2024 November

数学月刊

Mathematics Monthly

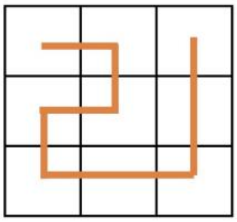
Author :
Guanzhong Yang(student)
Yan Wang(teacher)

Mathematics changes our lives

PREFACE

This month, we are going to talk about the following questions.

(a) On a 3×3 grid, draw a path that starts in the center of a square in the corner of the grid, and repeatedly moves from the center of one square to the center of an adjacent square, without visiting any squares twice, and such that every square in the grid is included. For example, the following is a path that meets these conditions:



What are the possible ending positions for a path drawn according to these rules? What if you started from an edge square or the center square of the grid? Explain.

(b) Repeat the above problem for a 4 × 4 grid.

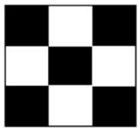
(c) Can you describe what happens in general for an n×n grid where n is a positive integer?

This is the admission question from 2024 SUMaC Program. If you have other brilliant ideas, email to anmiciuangray@163.com for surprising rewards!

(a)

Introduction

We can color squares black and white alternatively, as the pattern shown in the picture.



Notice that, since we can only move upwards, downwards, rightwards or leftwards, we must pass through the squares in the order of ' ..., white, black, white, black, ... '.

Solution

If we start from the corner of the grid, which is a black square, we'll pass through the squares in the order of ' black, white, black, white, black, white, black, white, black ', which means that we'll end up at a black square.
After trials, we find that we do can end up at any black square, except the starting square.

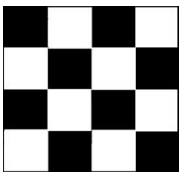
If we start from the center of the grid, which is a black square, similarly,we'll end up at a black square.
After trials, we find that we do can end up at any black square, except the starting square.

If we start from the edge of the grid, which is a white square,we'll pass through the squares in the order of ' white, black, white, black, white, black, white, black '.
We notice that we'll end up standing at black square and remaining one black square, so we cannot achieve the request.

(b)

Introduction

We can color squares black and white alternatively, as the pattern shown in the picture.



Solution

If we start from the corner of the grid, which is a black(or white) square, we'll end up at a white(or black) square.
After trials, we find that we do can end up at any white(or black) square.

If we start from the center of the grid, which is a black(or white) square, we'll end up at a white(or black) square.
After trials, we find that we do can end up at any white(or black) square.

If we start from the edge of the grid, which is a black(or white) square, we'll end up at a white(or black) square.
After trials, we find that we do can end up at any white(or black) square.

(c)

Introduction

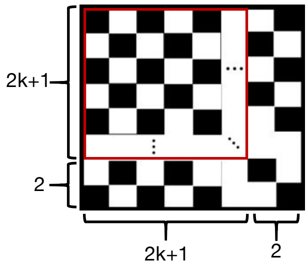
As what we can see, understandably, there are two conditions: either n is odd or n is even. Besides, we are asked to prove the result when n is an arbitrary positive number, meaning that we are asked to prove the result for all positive integers n.
So, the first method that comes into my mind is to use mathematical induction to prove it.

What we need to do is to prove:
If we make the square in the top left corner black, and color squares black and white alternatively.
When n is odd, we can start from any black square and end up at any black square.
When n is even, we can start from any black(or white) square and end up at any white(or black) square.

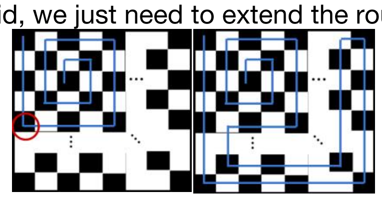
Part 1 when n is odd

Solution

Since we have prove the condition when n = 1, now let us just focus on the conditions when n = 2k + 1 and n = 2(k + 1) + 1.
Assuming the conclusion is true for n = 2k + 1, then, for n = 2(k + 1) + 1, we color the grid in the pattern shown in the picture.



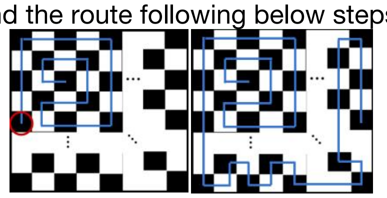
If we start at any black square inside the (2k + 1) × (2k + 1) grid(red grid) and end at any black square inside the red grid, we just need to extend the route following below steps:



(1) (2)

We firstly find a way passing through all the squares inside the red grid, like graph(1).
Notice than, we must turn around in at least two corners of the red grid, otherwise we cannot pass through all corners.
So, we can just simply extend the route at the turning point, like graph(2).

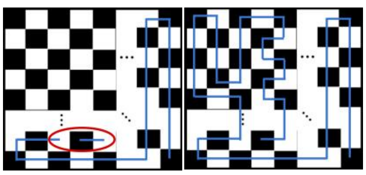
If we start at any black square inside the red grid and end at any black square outside the red grid, we just need to extend the route following below steps:



(3) (4)

We firstly find a way passing through all squares inside the red grid and ending at a corner, like graph(3).
Then we can extend the route, starting at this corner and ending at the square we want, like graph(4).

If we start at one black square outside the red grid and end at one black square outside the red grid, and at least one of these two black squares is not at the corner or the edge of the (2(k + 1) + 1) × (2(k + 1) + 1) grid(big grid), we just need to extend the route following below steps:

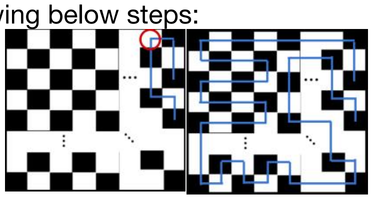


(5) (6)

We firstly focus on the starting or ending square which is not at the corner or the edge of the big grid. Empty the two white squares outside the red grid, and which are at the same row(or column) with the black square.
We then find the way from the starting or ending square which is at the corner or the edge of the big grid, to one of the empty white square, passing through all squares outside the red

grid but inside the big grid, except the starting or ending square which is not at the corner or the edge of the big grid and the other empty white square, like graph(5).
Then, we have to find two black squares, inside the red grid, each of which is next to one of the two empty white squares. Subsequently, find a way starting with one of these black square and ending at the other one, like graph(6).
Eventually, connect the routes together.

If we start at one black square outside the red grid and end at one black square outside the red grid, and these two black squares are at the corner or the edge of the big grid, we just need to extend the route following below steps:



(7) (8)

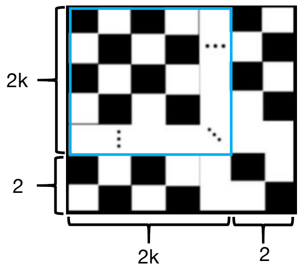
We firstly find a short way, starting at one of the black square and ending at the other black square, only passing through some(not necessarily all) squares outside the red grid.
We make the route passing through at least one inner corner, like graph(7).
Then we could just extend the route at the corner, like graph(8).

So, since n = 1 is true and n = 2k + 1 → n = 2(k + 1) + 1, by mathematical induction, when n is odd, we can start from any black square and end up at any black square.

Part 2 when n is even

Solution

Since we have prove the condition when n = 2, now let us just focus on the conditions when n = 2k and n = 2(k + 1).
Assuming the conclusion is true for n = 2k, then, for n = 2(k + 1), we color the grid in the pattern shown in the picture.



Similarly, we can still prove this condition by dividing it into four similar situations:
①we start at any black(or white) square inside the 2k × 2k grid(blue grid) and end at any white(or black) square inside the blue grid.
②we start at any black(or white) square inside the blue grid and end at any white(or black) square outside the blue grid.
③we start at one black(or white) square outside the blue grid and end at one white(or black) square outside the blue grid, and at least one of these two squares is not at the corner or the edge of the 2(k + 1) × 2(k + 1) grid(bigger grid).
④we start at one black(or white) square outside the blue grid and end at one white(or black) square outside the blue grid, and these two squares are at the corner or the edge of the bigger grid.

Actually, all these situations can be proven by the similar processes we used in part 1.
So, since n = 2 is true and n = 2k → n = 2(k + 1), by mathematical induction, when n is even, we can start from any black(or white) square and end up at any white(or black) square.

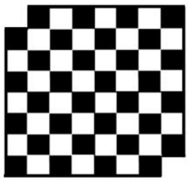
Afterword

I firstly encountered this method in primary school. At that time, I was crazy about a series of comic books about mathematics, called *Fantasy Math War*.
In one of this series, there is a question:

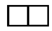
You have a chessboard with 8 × 8 squares. Now you remove the squares in the top left and bottom right corners, so 62 squares remained. Assuming that you have 31 rectangles with 1 × 2 squares, can you cover the modified chessboard with these 31 rectangles?

In the book, writer provided us with a brilliant way to resolve this question:

We can color the chessboard black and white alternatively, as the pattern shown in the picture.



modified chessboard



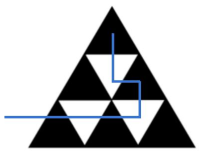
rectangle

So, there are 32 black squares and 30 white squares.
We could notice that, since every rectangle can only cover two adjacent squares, meaning that every rectangle can only cover one black square and one white square.
Whereas, there are 32 black squares and 30 white squares, meaning that it is impossible for us to achieve the requirement.

The second time is when I was preparing for BMO. I found another question using this method. Here's the question.

The equilateral triangle ABC has sides of integer length N. The triangle is completely divided (by drawing lines parallel to the sides of the triangle) into equilateral triangular cells of side length 1. A continuous route is chosen, starting inside the cell with vertex A and always crossing from one cell to another through an edge shared by the two cells. No cell is visited more than once. Find the greatest number of cells which can be visited.

The key of this question is to color the cell black and white alternatively and notice that we could only pass the white cells and black cells alternatively.



Thanks to these two experiences, at the time I saw this question, this method just immediately came into my mind.