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Mathematics Monthly

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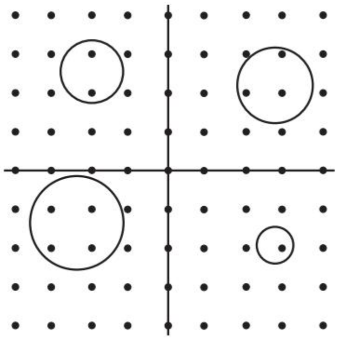
Mathematics changes our lives

PREFACE

This month, we are going to talk about the following questions.

Q1

A lattice point is a point (x, y) in the plane, both of whose coordinates are integers. It is easy to see that every lattice point can be surrounded by a small circle which excludes all other lattice points from its interior. It is not much harder to see that it is possible to draw a circle which has exactly two lattice points in its interior, or exactly 3, or exactly 4, as shown in the picture below



Do you think that for every positive integer n there is a circle in the plane which contains exactly n lattice points in its interior? Justify your answer.

Q2

The set S contains some real numbers, according to the following three rules.

(i) $\frac{1}{1}$ *is in S.*

(ii) If $\frac{a}{b}$ *is in S, where* $\frac{a}{b}$ *is written in lowest terms (that is, a and b have highest common factor 1), then* $\frac{b}{2a}$ *is in S.*

(iii) If $\frac{a}{b}$ *and* $\frac{c}{d}$ *are in S, where they are written in lowest terms, then* $\frac{a+c}{b+d}$ *is in S.*

These rules are exhaustive: if these rules do not imply that a number is in S, then that number is not in S. Can you describe which numbers are in S? For example, by (i), $\frac{1}{1}$ *is in S. By (ii), since* $\frac{1}{1}$ *is in S,* $\frac{1}{2}$ *is in S. Since both* $\frac{1}{1}$ *and* $\frac{1}{2}$ *are in S, (iii) tells us* $\frac{2}{3}$ *is in S.*

Q3

Let P_0 be an equilateral triangle of area 10. Each side of P_0 is trisected into three segments of equal length, and the corners of P_0 are snipped off, creating a new polygon (in fact, a hexagon) P_1. What is the area of P_1? Now repeat the process to P_1 — i.e. trisect each side and snip off the corners — to obtain a new polygon P_2. What is the area of P_2? Now repeat this process infinitely to create an object P_∞. What can you say about the shape P_∞? What is the area of P_∞?

This is the admission question from 2024 PROMYS Program. If you have other brilliant ideas, email to anmiciuangray@163.com for surprising rewards!

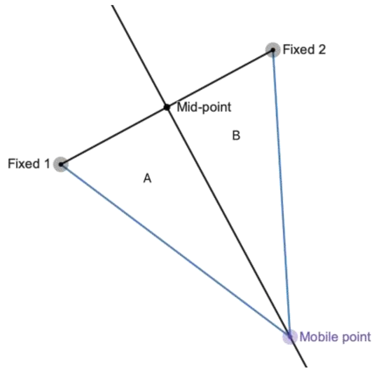
1

Some Auxiliary Results

Theory 4.1 There are infinite lines on a plane.

Lemma 4.2 Given two fixed points on a plane, if there is another point satisfying that the distances between this point and each of two fixed points are the same, then the point must be on the perpendicular bisector of the line with these two fixed points as vertices.

Proof.



We can prove it through congruent triangles A and B.

Lemma 4.3 Given finite fixed points on a plane, we could find another point, satisfying that the distances between this point and every fixed point are different.

Proof.

We firstly draw all the lines with arbitrary two fixed point as vertices, and then draw the perpendicular bisectors of these lines. Then, based on Lemma 4.2, as long as the point we found isn't on any of these perpendicular bisectors, the distances between this point and every fixed point are different.

According to Theory 4.1, there are infinite lines on a plane, meaning that we fail to use finite lines to cover a plane. Thus, we do can find a point satisfying that the distances between this point and every fixed point are different.

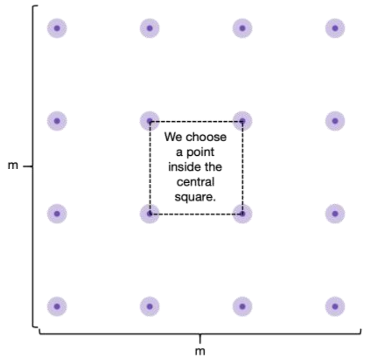
Theory 4.4 If a point is inside a circle with radius r, then the distance between the point and the center of the circle must be less than r.

Solution

Given the value of n, we can find the minimum value of even number m satisfying

$$m \geq n.$$

we find a region like below.



Then, we just need to find a point inside the central square, satisfying that the distances between this point and every fixed lattice point are different.
Based on Lemma 4.3, we are confident that this point exists.
Since the distances between this point and every fixed lattice point are different, if we consider this point as the center of the circle, we can thus control the radius of circle in order to make different numbers of lattice points inside the circle.
If we sort the lengths of distances between this point and every fixed lattice point in ascending order, namely

$$r_1, r_2, r_3, \cdots, r_n, r_{n+1}, \cdots, r_{m^2}.$$

According to Theory 4.4, in order to make n lattice points inside the circle, we just need to let the radius of the center r be

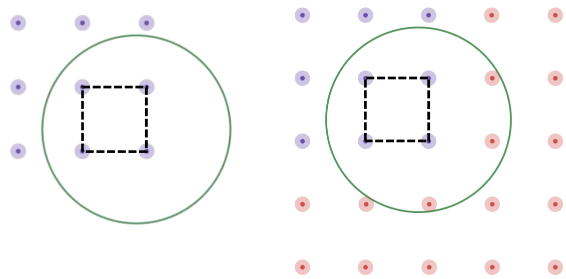
$$r_n < r \leq r_{n+1}.$$

That be said, for every positive integer n, there is a circle in the plane which contains exactly n lattice points in its interior.

Afterword

The reason why we want to choose a point inside the central square is that, otherwise, there might be other points which are inside the circle we find.

For example.

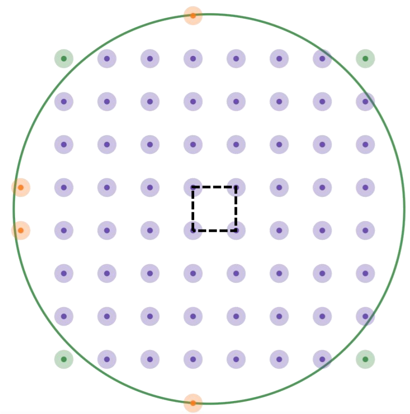


In the above example, we merely want to find a circle with 4 lattice points inside, but we find a circle with a center inside the dashed square.
Although if we merely consider the 3 ×3 points(blue points), it seems like that we successfully find what we want, if we consider the other points(red points), it points out the fact that we fail to find the circle we want.

Thus, in order to avoid this situation, we must find the center inside the central square.

But why let m ≥ n? Well, in fact, m no longer needs to be greater than or equal to n, as long as using m×m points can avoid the aforementioned problem.

For example.



We want to find 60 lattice points inside a circle. But if we find this circle using 8×8 points, we will fail because if we merely consider this 8×8 points, some points(green points) are outside the circle, whereas some points outside these 8×8 points(orange points) are actually inside the circle.

This is because the distance from the center to the corner of the n×n points is too far, so we should not plan to include these ‘far’ points into the circle.
Thus, by finding a circle inside the m×m points, there are just enough points on single row or column, no to mention the number of points inside the circle with the center inside the central square, whose diameter is m-1 ≤ r ≤ m, meaning that we are sure that the aforementioned problem can be avoided.

2.

Some Auxiliary Results

Lemma 5.1 All numbers in *S* can be expressed as results of finite times of operation relating to $\frac{1}{1}$ and $\frac{1}{2}$.

Proof.

Understandably, because that how we get a number $\frac{p}{q}$, if it exists in *S*.

$$\frac{p}{q}=\frac{\overbrace{1+1+1+\cdots+1}^{(n+m)\text{ times of 1}}}{\underbrace{1+1+\cdots}_{n\text{ times of 1}}+\underbrace{2+2+\cdots}_{m\text{ times of 2}}}.$$

Lemma 5.2 All numbers in *S* are smaller than or equal to 1, and only $\frac{1}{1}$ and $\frac{2}{2}$ are equal to 1.

Proof.

Except for $\frac{1}{1}$ and $\frac{2}{2}$, based on Lemma 5.1, other numbers in *S* can be expressed as results of the finite times of operation relating to $\frac{1}{1}$ and $\frac{1}{2}$, and the operation consisting at least one $\frac{1}{2}$, making the final results smaller than 1.

Lemma 5.3 All numbers in *S* are greater than or equal to $\frac{1}{2}$, and only $\frac{1}{2}$ is equal to $\frac{1}{2}$.

Proof.

Except for $\frac{1}{2}$, based on Lemma 5.1, other numbers in *S* can be expressed as results of the finite times of operation relating to $\frac{1}{1}$ and $\frac{1}{2}$, and the operation consisting at least one $\frac{1}{1}$, making the final results greater than $\frac{1}{2}$.

Lemma 5.4

$$\gcd\text{f}(a,a-1) = 1$$

Proof.

This can be, to some extent, proved by Euclidean Algorithm.

Lemma 5.5

$$\text{gcf}(a,2a-1) = 1$$

Proof.

Assume

$$\text{gcf}(a,2a-1) = m,$$

where $m \neq 1$.

Then

$$a \equiv 0 \pmod{m},$$

But

$$2a - 1 \equiv -1 \pmod{m}.$$

So by contradiction, gcf(a,2a-1) = 1.

Solution

When solving a problem which involves a plenty of different terms, it is always helpful to find a few terms in advance:

$$\boxed{\frac{1}{1}, \frac{1}{2},} \frac{2}{2}$$
$$\boxed{\frac{2}{3}, \frac{3}{4},} \frac{4}{6}$$
$$\frac{5}{7}, \frac{7}{10}, \frac{10}{14}$$
$$\cdots$$

graph 5.1

Here, we have two ways to go: to find out which numbers are in *S*, which you will see in the afterword, or to find out which numbers are not in *S*.
Since I fail to use the former method to find out the answer, I'll begin with the latter.

We may notice that, after several trials, the numbers in *S* are smaller than or equal to 1 and greater than or equal to $\frac{1}{2}$, which can be proven by Lemma 5.2 and Lemma 5.3. In this way, we are sure that all numbers are in the range of $[\frac{1}{2},1]$(only $\frac{1}{2} = \frac{1}{2}$ and $\frac{1}{1}, \frac{2}{2} = 1$).
So now, we just need to prove that all numbers, which are in the range of $[\frac{1}{2},1]$(only $\frac{1}{2} = \frac{1}{2}$ and $\frac{1}{1}, \frac{2}{2} = 1$), are in *S*.

denominator	numbers
1	$\frac{1}{1}$
2	$\frac{1}{2}, \frac{2}{2}$
3	$\frac{2}{3}$
4	$\frac{3}{4}$
5	$\frac{3}{5}, \frac{4}{5}$

After trials, we can find out the general pattern of it:
①For numbers $\frac{a}{b}$ where a = 2b - 1, based on Lemma 5.5, we can get them through the operations between the operations of $\frac{a-1}{b-2}$ and $\frac{1}{2}$.
②For most of the numbers $\frac{a}{b}$ in the range of $[\frac{1}{2},1]$(only $\frac{1}{2} = \frac{1}{2}$ and $\frac{1}{1}, \frac{2}{2} = 1$), where gcf(a-1,b-1) = 1, we can get them through the operations of $\frac{a-1}{b-1}$ and $\frac{1}{1}$.
③For some numbers $\frac{a}{b}$ in the range of $[\frac{1}{2},1]$(only $\frac{1}{2} = \frac{1}{2}$ and $\frac{1}{1}, \frac{2}{2} = 1$), where gcf(a-1,b-1) $\neq$ 1, based on Lemma 5.4, we can get them through the operations of $\frac{p'}{p}$ and $\frac{q-1}{q}$, where p is the possible minimum prime and p' and q are both integers.

For example.

$$①\frac{4}{7} = \frac{3+1}{5+2}$$
$$②\frac{6}{7} = \frac{5+1}{6+1}.$$
$$③\frac{5}{7} = \frac{2+3}{3+4}$$

In conclusion, all fractions that are in the range of $[\frac{1}{2},1]$(only $\frac{1}{2} = \frac{1}{2}$ and $\frac{1}{1}, \frac{2}{2} = 1$) are in *S*.

Afterword

Piece of Process Using Method 2

Actually when I firstly dealt with type ③ fractures, I wanted to express them as the results of the operations of $\frac{m'}{m}$ and $\frac{n-1}{n}$, where m is no necessary a prime, but m and m' and n are all integers.

Since I knew that, based on Lemma 5.4, $\frac{n-1}{n}$ is absolutely the lowest form, I wanted to find a way to ensure that the $\frac{m'}{m}$ is also the lowest form.

Firstly, I thought that it might involve some processes about Congruence theorem.

If

	m'	m
mod 2	r_1'	r_1
mod 3	r_2'	r_2
mod 5	r_3'	r_3
⋮	⋮	⋮

I just needed to prove that there exists a pair of m' and m, which $(r_n', r_n) \neq (0,0)$ and n = 1,2,3,⋯.

But this method is still quite tough. Even worse, how should I ensure that $\frac{m'}{m}$ is in the range of $[\frac{1}{2},1]$(only $\frac{1}{2}=\frac{1}{2}$ and $\frac{1}{1}$, $\frac{2}{2}=1$)?

Emmmm, this idea isn't suitable, maybe I should find another one.

I then re-considered the goal we want to achieve: $\frac{m'}{m}$ is the lowest form, meaning that m' and m are co-prime.

Co-prime? And m is greater than m'? So what about making m be a prime: in this way, m' and m will definitely be co-prime!

Yeah! That's a brilliant strategy! But could we ensure that $\frac{m'}{m}$ is in the range of $[\frac{1}{2},1]$(only $\frac{1}{2}=\frac{1}{2}$ and $\frac{1}{1}$, $\frac{2}{2}=1$)?

Well, it is hard for us to know the values of primes when they are really large, but we know the values of primes when they are smaller: in this way, by making m as smaller as possible, we can successfully find out a $\frac{m'}{m}$ satisfying the condition.

Method 1 Find General Forms, Use General Forms, Find More General Forms

I firstly wanted to express some of the numbers in *S* in general form, and then use those numbers expressed in general form to derive more terms.

In graph 5.1, the numbers in the leftmost column can be expressed as:

$$\frac{2a_{n-1}+a_{n-2}}{3a_{n-1}+a_{n-2}}$$

where a_n is the numerator of the nth number in the leftmost column.

Similarly, we can find out the corresponding expression of other numbers in graph and can then be used to find other numbers.

But, obviously, this method is still too complex so I eventually gave up.

3.
Some Auxiliary Results

Theory 6.1 A circle is essentially a polygon with infinite edges.

Lemma 6.2 The points which are the mid-points of the sides of equilateral triangle, called M_1, M_2 and M_3, always on the of edges of the new polygon and acts as mid-points during the process.

Proof.

During the process, every side is divided into three part and the middle part remains, meaning that, no matter how short the edge is, the point which is the mid-point of the side of equilateral triangle always on the of edges of the new polygon and acts as mid-points.

Theory 6.3 Given the locations of three points on a circle, we can then determine that circle.

Solution

Thankfully, the areas of P_1 and P_2 can be calculated by us step by step.
In order to simplify, we firstly make the edge of equilateral triangle be 1.

Transition	The number of removed triangles	Length of two sides of the removed triangles	The angle between the two sides	Total removed area, calculating by $\frac{absinC}{2}$	Length of new side, calculating by $\sqrt{a^2+b^2-2abcosC}$
$P_0 \to P_1$	3	$(\frac{1}{3}, \frac{1}{3})$	60°	$\frac{\sqrt{3}}{12}$	$\frac{1}{3}$
$P_1 \to P_2$	6	$(\frac{1}{9}, \frac{1}{9})$	120°	$\frac{\sqrt{3}}{54}$	$\frac{\sqrt{3}}{9}$
$P_2 \to P_3$	12	$(\frac{1}{27}, \frac{\sqrt{3}}{27})$	150°	$\frac{\sqrt{3}}{243}$	Who cares?

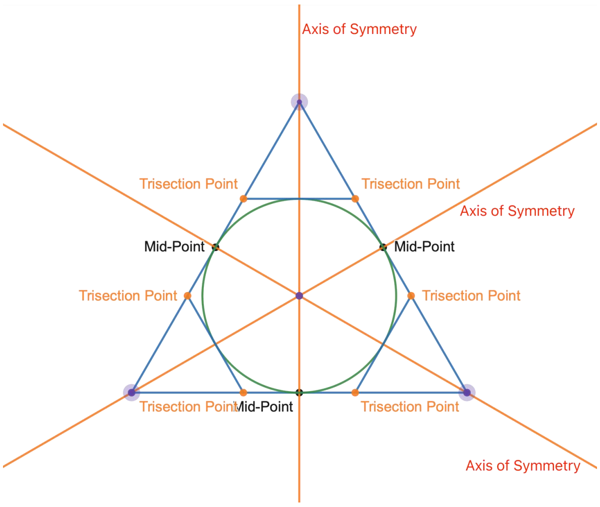
So, we proportionally expand the areas.

$$P_0: \frac{\sqrt{3}}{4} \to 10$$

$$P_1: (\frac{\sqrt{3}}{4} - \frac{\sqrt{3}}{12}) \to \frac{20}{3}$$

$$P_2: (\frac{\sqrt{3}}{4} - \frac{\sqrt{3}}{12} - \frac{\sqrt{3}}{54}) \to \frac{160}{27}$$

Now let's focus on the next two questions.
Understandably, during the process, the edges will double again and again, resulting in a circle, which we can check through Theory 6.1.
But why is it a circle?



We put P_0, P_1 and P_∞ at the same picture and add three axes of symmetry, just like above.
Actually, these three axes of symmetry always exist because the operations that we do(cutting edges) is always symmetric.
Thus, the final shape shall be a circle.

But how to find the area of the circle?

Well, at first I thought that the removed areas will be a geometric series, since $\frac{\sqrt{3}}{12}, \frac{\sqrt{3}}{54}, \frac{\sqrt{3}}{243}$ does form a geometric series. But, obviously, it's hard to prove whether it is right or not.
Then I wanted to find the pattern of the change in the circumference, since once we fix the perimeter of a circle, its area is also fixed. But also, it is hard to achieve this goal because this involved too much calculation.
What else could be used to calculate the area of a circle? Well, its radius! Based on Lemma 6.2 and Theory 6.3, we can fix the circle thanks to three points M_1, M_2 and

M_3 and its radius, which is just the radius of the inscribed circle of the equilateral triangle, $\frac{1}{2\sqrt{3}}$.

We proportionally expand the area.

$$P_0: \frac{\sqrt{3}}{4} \rightarrow 10$$
$$P_\infty: \frac{\pi}{12} \rightarrow \frac{10\pi}{3\sqrt{3}}$$

In conclusion, the area of the first polygon is $\frac{20}{3}$ and that of the second one is $\frac{160}{27}$, and the final one, which is a circle, $\frac{10\pi}{3\sqrt{3}}$.