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Mathematics changes our lives

PREFACE

This month, we are going to talk about the following questions.

A polynomial is integral when it has integer coefficients. The square root of 2 is a solution to the integral polynomial equation $x^2 - 2 = 0$.

A number is rational when it can be expressed as $\frac{a}{b}$ for integers a and b (with $b \neq 0$).

A number is irrational when it is not rational.

(a) Suppose c is a non–square integer. (That is, $c \neq n^2$ for any n.) Explain why $\sqrt{c}$ is not rational. Similarly, if c is a non–cube integer, does it follow that $\sqrt[3]{c}$ is irrational?

(b) Find an integral polynomial equation that has $\alpha = \sqrt{3} + \sqrt{5}$ as a solution. Show that α is irrational.

(c) Let a and b be integers. Find an integral polynomial equation which has $\sqrt{a} + \sqrt{b}$ as a solution. Must $\sqrt{a} + \sqrt{b}$ be irrational? If $a \neq b$, must $\sqrt{a} - \sqrt{b}$ be irrational?

(d) Formulate some generalizations. As a starting point, is $\beta = \sqrt{3} + \sqrt{5} + \sqrt{7}$ irrational? What about numbers like $\gamma = 3\sqrt{2} - 2\sqrt{3} - 3\sqrt{5} + \sqrt{6}$ and $\delta = \sqrt[3]{5} - \sqrt{2}$?

This is the admission question from 2024 ROSS Program. If you have other brilliant ideas, email to anmiciuangray@163.com **for surprising rewards!**

1.(a)

Some Auxiliary Results

Lemma 1.1.1 if

$$n^k = p_1^{a_1} \bullet p_2^{a_2} \bullet \cdots \bullet p_n^{a_n}$$

where p_i is a prime and a_i is an integer for i = 1,2,3,⋯,n,
then $k|a_i$ for i = 1,2,3,⋯.

Proof.

If

$$n = p_1^{b_1} \bullet p_2^{b_2} \bullet \cdots \bullet p_n^{b_n}$$

where p_i is a prime and b_i is an integer for i = 1,2,3,⋯,
then

$$n^k = p_1^{kb_1} \bullet p_2^{kb_2} \bullet \cdots \bullet p_n^{kb_n}.$$

Solution

We will prove that $\sqrt{c}$ and $\sqrt[3]{c}$ are not rational through contradiction.

If $\sqrt{c}$ is rational, then

$$\sqrt{c} = \frac{a}{b}$$

where a and b are integers,

$$c = \frac{a^2}{b^2},$$

$$b^2 \bullet c = a^2.$$

Notice that, c is a non-square integer, meaning that at least one of its prime factors has a degree that is not a multiple of 2.
Whereas, based on Lemma 1.1.1, the prime factors of both a^2 and b^2 have a degree that is a multiple of 2, making the equality impossible.
So $\sqrt{c}$ is irrational.

If $\sqrt[3]{c}$ is rational, then

$$\sqrt[3]{c} = \frac{a}{b}$$

where a and b are integers,

$$c = \frac{a^3}{b^3},$$

$$b^3 \bullet c = a^3.$$

Notice that, c is a non-cube integer, meaning that at least one of its prime factors has a degree that is not a multiple of 3.
Whereas, based on Lemma 1.1.1, the prime factors of both b^3 and a^3 have a degree that is a multiple of 3, making the equality impossible.
So $\sqrt[3]{c}$ is irrational.

1.(b)

Some Auxiliary Results

Lemma 1.2.1 The difference of two rational numbers should be a rational number.
Proof.

$$\frac{a}{b} - \frac{c}{d} = \frac{ad-bc}{bd}$$

where a, b, c and d are integers.

Solution

We can easily find out an integral polynomial equation with the root of ($\sqrt{3} + \sqrt{5}$):

$$f(x) = (x + \sqrt{3} + \sqrt{5})(x - \sqrt{3} + \sqrt{5})(x + \sqrt{3} - \sqrt{5})(x - \sqrt{3} - \sqrt{5})$$
$$= ((x + \sqrt{3})^2 - 5)((x - \sqrt{3})^2 - 5)$$
$$= (x^2 + 2\sqrt{3}x - 2)(x^2 - 2\sqrt{3}x - 2)$$
$$= (x^2 - 2)^2 - 12x^2$$
$$= x^4 - 16x^2 + 4$$

Now, let's prove that ($\sqrt{3} + \sqrt{5}$) is irrational through contradiction.
If

$$\sqrt{3} + \sqrt{5} = \frac{a}{b}$$

where a and b are integers,

$$8 + 2\sqrt{15} = \frac{a^2}{b^2},$$
$$8b^2 + 2\sqrt{15}b^2 = a^2.$$

Notice that, both a^2 and $8b^2$ are all rational, so the equality might hold when $2\sqrt{15}b^2$ is also rational, based on Lemma 1.2.1.
If

$$2\sqrt{15}b^2 = \frac{c}{d}$$

where c and d are integers,

$$60b^4 = \frac{c^2}{d^2},$$
$$60b^4d^2 = c^2.$$

Based on Lemma 1.1.1, the prime factors of b^4, d^2 and c^2 have a degree that is a multiple of 3, whereas $60 = 2^2 \times 3^1 \times 5^1$, making the equality impossible.
So ($\sqrt{3} + \sqrt{5}$) is irrational.

Afterword

After finishing 1(d), I find an easier solution.
We find an integral polynomial equation with root of $x = \sqrt{3}$,

$$(x + \sqrt{3})(x - \sqrt{3}) = 0,$$

and let

$$m = \sqrt{3} + \sqrt{5},$$
$$m - \sqrt{5} = \sqrt{3}.$$

If we assume that m is rational, since $x = \sqrt{3}$ is a root, so $x = m - \sqrt{5}$ is a root, based on Lemma 1.4.1, $x = m + \sqrt{5}$ is also a root.

$$f(m + \sqrt{5}) = (m + \sqrt{5} + \sqrt{3})(m + \sqrt{5} - \sqrt{3})$$
$$= (\sqrt{3} + \sqrt{5} + \sqrt{5} + \sqrt{3})(\sqrt{3} + \sqrt{5} + \sqrt{5} - \sqrt{3})$$
$$\neq 0,$$

meaning that our assumption is not correct, so m is irrational.

1.(c)

Solution

We can easily find out an integral polynomial equation with the root of $(\sqrt{a}+\sqrt{b})$:

$$\begin{aligned} f(x) &= (x+\sqrt{a}+\sqrt{b})(x-\sqrt{a}+\sqrt{b})(x+\sqrt{a}-\sqrt{b})(x-\sqrt{a}-\sqrt{b}) \\ &= ((x+\sqrt{a})^2-b)((x-\sqrt{a})^2-b) \\ &= (x^2+2\sqrt{a}x+a-b)(x^2-2\sqrt{a}x+a-b) \\ &= (x^2+a-b)^2-4ax^2 \\ &= x^4-2(a+b)x^2+(a-b)^2 \end{aligned}$$

Now, let's firstly assume that $(\sqrt{a}+\sqrt{b})$ is rational.
If

$$\sqrt{a}+\sqrt{b}=\frac{c}{d}$$

where c and d are integers,

$$a+b+2\sqrt{ab}=\frac{c^2}{d^2},$$

$$ad^2+bd^2+2d^2\sqrt{ab}=c^2.$$

Based on Lemma 1.2.1, since (ad^2+bd^2) and c^2 are rational, the equality might hold when $2d^2\sqrt{ab}$ is rational.
If

$$2d^2\sqrt{ab}=\frac{e}{f}$$

where e and f are integers,

$$4abd^4=\frac{e^2}{f^2},$$

$$4abd^4f^2=e^2$$

Based on Lemma 1.1.1, if one prime factor of ab has a degree that is not a multiple of 2, then $\sqrt{a}+\sqrt{b}$ is irrational.
If the the prime factors of ab all have a degree that is a multiple of 2, it does not necessarily mean that $\sqrt{a}+\sqrt{b}$ is rational---above all, the aforementioned method is just a way to find irrational.
Then we need to focus on the exact prime factors of a and b.
If the prime factors of both a and b have a degree that is a multiple of 2, then understandably, $\sqrt{a}+\sqrt{b}$ is rational.
If at least one prime factor of both a and b have a degree that is not a multiple of 2, meaning that

$$a=p_1^{a_1}\bullet p_2^{a_2}\bullet\cdots\bullet p_n^{a_n},\ b=p_1^{b_1}\bullet p_2^{b_2}\bullet\cdots\bullet p_n^{b_n}$$

where where p_i is a prime and a_i and b_i are all integers for i = 1,2,3,⋯,n, and $2|a_i$ and $2|b_i$ for only some i, not all.
Assume that when $i=c_1, c_2, \cdots$, $2|a_i$ and $2|b_i$; when $i=d_1, d_2, \cdots$, $2\nmid a_i$ and $2\nmid b_i$.
Rearrange,

$$\begin{aligned} \sqrt{a}+\sqrt{b} &= (p_{c_1}^{a_{c_1}}\bullet p_{c_2}^{a_{c_2}}\bullet\cdots\bullet p_{d_1}^{a_{d_1}}\bullet p_{d_2}^{a_{d_2}}\bullet\cdots)^{\frac{1}{2}}+(p_{c_1}^{b_{c_1}}\bullet p_{c_2}^{b_{c_2}}\bullet\cdots\bullet p_{d_1}^{b_{d_1}}\bullet p_{d_2}^{b_{d_2}}\bullet\cdots)^{\frac{1}{2}} \\ &= (p_{d_1}\bullet p_2\bullet\cdots)^{\frac{1}{2}}\bullet \\ &\quad ((p_{c_1}^{a_{c_1}}\bullet p_{c_2}^{a_{c_2}}\bullet\cdots\bullet p_{d_1}^{a_{d_1}-1}\bullet p_{d_2}^{a_{d_2}-1}\bullet\cdots)^{\frac{1}{2}}+(p_{c_1}^{b_{c_1}}\bullet p_{c_2}^{b_{c_2}}\bullet\cdots\bullet p_{d_1}^{b_{d_1}-1}\bullet p_{d_2}^{b_{d_2}-1}\bullet\cdots)^{\frac{1}{2}}) \end{aligned}$$

Notice that, degrees of the prime factors in the highlighted part are all even, making the highlighted part being an integer.
In conclusion, in this way,

$$\sqrt{a}+\sqrt{b}=A\sqrt{B}$$

where B is a non-square integer and A is an integer. Obviously, it is irrational.
In conclusion, for $(\sqrt{a}+\sqrt{b})$, when a and b are all square integers, it is rational, otherwise it is irrational.

Similarly, for $\sqrt{a}$ - $\sqrt{b}$, we firstly assume that it is rational.

$$\sqrt{a} - \sqrt{b} = \frac{c}{d}$$

where c and d are integers,

$$a + b - 2\sqrt{ab} = \frac{c^2}{d^2},$$

$$ad^2 + bd^2 - 2d^2\sqrt{ab} = c^2.$$

Based on Lemma 1.2.1, since ($ad^2 + bd^2$) and c^2 are rational, the equality might hold when $2d^2\sqrt{ab}$ is rational.

After that, we can use the similar strategy to find out the answer.

In conclusion, for ($\sqrt{a}$ - $\sqrt{b}$), when a and b are all square integers, it is rational, otherwise it is irrational.

Afterword

After finishing 1(d), I find an easier solution.

When a and b are both square integers, understandably, ($\sqrt{a}$ + $\sqrt{b}$) is rational.

Similarly, when one of a and b is a square integer, obviously, ($\sqrt{a}$ + $\sqrt{b}$) is irrational based on Lemma 1.2.1.

Now, we will focus that condition when both a and b are non-square integers.

We find an integral polynomial equation with root of x = $\sqrt{a}$,

$$(x + \sqrt{a})(x - \sqrt{a}) = 0,$$

and let

$$m = \sqrt{a} + \sqrt{b},$$

$$m - \sqrt{a} = \sqrt{b}.$$

If we assume that m is rational, since x = $\sqrt{a}$ is a root, so x = m - $\sqrt{b}$ is a root, based on Lemma 1.4.1, x = m + $\sqrt{b}$ is also a root.

$$f(m + \sqrt{b}) = (m + \sqrt{b} + \sqrt{a})(m + \sqrt{b} - \sqrt{a})$$

$$= (\sqrt{a} + \sqrt{b} + \sqrt{b} + \sqrt{a})(\sqrt{a} + \sqrt{b} + \sqrt{b} - \sqrt{a})$$

$$\neq 0,$$

meaning that our assumption is not correct, so m is irrational.

In conclusion, for ($\sqrt{a}$ + $\sqrt{b}$), when a and b are all square integers, it is rational, otherwise it is irrational.

Similarly, for ($\sqrt{a}$ - $\sqrt{b}$), when a and b are all square integers, it is rational, otherwise it is irrational.

1.(d)

Some Auxiliary Results

Lemma 1.4.1 For an integral polynomial, if

$$x = a \pm \sqrt{b}$$

where a is integer and b is a non-square integer, is a root, then

$$x = a \mp \sqrt{b}$$

is also a root.

Proof.

For an integral polynomial f(x), if

$$f(a + \sqrt{b}) = p + q\sqrt{b} = 0,$$

so

$$p = q = 0.$$

Thus

$$f(a - b\sqrt{c}) = p - q\sqrt{b} = 0.$$

Similarly,if

$$f(a - \sqrt{b}) = p - q\sqrt{b} = 0,$$

then

$$f(a + \sqrt{b}) = p + q\sqrt{b} = 0.$$

Lemma 1.4.2 We define $N_{(i,j)}$ as the jth digit, from right to left, of $I|_2$, where $I|_2 = i$.

For example.

$N_{(10,2)}$ is the 2nd digit of $1010|_2$, since $1010|_2 = 10$, so $N_{(10,2)} = 1$.

$N_{(10,5)}$ is the 5th digit of $1010|_2$, so $N_{(10,5)} = 0$.

Then

$$f(x) = \prod_{i=1}^{2^n} (x + \sum_{j=1}^{n} ((-1)^{N(i,j)} \sqrt{p_j}))$$

where p_j is a non-square integer for j = 1,2,⋯,n, and $p_{j_1} \neq p_{j_2}$ for $1 \leq j_1 < j_2 \leq n$, must be an integral polygon.

Proof.

Here, I would like to show you one proof that I create. Although I fail to express it step by step using mathematical formulas, I think that this basic strategy is useful.

In order to help your understanding, I would like to tell you the source of my inspiration: in the complex plane, the multiplication of two conjugate functions is a real number.

We firstly create a (n+1) dimensional space, define that $((-1)^{N(i,j)}\sqrt{p_j})$ is on the jth axis, with value of $(-1)^{N(i,j)}$, and the rational number is on the (n+1)th axis.

Then, for

$$\sum_{j=1}^{n} ((-1)^{N(i,j)} \sqrt{p_j})$$

where i = 1,2,⋯,2^n,

we can express it as

$$A \prod_{j=1}^{n} f_j(\theta_{(1,i,j)}),$$

where A,

$$A = \sqrt{n},$$

is the distance between the point and the origin, f_j is a special calculation acting as $e^{i\vartheta}$ in a complex plane and $\theta_{(w,i,j)}$ is the angle between the line, connection the point and the origin, and the jth axis, for i = 1,2,⋯,2^n and w = 1,2,⋯(just to distinguish different combinations).

Recall, just like complex numbers in a complex plane, we can also consider this point as a vector inside the (n+1) dimensional space.

We can consider x as an arbitrary point, or vector starting from origin, inside the (n+1) dimensional space.

Thus,

$$x = X\prod_{j=1}^{n+1} f_j(\theta_{(2,i,j)})$$

where X is the distance between the point and the origin. Notice that the values of i do not influence the value of x.

So,

$$x + \sum_{j=1}^{n} ((-1)^{N(i,j)}\sqrt{p_j}) = B\prod_{j=1}^{n+1} f_j(\theta_{(3,i,j)})$$

where F is the distance between the point and the origin.
Thus,

$$f(x) = \prod_{i=1}^{2^n}(x + \sum_{j=1}^{n}((-1)^{N(i,j)}\sqrt{p_j})) = \prod_{i=1}^{2^n}(B_i\prod_{j=1}^{n+1} f_j(\theta_{(3,i,j)}))$$
$$= (\prod_{i=1}^{2^n} B_i)(\prod_{i=1}^{2^n}\prod_{j=1}^{n+1} f_j(\theta_{(3,i,j)}))$$
$$= C(\prod_{i=1}^{2^n}\prod_{j=1}^{n+1} f_j(\theta_{(3,i,j)}))$$

Recall that

$$B_i = \sqrt{D_i}$$

where D_i is an integer for i = 1,2,⋯,2^n and

$$C = \prod_{i=1}^{2^n} B_i = \prod E_w^{\frac{F_w}{2}}$$

where $E_w \in \{B_i, 1 \le i \le 2^n\}$, $E_{w1} \neq E_{w2}$ for w1 ≠ w2 and F_w is even for any w, meaning that C is integer.

We now find an expression of the polynomial: for any given x, we can find a corresponding f(x), but how should we prove that it is an integral polynomial?
Well, since this polynomial has 2^n terms, we can use a group of 2^n equations to fix the polynomial.
Then, if we choose 2^n different rational numbers to act as 2^n different x and find 2^n different f(x), the values of which are all rational, then the coefficients of polynomial are all rational.
In another word, if we express f(x) as a point in the (n+1) dimensional space, if all the 2^n different f(x) are on the (n+1)th axis, then the coefficients of polynomial are all rational.
Actually, the 2^n different f(x) will definitely lie on the (n+1)th axis because, just like the multiplication of two conjugate functions in the complex plane, in the equation, some $\theta_{(3,i,j)}$ with same j but different i will have opposite values, leading to the result that the 2^n different f(x) will lie on the (n+1)th axis.

As long as we find out the polynomial with rational coefficients, we can find out the integral polynomial.
Whereas, I fail to prove that

$$f(x) = \prod_{i=1}^{2^n}(x + \sum_{j=1}^{n}((-1)^{N(i,j)}\sqrt{p_j}))$$

is not only a polynomial with rational coefficients, but also an integral polynomial, which is frustrating. Maybe this has to do with the value of C, I suppose?
But, I have to say, I hope that you enjoy this strategy since this is my 5th strategy and I do spent plenty of time on it!
You will have the opportunity to see my 2nd strategy, which is also a reasonable proof, on the afterword, behind the solution.

Solution

Assume that

$$\beta = \sqrt{3} + \sqrt{5} + \sqrt{7}$$

where β is rational,

$$\beta - \sqrt{7} = \sqrt{3} + \sqrt{5},$$
$$\beta^2 + 7 - 2\beta\sqrt{7} = 8 + 2\sqrt{15},$$
$$\beta^2 - 1 = 2\beta\sqrt{7} + 2\sqrt{15},$$

meaning that the assumption might hold if $(2\beta\sqrt{7} + 2\sqrt{15})$ is rational.
Assume that

$$\varepsilon = 2\beta\sqrt{7} + 2\sqrt{15},$$
$$\varepsilon^2 = 28\beta^4 + 60 + 8\beta\sqrt{105},$$
$$\varepsilon^2 - 28\beta^4 - 60 = 8\beta\sqrt{105}.$$

Based on Lemma 1.2.1, obviously, $8\beta\sqrt{105}$ is not rational, meaning that $(\sqrt{3} + \sqrt{5} + \sqrt{7})$ is irrational.

Assume that

$$\gamma = 3\sqrt{2} - 2\sqrt{3} - 3\sqrt{5} + \sqrt{6} = \sqrt{18} - \sqrt{12} - \sqrt{45} + \sqrt{6}$$

where γ is rational,

$$\gamma - \sqrt{6} = \sqrt{18} - \sqrt{12} - \sqrt{45}.$$

We then find an integral polynomial with a root of $\sqrt{18} - \sqrt{12} - \sqrt{45}$,

$$\begin{aligned} f(x) = {} & x^8 - 300x^6 + 21222x^4 - 430380x^2 + 408321 \\ = {} & (x - \sqrt{18} - \sqrt{12} - \sqrt{45})(x + \sqrt{18} - \sqrt{12} - \sqrt{45})(x - \sqrt{18} + \sqrt{12} - \sqrt{45})(x + \sqrt{18} + \sqrt{12} - \sqrt{45}) \\ & (x - \sqrt{18} - \sqrt{12} + \sqrt{45})(x + \sqrt{18} - \sqrt{12} + \sqrt{45})(x - \sqrt{18} + \sqrt{12} + \sqrt{45})(x + \sqrt{18} + \sqrt{12} + \sqrt{45}). \end{aligned}$$

Since $x = \sqrt{18} - \sqrt{12} - \sqrt{45}$ is one root, so does $x = \gamma - \sqrt{6}$ Based on Lemma 1.4.1, $x = \gamma + \sqrt{6}$ is also a root.

$$\begin{aligned} f(\gamma + \sqrt{6}) = {} & (\gamma + \sqrt{6} - \sqrt{18} - \sqrt{12} - \sqrt{45})(\gamma + \sqrt{6} + \sqrt{18} - \sqrt{12} - \sqrt{45}) \\ & (\gamma + \sqrt{6} - \sqrt{18} + \sqrt{12} - \sqrt{45})(\gamma + \sqrt{6} + \sqrt{18} + \sqrt{12} - \sqrt{45}) \\ & (\gamma + \sqrt{6} - \sqrt{18} - \sqrt{12} + \sqrt{45})(\gamma + \sqrt{6} + \sqrt{18} - \sqrt{12} + \sqrt{45}) \\ & (\gamma + \sqrt{6} - \sqrt{18} + \sqrt{12} + \sqrt{45})(\gamma + \sqrt{6} + \sqrt{18} + \sqrt{12} + \sqrt{45}) \\ = {} & (\sqrt{18} - \sqrt{12} - \sqrt{45} + \sqrt{6} + \sqrt{6} - \sqrt{18} - \sqrt{12} - \sqrt{45}) \\ & (\sqrt{18} - \sqrt{12} - \sqrt{45} + \sqrt{6} + \sqrt{6} + \sqrt{18} - \sqrt{12} - \sqrt{45}) \\ & (\sqrt{18} - \sqrt{12} - \sqrt{45} + \sqrt{6} + \sqrt{6} - \sqrt{18} + \sqrt{12} - \sqrt{45}) \\ & (\sqrt{18} - \sqrt{12} - \sqrt{45} + \sqrt{6} + \sqrt{6} + \sqrt{18} + \sqrt{12} - \sqrt{45}) \\ & (\sqrt{18} - \sqrt{12} - \sqrt{45} + \sqrt{6} + \sqrt{6} - \sqrt{18} - \sqrt{12} + \sqrt{45}) \\ & (\sqrt{18} - \sqrt{12} - \sqrt{45} + \sqrt{6} + \sqrt{6} + \sqrt{18} - \sqrt{12} + \sqrt{45}) \\ & (\sqrt{18} - \sqrt{12} - \sqrt{45} + \sqrt{6} + \sqrt{6} - \sqrt{18} + \sqrt{12} + \sqrt{45}) \\ & (\sqrt{18} - \sqrt{12} - \sqrt{45} + \sqrt{6} + \sqrt{6} + \sqrt{18} + \sqrt{12} + \sqrt{45}) \\ \neq {} & 0, \end{aligned}$$

which means that our assumption is not correct.
So $(3\sqrt{2} - 2\sqrt{3} - 3\sqrt{5} + \sqrt{6})$ is irrational.

Assume that

$$\delta = \sqrt[3]{5} - \sqrt{2}$$

where δ is rational,

$$\delta + \sqrt{2} = \sqrt[3]{5},$$
$$\delta^3 - 3\delta^2\sqrt{2} + 6\delta - 2\sqrt{2} = 5,$$
$$(3\delta^2 - 2)\sqrt{2} = 5.$$

Based on Lemma 1.2.1, $(3\delta^2 - 2)\sqrt{2}$ is irrational, meaning that our assumption is not correct.
So $\sqrt[3]{5} - \sqrt{2}$ is irrational.

Well, as far as I am concerned,

$$\sum_{i=1}^{n}(-1)^{a_i}\sqrt{p_i}$$

where $a_i \in \{0,1\}$, p_i is a non-square integer for i = 1,2,⋯,n, and $p_i \neq p_j$ for $1 \leq i < j \leq n$, is always irrational.

This can be proved through Lemma 1.4.1 and Lemma 1.4.2 by using the similar method when we want to prove that ($3\sqrt{2} - 2\sqrt{3} - 3\sqrt{5} + \sqrt{6}$) is irrational.

We firstly assume that

$$\mu = \sum_{i=1}^{n}(-1)^{a_i}\sqrt{p_i}$$

where μ is rational,

$$\mu - (-1)^{a_n}\sqrt{p_n} = \sum_{i=1}^{n-1}(-1)^{a_i}\sqrt{p_i}.$$

According to Lemma 1.4.2, we can find out an integral polynomial with a root of $x = \sum_{i=1}^{n-1}(-1)^{a_i}\sqrt{p_i}$:

$$f(x) = \prod_{i=1}^{2^{n-1}}\left(x + \sum_{j=1}^{n-1}\left((-1)^{N(i,j)}\sqrt{p_j}\right)\right).$$

Since $x = \sum_{i=1}^{n-1}(-1)^{a_i}\sqrt{p_i}$ is a root of integral polynomial, so does $x = \mu - (-1)^{a_n}\sqrt{p_n}$.

Then, based on Lemma 1.4.1, $x = \mu + (-1)^{a_n}\sqrt{p_n}$ is also a root.

When then can find that

$$f(\mu + (-1)^{a_n}\sqrt{p_n}) \neq 0$$

by replacing μ by

$$\sum_{i=1}^{n}(-1)^{a_i}\sqrt{p_i},$$

meaning that

$$\sum_{i=1}^{n}(-1)^{a_i}\sqrt{p_i}$$

Is irrational.

Afterword

At the time I prove Lemma 1.4.2, I firstly want to prove it through mathematical induction.
P_n:

$$f(x) = \prod_{i=1}^{2^n}(x + \sum_{j=1}^{n}((-1)^{N(i,j)}\sqrt{p_j}))$$

is an integral polygon.
Understandably, P_1 can be satisfied.
Then, assume that P_k is right, so

$$f(x) = \prod_{i=1}^{2^k}(x + \sum_{j=1}^{k}((-1)^{N(i,j)}\sqrt{p_j}))$$

where $N_{(i,j)}$ as the jth digit, from right to left, of $I|_2$, where $I|_2$ = i, and p_j is a non-square integer for j = 1,2,⋯,n, and $p_{j_1} \neq p_{j_2}$ for $1 \leq j_1 < j_2 \leq n$, is an integral polygon, then for P_{k+1}:

$$f(x) = \prod_{i=1}^{2^k}(x + \sum_{j=1}^{k}((-1)^{N(i,j)}\sqrt{p_j}) + \sqrt{p_{k+1}})\prod_{i=1}^{2^k}(x + \sum_{j=1}^{k}((-1)^{N(i,j)}\sqrt{p_j}) - \sqrt{p_{k+1}})$$

$$= \prod_{i=1}^{2^k}【(x + \sum_{j=1}^{k}((-1)^{N(i,j)}\sqrt{p_j}) + \sqrt{p_{k+1}})(x + \sum_{j=1}^{k}((-1)^{N(i,j)}\sqrt{p_j}) - \sqrt{p_{k+1}})】$$

$$= \prod_{i=1}^{2^k}【(x + \sum_{j=1}^{k}((-1)^{N(i,j)}\sqrt{p_j}))^2 - p_{k+1}】$$

$$=\prod_{i=1}^{2^k}(x^2 + 2x\sum_{j=1}^{k}((-1)^{N(i,j)}\sqrt{p_j}) + 【\sum_{j=1}^{k}p_j + 2\sum_{1\leq j_1<j_2\leq n}((-1)^{N(i,j_1)}\sqrt{p_{j_1}})((-1)^{N(i,j_2)}\sqrt{p_{j_2}}) - p_{k+1}】)$$

But, obviously, it is hard for us to turn the multiplication of sums into the sum of multiplications, or a form that is similar to P_k, so I soon give it up.

After finishing 1(d), I find an easier solution to prove ($\sqrt{3} + \sqrt{5} + \sqrt{7}$) is irrational
Assume that

$$\beta = \sqrt{3} + \sqrt{5} + \sqrt{7}$$

where β is rational,

$$\beta - \sqrt{7} = \sqrt{3} + \sqrt{5}.$$

We then find an integral polynomial with a root of $\sqrt{3} + \sqrt{5}$,

$$f(x) = x^4 - 16x^2 + 4 = (x + \sqrt{3} + \sqrt{5})(x - \sqrt{3} + \sqrt{5})(x + \sqrt{3} - \sqrt{5})(x - \sqrt{3} - \sqrt{5}).$$

Since x = $\sqrt{3} + \sqrt{5}$ is one root, so does x = $\beta - \sqrt{7}$. Based on Lemma 1.4.1, x = $\beta + \sqrt{7}$ is also a root.
Thus

$$\begin{aligned} f(\beta + \sqrt{7}) &= (\beta + \sqrt{7} + \sqrt{3} + \sqrt{5})(\beta + \sqrt{7} - \sqrt{3} + \sqrt{5})(\beta + \sqrt{7} + \sqrt{3} - \sqrt{5})(\beta + \sqrt{7} - \sqrt{3} - \sqrt{5}) \\ &= (\sqrt{3} + \sqrt{5} + \sqrt{7} + \sqrt{7} + \sqrt{3} + \sqrt{5})(\sqrt{3} + \sqrt{5} + \sqrt{7} + \sqrt{7} - \sqrt{3} + \sqrt{5}) \\ &\quad (\sqrt{3} + \sqrt{5} + \sqrt{7} + \sqrt{7} + \sqrt{3} - \sqrt{5})(\sqrt{3} + \sqrt{5} + \sqrt{7} + \sqrt{7} - \sqrt{3} - \sqrt{5}) \\ &\neq 0, \end{aligned}$$

which means that our assumption is not correct.
So, ($\sqrt{3} + \sqrt{5} + \sqrt{7}$) is irrational.

In fact, in this question, if we turn all the ‘integer’ into ‘number’, the whole solution still holds.