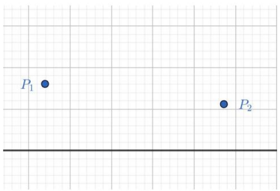
$$\frac{16}{64} = \frac{1}{4} = \frac{1\cancel{6}}{\cancel{6}4}.$$

(a) Find all pairs of two-digit misleading integers. Explain how you know you have found all possible misleading two-digit pairs.

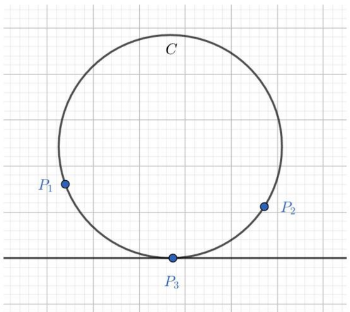
(b) Are there any misleading pairs of three-digit integers? For a three-digit pair to be misleading, when written as a fraction, "cancelling" the first digit of d with either the second or third digit of n (where the digits being canceled are equal) results in a fraction equal in value to the original fraction. Furthermore, the two integers cannot both end in 0 (otherwise, the pair could be obtained directly from a pair of misleading two-digit numbers).

Q2

Let $P_1 = (x_1, y_1)$ *and* $P_2 = (x_2, y_2)$ *be two points in the (x, y)—plane such that* $y > 0$ *and* $x_1 < x_2$.



We call a circle C a Yeppeun circle for P_1 *and* P_2 *if C contains* P_1 *and* P_2*, C is tangent to the x-axis at a point* $P_3 = (x_3, 0)$*, and the x-coordinate of the point of tangency lies between the x-coordinates of* P_1 *and* P_2*; that is,* $x_1 < x_3 < x_2$*. For example, the circle C in the following diagram is a Yeppeun circle for the given* P_1 *and* P_2.



Given any two such points P_1 *and* P_2 *above the x—axis, does a Yeppeun circle always exist? Explain.*

This is the admission question from 2024 SUMaC Program. If you have other brilliant ideas, email to anmiciuangray@163.com **for surprising rewards!**

1.(a)

Introduction

We start with expressing the question in the form of mathematical equation.
Assuming that n = 10A + B and d = 10B + C,

$$\frac{n}{d} = \frac{10A + B}{10B + C} = \frac{A}{C},$$

where that n ≠ d and A, B, C are positive one-digit integers.

Through observation, we can find that this question involves a lots of calculation which is almost impossible for us to calculate one by one. So what we do is to try to narrow down the scope of the answers.
I had already tried several methods, whose general strategies will be showed in the afterword. Now, I'll just show you the most time-saving measure.

Some Auxiliary Results

Lemma 1.1.1

$$\frac{90B}{81+9B} \leq C \leq \frac{90B}{81+B}.$$

Proof.
Rearrange:

$$C(10A + B) = A(10B + C).$$
$$9AC + BC = 10AB.$$
$$\frac{9}{B} + \frac{1}{A} = \frac{10}{C}.$$

Since $\frac{1}{9} \leq \frac{1}{A} \leq 1$:

$$\frac{10}{C} - 1 \leq \frac{9}{B} \leq \frac{10}{C} - \frac{1}{9}.$$
$$90B - 9BC \leq 81C \leq 90B - BC.$$

From 90B - 9AB ≤ 81C:

$$\frac{90B}{81+9B} \leq C.$$

From 81C ≤ 90B - BC:

$$C \leq \frac{90B}{81+B}.$$

So we can get:

$$\frac{90B}{81+9B} \leq C \leq \frac{90B}{81+B}.$$

Lemma 1.1.2

$$A = \frac{BC}{10B - 9C}.$$

Proof.

$$9AC + BC = 10AB.$$
$$A(9C - 10B) = -BC.$$
$$A = \frac{BC}{10B - 9C}.$$

Solution

Now, since A, B, C are positive one-digit integers, according to Lemma 1.1.1 and Lemma 1.1.2, we can gradually narrow down the scope of the answers.

The value of B.	The scope of C, according to Lemma 1.1.1.	The real answers of (A,B,C), according to Lemma 1.1.2.
1	1	none
2	2	none
3	3	none
4	4	none
5	4,5	none
6	4,5,6	(1,6,4) (2,6,5)
7	5,6,7	none
8	5,6,7,8	none
9	5,6,7,8,9	(1,9,5) (4,9,8)

So, eventually, the answers of (n,d) are:

(16,64) (26,65) (19,95) (49,98).

Since our method is logical and systematical, we are quite confidence about our result.

(In fact, I use another method(method 1 below) to somehow recheck my answer.)

Afterword

Here are some methods that I tried before.

Method 1

since

$$\frac{n}{d} = \frac{10A + B}{10B + C} = \frac{A}{C},$$

So

$$Ak = 10A + B,$$
$$Ck = 10B + C.$$

Thus

$$A(k - 10) = B,$$
$$C(k - 1) = 10B.$$

Because A, B, C are positive one-digit integers, $10 < k < 20$.

Moreover, $10 \mid C(k - 1)$, so either $C = 5$ or $5 \mid (k - 1)$.

When c = 5:

$$k - 1 = 2A(k - 10)$$
$$(2A - 1)k = 20A - 1 = 10(2A - 1) + 9$$

So

$$(2A - 1) \mid 9$$

A may be 1 or 2 or 5.

(A,B,C) may be (1,9,5) (2,6,5) and (5,5,5)[ignore]

When $5 \mid (k - 1)$:

Since $10 < k < 20$, k may be 11 or 16.

So (A,B,C) may be (1,6,4) and (x,x,x)[ignore]

This seems like a logic method, but here's an important note:
k no longer need to be an integer as long as Ak and Ck are integers!
Obviously, it is hard to take this note into consideration if we want to use this method since in this method, we bravely assume that k is an integer.

Method 2
Rearrange:

$$C(9A + B) = 10AB.$$

Because A, B, C are positive one-digit integers, either C = 5 or 5 | (9A + B).
It's easy to find out the solution when C = 5:

$$9A + B = 2AB.$$
$$A(9 - 2B) = -B.$$
$$A = \frac{-B}{9 - 2B}.$$

Since A is a positive integer, thus (9 - 2B) is negative, B ≥ 5.
After trials, we can find that (A,B) = (5,5) [ignore] or (1,9).

Now we consider the situation that 5 | (9A + B):
Assume p is an integer that 1 ≤ p ≤ 18,

$$9A + B = 5p,$$
$$pC = 2AB.$$

So

$$pC = 2A(5p-9A).$$
$$-18A^2 + 10pA - pC = 0.$$
$$A = \frac{-10p \pm \sqrt{100p^2 - 72pC}}{-36} = \frac{-5p \pm 2\sqrt{25p^2 - 18pC}}{-18}.$$

Obviously, it's definitely impossible for us to find out all the answers if we want to use this method.

1.(b)

Introduction

In this question, we need to divide all the situations into two parts.

Part 1:
Assuming that n = 10A + B and d = 100B + C,

$$\frac{n}{d} = \frac{10A + B}{100B + C} = \frac{A}{C},$$

where that n ≠ d and B is positive one-digit integer and A, C are positive two-digit integers.

Part 2:
Assuming that n = 100A + 10B + D and d = 100B + 10C + E,

$$\frac{n}{d} = \frac{100A + 10B + D}{100B + 10C + E} = \frac{10A + D}{10C + E},$$

where that n ≠ d and A, B, C, D, E are positive one-digit integers.

Similarly, through observation, we can find that it is even more impossible for us to calculate all the possibilities one by one.
But, thankfully, in this question, we no longer need to find out all the answers -- just to prove that there are some misleading pairs of three-digit integers.
So as long as we find some misleading pairs of three-digit integers, we complete this question.
Here's two ways to do this: one is to use the method we afore-mentioned, and another is to find out answers based on what we get in the previous question

Part 1-similar method:

Some Auxiliary Results

We shall firstly use the afore-mentioned method to consider the part 1.

Lemma 1.2.1

$$\frac{99000\mathrm{B}}{8910+99\mathrm{B}} \le \mathrm{C} \le \frac{99000\mathrm{B}}{8910+10\mathrm{B}}.$$

Proof.

Rearrange Part 1:

$$\mathrm{C}\ (10\mathrm{A} + \mathrm{B}) = \mathrm{A}\ (100\mathrm{B} + \mathrm{C}).$$

$$9\mathrm{AC} + \mathrm{BC} = 100\mathrm{AB}.$$

$$\frac{9}{\mathrm{B}} + \frac{1}{\mathrm{A}} = \frac{100}{\mathrm{C}}.$$

Since $\frac{1}{99} \le \frac{1}{\mathrm{A}} \le \frac{1}{10}$:

$$\frac{100}{\mathrm{C}} - \frac{1}{10} \le \frac{9}{\mathrm{B}} \le \frac{100}{\mathrm{C}} - \frac{1}{99}.$$

$$99000\mathrm{B} - 99\mathrm{BC} \le 8910\mathrm{C} \le 99000\mathrm{B} - 10\mathrm{BC}.$$

From 9900B - 99BC ≤ 8910C:

$$\frac{99000\mathrm{B}}{8910+99\mathrm{B}} \le \mathrm{C}.$$

From 8910C ≤ 990B - 10BC:

$$\mathrm{C} \le \frac{99000\mathrm{B}}{8910+10\mathrm{B}}.$$

Lemma 1.2.2

$$\mathrm{A} = \frac{\mathrm{BC}}{100\mathrm{B} - 9\mathrm{C}}.$$

Proof.

$$9\mathrm{AC} + \mathrm{BC} = 100\mathrm{AB}.$$

$$\mathrm{A}\ (9\mathrm{C} - 100\mathrm{B}) = -\mathrm{BC}$$

$$\mathrm{A} = \frac{\mathrm{BC}}{100\mathrm{B} - 9\mathrm{C}}.$$

Solution

We firstly talk about part 1.

Now, since B is positive one-digit integer and A, C are positive two-digit integers, according to Lemma 1.2.1 and Lemma 1.2.2, we can gradually narrow down the scope of the answers.

The value of B.	The scope of C, according to Lemma 1.2.1.	The real answers of (A,B,C), according to Lemma 1.2.2.
1	11	none
2	22	none
3	33	none
4	43,44	none
5	53,54,55	none
6	63,64,65,66	(16,6,64) (26,6,65)
7	73,74,75,76,77	(21,7,75)
8	82,83,84,85,86,87,88	none
9	91,92,93,94,95,96,97,98,99	(19,9,95) (24,9,96) (49,9,98)

Eventually, the answers of (n,d) are:

(166,664) (266,665) (217,775) (199,995) (249,996) (499,998)

So we successfully prove that there are some pairs of three-digits misleading integers.

Part 2-based on question 1.(a):

Some Auxiliary Results

Lemma 1.3.1 Given gcf(P,Q)=1, kP and kQ are single-digit integers, and

$$\frac{10A+B}{10B+C}=\frac{A}{C}=\frac{P}{Q},$$

so

$$\frac{n}{d}=\frac{100A+10B+kP}{100B+10C+kQ}=\frac{10A+kP}{10C+kQ}.$$

Proof.

Based on the basic properties of fractions:

$$\frac{a}{b}=\frac{c}{d}=\frac{a+c}{b+d}.$$

Thus

$$\frac{A}{C}=\frac{10\ (10A+B)+kP}{10\ (10A+C)+kQ}=\frac{10A+kP}{10C+kQ}.$$

Solution

The mathematical meaning of Lemma 1.3.1 is that: given a pair of two-digit misleading integers, we could find a pair of three-digits misleading integers by adding certain things behind.

$$\frac{16}{64}=\frac{1}{4}=\frac{161}{644}$$
$$\frac{16}{64}=\frac{2}{8}=\frac{162}{648}$$

Similarly, we could also find a lot of pairs of three-digits misleading integers through this method.

The answers of (n,d) can be:

(161,644) (162,648) (262,655) (191,955) (491,982) (492,984) (493,986) (494,988)

So we successfully prove that there are some pairs of three-digits misleading integers.

But, we should notice that, there are far more answers in part 2.

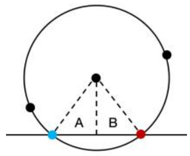
Above all, some pairs of three-digit misleading integers getting from part 1 also satisfies the condition of part 2, but they cannot be found through this method, like:

(166,664) (266,665) (199,995) (499,998).

2.

Some Auxiliary Results

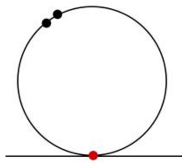
Lemma 4.1 the x-coordinate of the center of Yeppeun circle must be x_3.
Proof.
We can prove this by contradiction.
Like graph, if we assume that the blue point is P_3.
Then, if the x-coordinate of the center of Yeppeun circle is not x_3, then we can find another red point which lies on x-axis, we can prove this by proving that ΔA and ΔB are the same. Understandably, in this situation, C is not tangent to the x-axis, which contraries to our hypothesis.



So, the x-coordinate of the center of Yeppeun circle must be x_3.

Solution

Obviously, Yeppeun circle doesn't always exist, and we could prove this by giving an extreme counterexample, like graph:



But I would like to think it more logically, thus I'll use mathematical method to support my conclusion.

Since the perpendicular bisector of any two points on the circle passes through the center of circle, we could use the perpendicular bisector of P_1P_3 and the perpendicular bisector of P_2P_3 to find the center.
After calculation, we find that:
the equation of the perpendicular bisector of P_1P_3 is

$$y = -\frac{x_1 - x_3}{y_1} x + \frac{x_1^2 - x_3^2}{2y_1} + \frac{y_1}{2},$$

and equation of the perpendicular bisector of P_2P_3 is

$$y = -\frac{x_2 - x_3}{y_2} x + \frac{x_2^2 - x_3^2}{2y_2} + \frac{y_2}{2}.$$

Actually, for any x, we can find a corresponding circle. But the circle we want, according to Lemma 4.1, has the feature that the x-coordinate of the center of Yeppeun circle is x_3, meaning that these two perpendicular bisectors intersect when $x = x_3$.
So

$$-\frac{x_1 - x_3}{y_1} x_3 + \frac{x_1^2 - x_3^2}{2y_1} + \frac{y_1}{2} = -\frac{x_2 - x_3}{y_2} x_3 + \frac{x_2^2 - x_3^2}{2y_2} + \frac{y_2}{2}.$$

$$\frac{(x_2 - x_3)^2 + y_2^2}{2y_2} = \frac{(x_1 - x_3)^2 + y_1^2}{2y_1}.$$

Notice that, since we are given the coordinates of P_1 and P_2, meaning that we know the value of x_1, y_1, x_2 and y_2, we can find the value of x_3.
After trials, we'll find that, given the coordinates of P_1 and P_2, not all x_3 satisfies $x_1 < x_3 < x_2$.
So the conclusion is: Yeppeun circle doesn't always exist.