2024 May

数学月刊

Mathematics Monthly

Author :
Guanzhong Yang(student)
Yan Wang(teacher)

Mathematics changes our lives

PREFACE

This month, we are going to talk about the following questions.

Q1

A unit fraction is a fraction of the form $\frac{1}{n}$ *where n is a positive integer. Note that the unit fraction* $\frac{1}{11}$ *can be written as the sum of two unit fractions in the following three ways:*

$$\frac{1}{11} = \frac{1}{12} + \frac{1}{132} = \frac{1}{22} + \frac{1}{22} = \frac{1}{132} + \frac{1}{12}.$$

Are there any other ways of decomposing $\frac{1}{11}$ *into the sum of two unit fractions?*

In how many ways can we write $\frac{1}{60}$ *as the sum of two unit fractions?*

More generally, in how many ways can the unit fraction $\frac{1}{n}$ *be written as the sum of two unit fractions? In other words, how many ordered pairs (a, b) of positive integers a, b are there for which*

$$\frac{1}{n} = \frac{1}{a} + \frac{1}{b}?$$

Q2

Start with a positive integer, then choose a negative integer. We'll use these two numbers to generate a sequence using the following rule: create the next term in the sequence by adding the previous two. For example, if we started with 6 and −5, we would get the sequence

$$\underbrace{6, -5, 1, -4}_{\text{alternating part}}, -3, -7, -10, -17, -27, \ldots$$

which starts with 4 elements that alternate sign before the terms are all negative. If we started with 3 and −2, we would get the sequence

$$\underbrace{3, -2, 1, -1}_{\text{alternating part}}, 0, -1, -1, -2, -3, \ldots$$

which also starts with 4 elements that alternate sign before the terms are all non-positive (we don't count 0 in the alternating part).

(a) Can you find a sequence of this type that starts with 5 elements that alternate sign? With 10 elements that alternate sign? Can you find a sequence with any number of elements that alternate sign?

(b) Given a particular starting integer, what negative number should you choose to make the alternating part of the sequence as long as possible? For example, if your sequence started with 8, what negative number would give the longest alternating part? What if you started with 10? With n?

Q3

The tail of a giant hare is attached by a giant rubber band to a stake in the ground. A flea is sitting on top of the stake eyeing the hare (hungrily). Seeing the flea, the hare leaps into the air and lands one kilometer from the stake (with its tail still attached to the stake by the rubber band). The flea does not give up the chase but leaps into the air and lands on the stretched rubber band one centimeter from the stake. The giant hare, seeing this, again leaps into the air and lands another kilometer from the stake (i.e., a total of two kilometers from the stake). The flea is undaunted and leaps into the air again, landing on the rubber band one centimeter further along. Once again the giant hare jumps another kilometer. The flea again leaps bravely into the air and lands another centimeter along the rubber band. If this continues indefinitely, will the flea ever catch the hare? (Assume the earth is flat and continues indefinitely in all directions.)

This is the admission question from 2024 PROMYS Program. If you have other brilliant ideas, email to anmiciuangray@163.com for surprising rewards!

1.
Some Auxiliary Results

Lemma 1.1 Given that

$$n = p_1^{A_1} \bullet p_2^{A_2} \bullet \cdots \bullet p_n^{A_n}$$

where p_i is a prime number and A_i is an integer for i = 1,2,3,⋯,n,
then the number of factors of n is

$$(A_1+1)(A_2+1)\cdots(A_n+1).$$

Proof.
Given a factor of n, m, then

$$m = p_1^{B_1} \bullet p_2^{B_2} \bullet \cdots \bullet p_n^{B_n}$$

where $0 \le B_1 \le A_1$, $0 \le B_2 \le A_2$, ⋯, $0 \le B_n \le A_n$.
That be said, there are (A_1+1) possibilities of $p_1^{B_1}$, (A_2+1) possibilities of $p_2^{B_2}$, ⋯, (A_n+1) possibilities of $p_n^{B_n}$.
In total, there are

$$(A_1+1)(A_2+1)\cdots(A_n+1)$$

different m.

Solution

We may find that there are so many different situations if we try all the possibilities one by one. Thus, it might be easier if we firstly find out the general strategy.
Given that

$$\frac{1}{a}+\frac{1}{b}=\frac{1}{n}.$$

Rearrange,

$$\frac{a+b}{ab}=\frac{1}{n}.$$

$$ab = an + bn.$$

$$ab - an - bn + n^2 = n^2.$$

$$(a - n)(b - n) = n^2.$$

If

$$n = p_1^{A_1} \bullet p_2^{A_2} \bullet \cdots \bullet p_n^{A_n},$$

then the number of factors of n^2 is

$$(2A_1+1)(2A_2+1)\cdots(2A_n+1).$$

We notice that, for each factor of n^2, assuming that it is (a - n), we could find a corresponding (b - n), satisfying the equation.
Because

$$\frac{1}{a}+\frac{1}{b}=\frac{1}{n}$$

and a and b are positive, thus, understandably, a and b are bigger than n, resulting in that (a - n) and (b - n) are positive.
Notice that, for each pair of (a - n) and (b - n), we can get a corresponding (a,b) satisfying

$$\frac{1}{a}+\frac{1}{b}=\frac{1}{n}.$$

In conclusion, if

$$n = p_1^{A_1} \bullet p_2^{A_2} \bullet \cdots \bullet p_n^{A_n},$$

then there are

$$(2A_1+1)(2A_2+1)\cdots(2A_n+1)$$

pairs of (a,b) satisfying

$$\frac{1}{a}+\frac{1}{b}=\frac{1}{n}.$$

Now we have the general conclusion, and we can use it to resolve some specific problems.
When n = 11, there are 3 ways of decomposing $\frac{1}{11}$ into the sum of two unit fractions, meaning no other ways.
When n = 60, there are 45 ways of decomposing $\frac{1}{60}$ into the sum of two unit fractions, since 60 = $2^2 \bullet 3^1 \bullet 5^1$.

2. (a)

Some Auxiliary Results

Lemma 2.1.1 For a sequence with n elements that alternate sign, if n is odd, then

$$\frac{F_{n-2}}{F_{n-3}} b > a \geq \frac{F_n}{F_{n-1}} b;$$

if n is even, then

$$\frac{F_{n-2}}{F_{n-3}} b < a \leq \frac{F_n}{F_{n-1}} b,$$

where F_n is the nth term in the Fibonacci sequence(1, 1, 2, 3, 5, 8, ⋯).

Proof.

We first list the a few terms of the equation to find the pattern.

Assume that both a and b are positive integers.

	1st	2nd	3rd	4th	5th	6th
term	a	-b	a-b	a-2b	2a-3b	3a-5b
relationship between a and b			a > 1b	a < 2b	a > $\frac{3}{2}$b	a < $\frac{5}{3}$b
coefficient of b			$1 = \frac{F_2}{F_1}$	$2 = \frac{F_3}{F_2}$	$\frac{3}{2} = \frac{F_4}{F_3}$	$\frac{5}{3} = \frac{F_5}{F_4}$

Where F_n is the nth term in the Fibonacci sequence.

Obviously, we may find that the coefficient of b in nth term of relationship between a and b(highlighted part) is $\frac{F_{n-1}}{F_{n-2}}$. This can be proved through mathematical induction.

P_n: the coefficient of b in nth term of relationship between a and b(highlighted part) is $\frac{F_{n-1}}{F_{n-2}}$.

When n = 3 and 4, understandably, P_3 and P_4 are true.

Now we assume that P_{k-1} and P_k are true, the coefficient of b in (k-1)th term is $\frac{F_{k-2}}{F_{k-3}}$ and in kth term is $\frac{F_{k-1}}{F_{k-2}}$, then for P_k:

The (k-1)th term in sequence is (F_{k-3}a - F_{k-2}b), and the kth term is (F_{k-2}a - F_{k-1}b), so the (k+1) th term in sequence is (F_{k-1}a - F_kb), meaning that the coefficient of b in (k+1)th term is $\frac{F_k}{F_{k-1}}$, so P_{k-1} & $P_k \rightarrow P_{k+1}$.

Since P_3 and P_4 are true and P_{k-1} & $P_k \rightarrow P_{k+1}$, by mathematical induction, the coefficient of b in nth term of relationship between a and b(highlighted part) is $\frac{F_{n-1}}{F_{n-2}}$.

Notice that, if there are n elements that alternate sign, meaning that a satisfies the relationship between a and b of (n-1)th term, but not satisfies the relationship between a and b of (n+1)th term.

That be said, if n is odd, then

$$\frac{F_{n-2}}{F_{n-3}} b > a \geq \frac{F_n}{F_{n-1}} b;$$

if n is even, then

$$\frac{F_{n-2}}{F_{n-3}} b < a \leq \frac{F_n}{F_{n-1}} b.$$

Solution

Again, if we want to find out the answer by trying, it will be a bit too reliant on luck. Thus, based on general conclusion Lemma 2.1.1, we can find out the specific answers.

When n = 5, $\frac{F_3}{F_2}b = 2b > a \geq \frac{5}{3}b = \frac{F_5}{F_4}b$. If b = 3, a = 5, the sequence is

5, -3, 2, -1, 1, 0, 1, 1, ⋯

When n = 10, $\frac{F_8}{F_7}b = \frac{21}{13}b < a \leq \frac{55}{34}b = \frac{F_{10}}{F_9}b$. If b = 442, a = 715, the sequence is

715, -442, 273, -169, 104, -65, 39, -26, 13, -13, 0, -13, -13, ⋯

Similarly, we can use Lemma 2.1.1 to find any sequences with n elements that alternating sign.

2.(b)

Solution

After trying several different methods(which you can find in the afterword), I fail to find out a way that can find out the maximum value of n directly. Thus, I guess maybe we still need some trials in order to get the maximum value of n.

Rearrange what we conclude in part(a).
If n is odd, then

$$\frac{F_{n-1}}{F_n}a \geq b > \frac{F_{n-3}}{F_{n-2}}a;$$

if n is even, then

$$\frac{F_{n-1}}{F_n}a \leq b < \frac{F_{n-3}}{F_{n-2}}a.$$

So, given the value of a, if there exists a sequence, starting with a, with n elements that alternate sign, there must exist at least one possible value of b.
Recall that b is a positive integer.
In this way, given the value of a, as long as we find out the minimum value of n which makes corresponding integer b not existed through logic trials, we then could find the maximum value of n and corresponding b.

We can use the conditions that question mentions to illustrate my method.
Given that a = 8, when n = 6,

$$\frac{F_{n-1}}{F_n}a = 5 \leq b < \frac{16}{3} = \frac{F_{n-3}}{F_{n-2}}a,\ b = 5;$$

when n = 7,

$$\frac{F_{n-1}}{F_n}a = \frac{64}{13} \geq b > \frac{21}{5} = \frac{F_{n-3}}{F_{n-2}}a,$$

there's no integer b satisfying this inequality.
So, maximum value of n is 6, and I should choose -5 as the second term in order to make the alternating part of the sequence as long as possible.
Given that a = 10, when n = 5,

$$\frac{F_{n-1}}{F_n}a = 6 \geq b > 5 = \frac{F_{n-3}}{F_{n-2}}a,\ b = 6;$$

when n = 6,

$$\frac{F_{n-1}}{F_n}a = \frac{25}{4} \leq b < \frac{20}{3} = \frac{F_{n-3}}{F_{n-2}}a,$$

there's no integer b satisfying this inequality.
So, maximum value of n is 5, and I should choose -6 as the second term in order to make the alternating part of the sequence as long as possible.

Although this method seems like quite complex, in fact, we don't need to try all the values of n. We can just try several values of n with some intervals between each other in order to narrow the range. Moreover, in afterword, method 2, I also provide you with another way to narrow the range of n.

Afterword

Here are some methods that I tried before but failed eventually.

Method 1 Euclidean Algorithm

I firstly observed the sequences we got in part(a):

5, -3, 2, -1, 1, 0, 1, 1, ⋯

715, -442, 273, -169, 104, -65, 39, -26, 13, -13, 0, -13, -13, ⋯

Rearrange,

kF_n, $-kF_{n-1}$, ⋯, $(-1)^{n-2}kF_2$, $(-1)^{n-1}kF_1$, 0, $(-1)^{n-1}kF_1$, $(-1)^{n-1}kF_1$, ⋯

Where k is the greatest common factor of a and b.

So, I thought that, given the value of a, the sequence with the longest alternating part is just another form of Euclidean Algorithm. So we just need to prove that, in a the sequence with the longest alternating part, there is always a term of 0 behind the alternating part.

Whereas, I found my assumption is wrong. Because when a = 5, the sequence with the longest alternating part is

5, -4, 1, -3, -2, -5, ⋯

Method 2 Similar with Solution, Narrow Range Using Mathematical Induction

I re-observed the method we used in part(a).

Rearrange, if n is odd, then

$$\frac{F_{n-1}F_{n-2}}{F_nF_{n-2}}a \geq b > \frac{F_nF_{n-3}}{F_nF_{n-2}}a;$$

if n is even, then

$$\frac{F_{n-1}F_{n-2}}{F_nF_{n-2}}a \leq b < \frac{F_nF_{n-3}}{F_nF_{n-2}}a.$$

I found that,

$$F_{n-1}F_{n-2} - F_nF_{n-3} = (-1)^{n+1}.$$

This can be proved through mathematical induction.

Assume

$$P_n : F_{n-1}F_{n-2} - F_nF_{n-3} = (-1)^{n+1}.$$

When n = 4, $F_3F_2 - F_1F_4 = 2 - 3 = -1 = (-1)^{4+1}$, P_4 is true.

Assume that P_k is true:

$$F_{k-1}F_{k-2} - F_kF_{k-3} = (-1)^{k+1},$$

for P_{k+1}:

$$F_kF_{k-1} - F_{k+1}F_{k-2}$$
$$= (F_{k-1} + F_{k-2})(F_k - F_{k-2}) - (F_k + F_{k-1})(F_{k-1} - F_{k-3})$$
$$= F_kF_{k-1} - (F_{k-2})^2 - F_{k-1}F_{k-2} + F_kF_{k-2} - F_kF_{k-1} + F_kF_{k-3} - (F_{k-1})^2 + F_{k-1}F_{k-3}$$
$$= (-F_{k-1}F_{k-2} + F_kF_{k-3}) + (F_kF_{k-2} + F_{k-1}F_{k-3} - (F_{k-1})^2 - (F_{k-2})^2)$$
$$= (-1)^{k+2} + (F_kF_{k-2} - F_{k-2}(F_k - F_{k-1}) + F_{k-1}F_{k-3} - F_{k-1}(F_{k-2} + F_{k-3}))$$
$$= (-1)^{k+2}.$$

So $P_k \rightarrow P_{k+1}$.

Since P_4 is true and $P_k \rightarrow P_{k+1}$, by mathematical induction, we can find that $F_{n-1}F_{n-2} - F_nF_{n-3} = (-1)^{n+1}$ for all positive integers greater or equal to 4.

So now we have have find that, if n is odd, then

$$\frac{F_{n-1}F_{n-2}}{F_nF_{n-2}}a \geq b > \frac{F_{n-1}F_{n-2} - 1}{F_nF_{n-2}}a;$$

if n is even, then

$$\frac{F_{n-1}F_{n-2}}{F_nF_{n-2}}a \leq b < \frac{F_nF_{n-3} + 1}{F_nF_{n-2}}a.$$

Recall that b is an integer.

Since the difference of the upper limit and the lower limit of b is $\frac{1}{F_nF_{n-2}}a$, when $\frac{1}{F_nF_{n-2}}a \geq 1$, there must be at least one possible value of b.
In that way, we can just start with the value of n which satisfies $\frac{1}{F_nF_{n-2}}a < 1$ in order to find out the maximum value of n quickly.

Method 3 Useless Matrices Transformation Plus Eigenvalues and Eigenvectors
I noticed that, when we find out the value of next term, it just like that we are doing transformation.
If we use position vector to express them, the sequence

a , -b, a - b, a - 2b, 2a - 3b, 3a - 5b, ⋯

will be turned into

$$\begin{pmatrix} a \\ -b \end{pmatrix} \rightarrow \begin{pmatrix} a-b \\ -b \end{pmatrix} \rightarrow \begin{pmatrix} a-b \\ a-2b \end{pmatrix} \rightarrow \begin{pmatrix} 2a-3b \\ a-2b \end{pmatrix} \rightarrow \begin{pmatrix} 2a-3b \\ 3a-5b \end{pmatrix} \rightarrow \cdots$$

Recall that the sequence start with elements that alternate sign, meaning that a , a - b, 2a - 3b, ⋯ are positive and -b, a - 2b, 3a - 5b, ⋯ are negative.
Surprisingly, we find that all the position vectors are in the Fourth Quadrant, meaning that if one position vector isn’t in the Fourth Quadrant, the corresponding term isn’t in the initial part, consisting of elements that alternate sign.
Moreover, we can also find the matrices used to represent the transformation between two position vectors.

$$\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} a \\ -b \end{pmatrix} = \begin{pmatrix} a-b \\ -b \end{pmatrix}$$
$$\begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}\begin{pmatrix} a-b \\ -b \end{pmatrix} = \begin{pmatrix} a-b \\ a-2b \end{pmatrix}$$
$$\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} a-b \\ a-2b \end{pmatrix} = \begin{pmatrix} 2a-3b \\ a-2b \end{pmatrix}$$
$$\vdots$$

In conclusion, the position vector of nth term is

$$\underbrace{\cdots \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}}_{(n-1)\text{ terms}}\begin{pmatrix} a \\ -b \end{pmatrix}$$

and we can find when the initial part, consisting of elements that alternate sign, ends by finding out the corresponding position vector of which term isn’t in the Fourth Quadrant.
Next, we hope to find the pattern of those position vectors in the process of transformations by finding out the eigenvalues and eigenvectors of the transformations. Whereas, disappointingly, shear doesn’t have eigenvalues and eigenvectors except x-axis or y-axis.
That be said, we cannot simplify the process of transformations but to trace the position vectors one by one in detail, making this method useless.

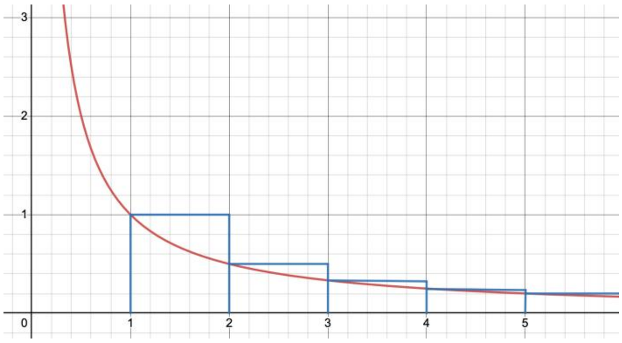
3.

Some Auxiliary Results

Lemma 3.1

$$\lim_{N\to\infty} \sum_{i=1}^{N} \frac{1}{i} = \infty.$$

Proof.



By finding the lower limit of $\sum_{i=1}^{N} \frac{1}{i}$ is ∞, we can prove that

$$\lim_{N\to\infty} \sum_{i=1}^{N} \frac{1}{i} = \infty.$$

Just like graph, the lower limit of $\sum_{i=1}^{N} \frac{1}{i}$ is

$$\lim_{N\to\infty} \int_{1}^{N+1} \frac{1}{x} dx = \lim_{N\to\infty} [\ \ln x\ ]_{1}^{N+1} = \lim_{N\to\infty} \ln(N+1) = \infty.$$

So we are done.

Solution

It is hard for us to compare the 'magnitudes' of two 'infinite's, so it will be useful if we fix one of them.

If we keep the total rubber band length constant, after the conversion of the length of movement of flea based on proportion, the flea moves forward $\frac{1}{1}$cm on the first jump, $\frac{1}{2}$cm on the second jump, $\frac{1}{3}$cm on the third jump... and $\frac{1}{n}$cm on the nth jump....

So the total distance that the flea moves is $(\sum_{i=1}^{N} \frac{1}{i})$cm.

Based on Lemma 3.1, as long as N is large enough, the flea will move far enough to reach the destination.