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Mathematics changes our lives

PREFACE

This month, we are going to talk about the following questions.

Q1

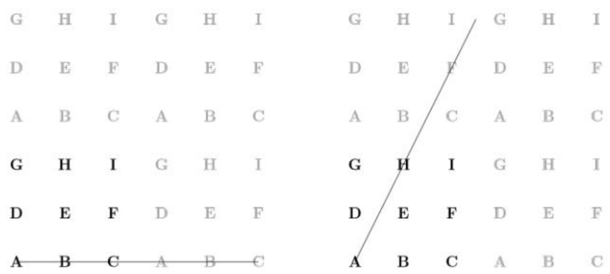
A monkey has filled in a 3 × 3 grid with the numbers 1, 2, . . . , 9. A cat writes down the three numbers obtained by multiplying the numbers in each horizontal row. A dog writes down the three numbers obtained by multiplying the numbers in each vertical column. Can the monkey fill in the grid in such a way that the cat and dog obtain the same lists of three numbers? What if the monkey writes the numbers 1, 2, . . . , 25 in a 5×5 grid? Or 1, 2, . . . , 121 in a 11 × 11 grid? Can you find any conditions on n that guarantee that it is possible or any conditions that guarantee that it is impossible for the monkey to write the numbers 1, 2, . . . , n^2 in an n × n grid so that the cat and the dog obtain the same lists of numbers?

Q2

Consider the grid of letters that represent points below, which is formed by repeating the bold 3 × 3 grid. (The points are labeled by the letters A through I.)

G	H	I	G	H	I
D	E	F	D	E	F
A	B	C	A	B	C
G	**H**	**I**	G	H	I
D	**E**	**F**	D	E	F
A	**B**	**C**	A	B	C

We'll draw lines through the bottom left point (the bold A) and at least one other bold letter, then see what set of letters the line hits. We've drawn two example lines for the repeating 3 × 3 grid below. For the example on the left, the set of letters is {A, B, C}, and for the right, the set of letters is {A, F, H}.



Assuming the 3 × 3 grid repeats forever in every direction, do any of these lines ever pass through more than 3 different letters? Can you get the same set of letters from two different lines? Find the four different sets of letters that you can get from drawing lines in this grid.

What would happen if you had a repeated 5×5 grid of letters (and still had to draw lines through the bottom left point and at least one other bold point)? Can you predict what would happen with a repeated 7 × 7 grid? Does your prediction also work for 6 × 6? Can you justify your predictions?

Q3

For an arbitrary set of numbers S, define S + S to be the set of all sums x + y where x and y are in S. Let A, B, C, D be sets of integers such that (A ∪ B) + (A ∪ B) = (C ∪ D) + (C ∪ D). For example, if A = B = C = D = {1, 2} then

$$(A \cup B) + (A \cup B) = (C \cup D) + (C \cup D) = \{1 + 1, 1 + 2, 2 + 1, 2 + 2\} = \{2, 3, 4\}.$$

This is the admission question from 2024 PROMYS Program and 2024 SUMaC Program. If you have other brilliant ideas, email to anmiciuangray@163.com for surprising rewards!

1.

Some Auxiliary Results

Lemma 7.1 If we define the 'unique primes' as the numbers that are co-prime with any other numbers and define that 1 is not an 'unique prime', then if there are two 'unique primes' in the same row(or column), the condition of question cannot be satisfied.

Proof.

I firstly would like to give you some examples of 'unique primes'.

For example.
'unique primes' in 1~9 are 5, 7.
'unique primes' in 1~25 are 13, 17, 19, 23.
'unique primes' in 1~121 are 61, 67, 71, 73, 79, 83, 89, 97, 101, 103, 107, 109, 113.

We then analysis the question: When we calculate the multiplication of numbers, we are essentially multiplying the prime factors of the numbers.
For every unique prime, it has one prime factor that is unique among all the numbers.
If there are two 'unique primes' in the same row(or column), there will be two unique prime factors in this row(or column).
But in this way, there will not exist a column(or row), where there are the same two unique prime factors, because two points will fix a line.
Not having the same prime factors is just equal to not having the same multiples.
That be said, once there are two 'unique primes' in the same row or column, the condition of question cannot be satisfied.

Solution

Based on Lemma 7.1, we can find that, in 3×3 grids and 5×5 grids, we can put those 'unique primes' in ways that there are not two 'unique primes' in the same row(or column), which means that these two situations can possibly be satisfied.

For example.

5	4	6
8	7	1
3	2	9

In 3×3 grids, we can put the numbers as this.
Disappointingly, I fail to find out the way of putting the numbers into 5×5 grids, I think that this process involves much calculation as well as luck. But, I deem that we can put the numbers into 5×5 grids if we are lucky.

Whereas, in 11×11 grids, we cannot put those 'unique primes' in ways that there are not two 'unique primes' in the same row(or column), which means that this situation is definitely not satisfied.
Similarly, we can check the situation of n×n grids by using the same method.

2.

Introduction

There is a brilliant idea to resolve this problem, and it can explain almost all situations no matter how big the grid is. In order to explain this idea to you, we need to firstly transform this question and express it in a mathematical way.
In a n×n grid, for a letter which locates at {p,q}, where p represents that the letter is in pth column and q represents that the letter is in qth row. We use (x,y) to express this kind of letter, where

$$x \equiv p-1 \pmod{n},\ y \equiv q-1 \pmod{n}$$

For example.
In a 3×3 grid, A is always (0,0), D is always (0,1), I is always (2,2).

And for every line, we can express it as [X,Y], where X represents that the difference of columns of two adjacent letters on line is X, Y represents that the difference of rows of two adjacent letters on line is Y.

For example.
In a 3×3 grid, the line passing through A→D can be expressed as [0,1].
Similarly, the line passing through A→H can be expressed as [1,2].

Now, given the expression of the line, we can easily find out the letters it passing through.

For example.
In a 3×3 grid, the line with expression [2,1] passing through letters with coordinates:

$$\{1,1\} \to \{3,2\} \to \{5,3\} \to \{7,4\} \to \{9,5\} \to \{11,6\} \to \cdots$$

We then find out the corresponding letters:

$$(0,0) \to (2,1) \to (1,2) \to (0,0) \to (2,1) \to (1,2) \to \cdots$$

$$A \to F \to H \to A \to F \to H \to \cdots$$

Some Auxiliary Results

Lemma 8.1 If a line, after passing through a series of letters, passes the same kind of letter which is the initial letter of the series, then the series starts again.
Proof.
The same initial point and the same expression of the line, understandably, the series of letters passed through will be the same.

For example.
In a 3×3 grid, the line with expression [2,1] passing through letters A → F → H, after that, it passes through A → F → H again and again.

Lemma 8.2 In a n×n grid, the number of letters that a line can passing through is equal to or less than n.
Proof.
Assume that the expression of the line is [a,b], then, in a n×n grid, the coordinates of the letters passed through are:

$$\{1,1\} \to \{1+a,1+b\} \to \{1+2a,1+2b\} \to \cdots \to \{1+na,1+nb\} \to \cdots$$

We then find out the corresponding letters:

$$[0,0] \to [X_a,Y_b] \to [X_{2a},Y_{2b}] \to \cdots \to [0,0] \to \cdots$$

Based on Lemma 8.1, the series starts anew from [0,0], meaning that there are n or less than n different letters in the series.

Lemma 8.3 In a n×n grid, the line with expression of [a,b] passes through the same letter as the line with expression of [a+wn,b+vn], where w and v are integers.
Proof.
The line with expression of [a,b] passes through:

$$\{1,1\} \to \{1+a,1+b\} \to \{1+2a,1+2b\} \to \cdots \to \{1+na,1+nb\} \to \cdots$$
$$[0,0] \to [X_a,Y_b] \to [X_{2a},Y_{2b}] \to \cdots \to [0,0] \to \cdots$$

The line with expression of [a+wn,b+vn], where w and v are integers, passes through:

$$\{1,1\} \to \{1+a+wn,1+b+vn\} \to \cdots \to \{1+na+nwn,1+nb+nvn\} \to \cdots$$
$$[0,0] \to [X_a,Y_b] \to [X_{2a},Y_{2b}] \to \cdots \to [0,0] \to \cdots$$

Lemma 8.4 In a n×n grid, given that the expression of the line is [a,b], then the line with the expression of [ka,kb], where gcf(k,n) = 1, will pass through the same set of letters as the line with expression of [a,b] does.
Proof.
The line with expression of [a,b] passes through:

$$\{1,1\} \to \{1+a,1+b\} \to \{1+2a,1+2b\} \to \cdots \to \{1+na,1+nb\} \to \cdots$$
$$[0,0] \to [X_a,Y_b] \to [X_{2a},Y_{2b}] \to \cdots \to [0,0] \to \cdots$$

The line with expression of [ka,kb], where gcf(k,n) = 1, passes through:

$$\{1,1\} \to \{1+ka,1+kb\} \to \{1+2ka,1+2kb\} \to \cdots \to \{1+nka,1+nkb\} \to \cdots$$
$$[0,0] \to [X_{ka},Y_{kb}] \to [X_{2ka},Y_{2kb}] \to \cdots \to [0,0] \to \cdots$$

Notice that, since gcf(k,n) = 1, then $[X_{ika},Y_{ikb}] \neq [0,0]$, where i = 1,2,3,⋯,n-1.
That be said, the yellow part has the same set of the letters as the red part, but the order might be difference.
In conclusion, based on Lemma 8.1, they passes through the same set of letters.

For example.
The line with expression of [2,1] passes through:

$$\{1,1\} \to \{3,2\} \to \{5,3\} \to \{7,4\} \to \{9,5\} \to \{11,6\} \to \cdots$$
$$(0,0) \to (2,1) \to (1,2) \to (0,0) \to (2,1) \to (1,2) \to \cdots$$
$$A \to F \to H \to A \to F \to H \to \cdots$$

The line with expression of [1,2](actually it should be [4,2], but their effect are the same based on Lemma 8.3) passes through:

$$\{1,1\} \rightarrow \{2,3\} \rightarrow \{3,5\} \rightarrow \{4,7\} \rightarrow \{5,9\} \rightarrow \{6,11\} \rightarrow \cdots$$
$$(0,0) \rightarrow (1,2) \rightarrow (2,1) \rightarrow (0,0) \rightarrow (1,2) \rightarrow (2,1) \rightarrow \cdots$$
$$A \rightarrow H \rightarrow F \rightarrow A \rightarrow H \rightarrow F \rightarrow \cdots$$

Solution

Now, we can resolve the questions easily.

Based on Lemma 8.2, in a n×n grid, the number of letters that a line can passing through is equal to or less than n, no matter n is 3, 5, 6 or 7.

Based on Lemma 8.3 and Lemma 8.4, we can easily find out the lines which pass through the same set of letters, like in a 3×3 grid, the lines with the expressions of [1,2] and [2,1] pass through the same set of letters; in a 5×5 grid, the lines with the expressions of [1,3], [2,1], [3,4] and [4,2] all pass through the same set of letters.

In terms of finding the exact sets of letters, we can exactly find them step by step, like in a 3×3 grid, 4 sets of letters are ADG, AFH, AEI and ABC.

3.

Some Auxiliary Results

Lemma 2.1 If one set B, whose size is less than or equal to 5, can be express as A + A, where A is a set, then we can only find one set A satisfying A + A = B.
Proof.
Assume the set A is

$$\{ a_1 , a_2 , \dots , a_n \},$$

where $a_1 < a_2 < \dots < a_n$.
Assume the set B is

$$\{ b_1 , b_2 , \dots , b_m \},$$

where $b_1 < b_2 < \dots < b_m$ and $m \leq 5$.
Then

$$2a_1 = b_1,$$

because

$$a_1 + a_A > 2a_1x$$

where $A > 1$, and b_1 is the minimum value in set B.
Similarly,

$$a_1 + a_2 = b_2.$$
$$a_{n-1} + a_n = b_{m-1}.$$
$$2a_n = b_m.$$

Now we could solve these equations one by one to find the elements in A, and A, understandably, is fixed.

Solution

So if we bravely assume that

$|(A \cup B) + (A \cup B)| = |(C \cup D) + (C \cup D)| = |(A \cup C) + (A \cup C)| = |(B \cup D) + (B \cup D)| \leq 5.$

Then, since

$$(A \cup B) + (A \cup B) = (C \cup D) + (C \cup D) = (A \cup C) + (A \cup C) \neq (B \cup D) + (B \cup D),$$

according to Lemma 2.1, we can get

$$A \cup B = C \cup D = A \cup C \neq B \cup D.$$

The most obvious way to achieve this is

$$A = C,$$
$$B \subset A,$$
$$D \subset A,$$
$$|A| = |C| = 3,$$
$$B \cup D \neq A.$$

For instance:

$$A = C = \{1,2,3\},$$
$$B = \{1,2\},$$
$$D = \{1\}.$$

So

$$(A \cup B) + (A \cup B) = (C \cup D) + (C \cup D) = (A \cup C) + (A \cup C)$$
$$= \{1,2,3\} + \{1,2,3\} = \{2,3,4,5,6\},$$
$$(B \cup D) + (B \cup D) = \{1,2\} + \{1,2\} = \{2,3,4\}.$$

Hence

$$(A \cup B) + (A \cup B) = (C \cup D) + (C \cup D) = (A \cup C) + (A \cup C) \neq (B \cup D) + (B \cup D).$$

Afterword

At the beginning I want to prove the general hypothesis, where m is not necessary less than or equal to 5, but I soon realized:

$$\{1,2,3,4,5,6,7\} + \{1,2,3,4,5,6,7\} = \{1,2,3,5,6,7\} + \{1,2,3,5,6,7\}$$