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Mathematics changes our lives

PREFACE

This month, we are going to talk about the following questions.

Q1

Here's a (paraphrased) conversation that took place between two Ross students.

A: Hey, want to see a magic trick?

B: Sure, how does it go?

A: Think of any number. Any nonnegative integer, I should say.

B: Okay.

A: Multiply it by 3.

B: Okay….

A:

Now divide it by 2, and if you get a decimal, then round down. Tell me if you rounded down or not.

B: I did have to round down.

A:

Multiply it by 3 then divide it by 2 again. Again, tell me if you rounded down.

B: Okay, hang on… I did not have to round down this time.

A: Great, now just tell me: how many times does 9 go into this last number?

B: You want me to divide by 9?

A: Yes, just the quotient.

B: Um, the quotient is 4.

A: So your original number was 19.

B: That's right! How did you do that? Let me think….

(a) Figure out how A's magic trick works, and write up a clear, mathematical explanation of how to perform it.

(b) What variants of this trick can you come up with using the same principles? Can you change the numbers 2, 3, and 9 in the trick, or maybe the operations involved? What about the information you ask for?

Q2

It's hard to keep up with the fashionable trends of the day, and mathematical fashion is no exception. The fashion season of 2024 has the following unspoken fashion rules:

- *1 is fashionable.*
- *If a and b are fashionable numbers, then so is* $\frac{1}{a+b}$.
- *All other numbers are "so last year" and thus not fashionable.*

(a) Find the smallest and largest fashionable numbers, and prove that they are the smallest and largest.

(b) Which numbers are fashionable? Can you characterize the set of all fashionable numbers?

(c) Word on the street is that in 2025, the fashion rules will be the same except for "1 is fashionable". Explore how the set of fashionable numbers changes if we begin with a different starting set of numbers. For example, what if 2 and $\sqrt{3}$ *are the given fashionable numbers?*

This is the admission question from 2024 ROSS Program. If you have other brilliant ideas, email to anmiciuangray@163.com for surprising rewards!

1.(a)

Solution

We firstly assume that we have a value α.
We multiply it by 3 and then divide it by 2. If the result is a decimal, then round it down(if we need to round it down, we highlight the relevant block in the graph below), then we repeat the process again. Eventually, we divide the result by 9.
Actually,we can express this as below.

<table>
<tr><td colspan="4">start with α</td></tr>
<tr><td colspan="2">if $\alpha \equiv 1 \pmod 2$, we get $\frac{3\alpha-1}{2}$</td><td colspan="2">if $\alpha \equiv 0 \pmod 2$, we get $\frac{3\alpha}{2}$</td></tr>
<tr><td>if $\alpha \equiv 1 \pmod 4$,
we get $\frac{9\alpha-5}{4}$</td><td>if $\alpha \equiv 3 \pmod 4$,
we get $\frac{9\alpha-3}{4}$</td><td>if $\alpha \equiv 2 \pmod 4$,
we get $\frac{9\alpha-2}{4}$</td><td>if $\alpha \equiv 0 \pmod 4$,
we get $\frac{9\alpha}{4}$</td></tr>
<tr><td>we get $\frac{\alpha}{4}$ - $\frac{5}{36}$</td><td>we get $\frac{\alpha}{4}$ - $\frac{1}{12}$</td><td>we get $\frac{\alpha}{4}$ - $\frac{1}{18}$</td><td>we get $\frac{\alpha}{4}$</td></tr>
<tr><td colspan="4">we get $\lfloor\frac{\alpha}{4}\rfloor$, the quotient when α is divided by 4</td></tr>
</table>

Notice that, by knowing when we round the results down, We can then find the remainder when α is divided by 4.
Recall that we eventually get $\lfloor\frac{\alpha}{4}\rfloor$, the quotient when α is divided by 4.
Then, we can use the quotient and remainder when α is divided by 4 to find the value of α.

1.(b)

Introduction

The underlying logic of this magic trick is to determine a unique number through some of its properties.
In order to achieve this, using the property of quotient and remainder will be useful. Prime factorization may also sound reasonable, but we need to do infinite operations in order to fix a number, meaning that it is useless.

Solution

Speaking of the variants of this trick with the same principles, the easily one must be 'tell me the quotient and remainder when your number is divided by n'.

We can also change the numbers 3 and 9 in the trick into n and n^2, as long as when $\frac{n^2\alpha-5}{4}$ (or $\frac{n^2\alpha-3}{4}$, or $\frac{n^2\alpha-2}{4}$, or $\frac{n^2\alpha}{4}$) is an integer, all possible α have an unique and same remainder when divided by 4, and in different situations, the remainders are different.
But we cannot change the value of 2, because whether or not we round the result down represent only 2 conditions.
For instance, we can replace 3 and 9 by 7 and 49.

<table>
<tr><td colspan="4">start with α</td></tr>
<tr><td colspan="2">if $\alpha \equiv 1 \pmod 2$, we get $\frac{7\alpha-1}{2}$</td><td colspan="2">if $\alpha \equiv 0 \pmod 2$, we get $\frac{7\alpha}{2}$</td></tr>
<tr><td>if $\alpha \equiv 1 \pmod 4$, we get $\frac{49\alpha-5}{4}$</td><td>if $\alpha \equiv 3 \pmod 4$, we get $\frac{49\alpha-3}{4}$</td><td>if $\alpha \equiv 2 \pmod 4$, we get $\frac{49\alpha-2}{4}$</td><td>if $\alpha \equiv 0 \pmod 4$, we get $\frac{49\alpha}{4}$</td></tr>
<tr><td>we get $\frac{\alpha}{4}$ - $\frac{5}{196}$</td><td>we get $\frac{\alpha}{4}$ - $\frac{3}{196}$</td><td>we get $\frac{\alpha}{4}$ - $\frac{1}{98}$</td><td>we get $\frac{\alpha}{4}$</td></tr>
<tr><td colspan="4">we get $\lfloor\frac{\alpha}{4}\rfloor$, the quotient when α is divided by 4</td></tr>
</table>

We can also change the times of repeated operation. By repeating operation for m times, we can change the numbers 3 and 9 in the trick into n and n^2, as long as when $\frac{n^2\alpha-2^m+1}{2^m}$ (or $\frac{n^2\alpha-2^m+2}{2^m}$, $\cdots$, or $\frac{n^2\alpha-1}{2^m}$, or $\frac{n^2\alpha}{2^m}$) is an integer(recall that if $\frac{p+q2^m}{2^m}$, where p and q are all integers, is an integer, so does $\frac{p}{2^m}$), all possible α have an unique and same remainder when divided by 2^m, and in different situations, the remainders are different.

2.(a)

Solution

The smallest fashionable number is $\frac{1}{2}$ and the largest fashionable number is 1.

In order to prove this, we just need to show that all the fashionable numbers we get through $\frac{1}{a+b}$, where a and b are fashionable numbers, is in the range of $[\frac{1}{2},1]$, through mathematical induction.

P_n: the nth fashionable number we get is in the range of $[\frac{1}{2},1]$.

We firstly have $\frac{1}{2}$ ($\frac{1}{1+1}$) and 1 as fashionable numbers, and then get the first fashionable numbers $\frac{2}{3}$ ($\frac{1}{1+\frac{1}{2}}$), which is in the range of $[\frac{1}{2},1]$, meaning that P_1 is true.

Then, assume that P_i is true, meaning that ith fashionable number we get, F_i, is in the range of $[\frac{1}{2},1]$, where i = 1,2,3,⋯,k.

Then, for the (k+1)th fashionable number F_{k+1},

$$\frac{1}{2} = \frac{1}{1+1} \leq \frac{1}{F_{i_1} + F_{i_2}} = F_{k+1} \leq \frac{1}{\frac{1}{2}+\frac{1}{2}} = 1$$

where i = 1,2,3,⋯,k, meaning that P_{k+1} is true.

Since P_1 is true and $P_i \rightarrow P_{k+1}$ for i = 1,2,3,⋯,k, by mathematical induction, P_n is true for all positive integers.

Thus, the smallest fashionable number is $\frac{1}{2}$ and the largest fashionable number is 1.

2.(b)

Solution

In fact, all the rationals in the range of $[\frac{1}{2},1]$ are fashionable numbers, which can be prove through mathematical induction.
Recall that all the rationals can be expressed as $\frac{a}{b}$.
P_n: all the fractions, with n as their denominators and in the range of $[\frac{1}{2},1]$, are fashionable numbers.
Obviously, P_1 is correct($\frac{1}{1}=1$).
Then, assume that all P_i, where i = 1,2,⋯,k, are correct, for P_{k+1}:
①if $(k+1)$ is even, then we let

$$k+1=2a,$$

where a is an integer.
All the fractions, with 2a as their denominators and in the range of $[\frac{1}{2},1]$, are

$$\frac{a}{2a},\frac{a+1}{2a},\frac{a+2}{2a},\cdots,\frac{2a-2}{2a},\frac{2a-1}{2a},\frac{2a}{2a}.$$

Understandably, $\frac{a}{2a}$ $(\frac{1}{2})$ and $\frac{2a}{2a}$ (1) are included, so we need to show that the others are also fashionable numbers.
Generalise, we get a fraction

$$\frac{N}{2a}$$

where a + 1 ≤ N ≤ 2a - 1.
Since $\frac{1}{2}<\frac{a}{2a-1}\leq\frac{a}{N}\leq\frac{a}{a+1}<1$, based on our assumption, we have $\frac{a}{N}$ as a fashionable number.
Thus, we can get $\frac{N}{2a}$ through

$$\frac{N}{2a}=\frac{1}{\frac{a}{N}+\frac{a}{N}}.$$

②if $(k+1)$ is odd, then we let

$$k+1=2a+1,$$

where a is an integer.
All the fractions, with (2a + 1) as their denominators and in the range of $[\frac{1}{2},1]$, are

$$\frac{a+1}{2a+1},\frac{a+2}{2a+1},\cdots,\frac{2a-1}{2a+1},\frac{2a}{2a+1},\frac{2a+1}{2a+1}.$$

Understandably, $\frac{2a+1}{2a+1}$ (1) is included, so we need to show that the others are also fashionable numbers.
Generalise, we get a fraction

$$\frac{N}{2a+1}$$

where a + 1 ≤ N ≤ 2a.
Since $\frac{1}{2}=\frac{a}{2a}\leq\frac{a}{N}\leq\frac{a}{a+1}<1$ and $\frac{1}{2}<\frac{a+1}{2a}\leq\frac{a+1}{N}\leq\frac{a+1}{a+1}=1$, based on our assumption, we have $\frac{a}{N}$ and $\frac{a+1}{N}$ as a fashionable number.
Thus, we can get $\frac{N}{2a}$ through

$$\frac{N}{2a+1}=\frac{1}{\frac{a}{N}+\frac{a+1}{N}}.$$

In conclusion, since P_1 is true and $P_i\rightarrow P_{k+1}$ where i = 1,2,⋯,k, by mathematical induction, P_n is true for all positive integers.
Thus, all the rationals in the range of $[\frac{1}{2},1]$ are fashionable numbers.

2.(c)

Solution

Well, I have to admit that I fail to solve this question, so what I will do is to show you some of my basic ideas.
Obviously, if we are told that $x_1, x_1, \cdots, x_n$ are all fashion numbers, the the range of all fashion numbers is

$$[\frac{1}{2\cdot\max\{x_1,x_1,\cdots,x_n\}}, \max\{x_1,x_1,\cdots,x_n\}] \text{ or } [\max\{x_1,x_1,\cdots,x_n\}, \frac{1}{2\cdot\max\{x_1,x_1,\cdots,x_n\}}],$$

which can be proven through mathematical induction like 3(a).

So now, here's the next part: which specific numbers are in the set of all fashion numbers? This is an important question: if we want to find out the final result, we need to firstly have a 'range' and then to prove it. Whereas, I fail to find this 'range'.
Take the condition mentioned in 3(c) as an example, 2 and $\sqrt{3}$ are fashion numbers.
In this way, all the fashion numbers can be expressed as

$$2p + q\sqrt{3}$$

where p and q are all rationals, because, just like closure, we can not derive $\sqrt{5}$ through 2, $\sqrt{3}$ and $\frac{1}{a+b}$.
But here's the confusing point: we know that all fashion numbers should be in the range of $[\frac{1}{4}, 2]$, but when p = 0, these fashion numbers can only be in the range of $[\frac{\sqrt{3}}{6}, \sqrt{3}]$!
It seems like that we derive two contradictory conclusions, be if we take each of these conclusions into consideration, they are all correct, meaning that there must be some other regulations influencing the range of fashion numbers.
Disappointingly, we fail to find these extra regulations.

Another difficulty is that, if you analyse the mathematical induction we used in 3(b), the key point is to turn a fraction with n_1 as denominator into result of the operation of two fractions with n_2 as denominators, where $n_2 > n_1$.
This is reasonable: because the basic idea of mathematical induction is to gradually derive the general conclusion starting with a small value.
This method is helpful in 3(b) because all the fashion numbers are in the range of $[\frac{1}{2}, 1]$, thus any fractions with n_1 as denominator can be expressed as the result of the operation of two fractions with n_2 as denominators, where $n_2 > n_1$; whereas, in 3(c), if 2 and $\sqrt{3}$ are fashion numbers, then not all fractions with n_1 as denominator can be expressed as the result of the operation of two fractions with n_2 as denominators, where $n_2 > n_1$.

For example.

$$\frac{7}{4} = \frac{1}{\frac{2}{7} + \frac{2}{7}}$$

This means that we cannot use mathematical induction to prove the conclusion!
So now we have two ways to deal with this problem.
One is to 'narrow the range'. If 2 and $\sqrt{3}$ are fashion numbers, all fashion numbers in the range of [1,2] can be expressed by the result of the operation of two repeated fashion numbers in the range of $[\frac{1}{4}, \frac{1}{2}]$, meaning that we just need to prove that all

$$2p + q\sqrt{3},$$

where p and q are all rationals, in the range of $[\frac{1}{4}, 1]$ are fashion numbers. However, this strategy is still hard to prove because we cannot prove that $\frac{1}{1}$ is fashion number and, during the process, we need to involve some fashion numbers in the range of [1,2] if we want to prove our conclusion.
Another strategy is to use other way to prove the conclusion: maybe by contradiction? But other methods have one big mistake: they cannot show all

$$2p + q\sqrt{3},$$

where p and q are all rationals, in the range of $[\frac{1}{4},1]$ are fashion numbers unless we prove them one by one, because other methods only involve finite times of operations(or proofs, should I say, since we can only show P_i for almost specific i), but mathematical induction involves finite times of operations(or proofs, since we can show P_i for almost all i).
That be said, we fail to solve the question(sadly :( ).

In conclusion, if we know that some numbers are fashion numbers, we can find the range of all fashion numbers, but I fail to prove that all numbers in the range are fashion numbers.

2.

Afterword

I firstly saw this type of problem last year when I was preparing for PROMYS, and here's the question:

The set S contains some real numbers, according to the following three rules.
(i) $\frac{1}{1}$ *is in S.*
(ii) If $\frac{a}{b}$ *is in S, where* $\frac{a}{b}$ *is written in lowest terms (that is, a and b have highest common factor 1), then* $\frac{2b}{a}$ *is in S.*
(iii) If $\frac{a}{b}$ *and* $\frac{c}{d}$ *are in S, where they are written in lowest terms, then* $\frac{a+c}{b+d}$ *is in S.*
These rules are exhaustive, can you describe which numbers are in S?

Well, through such experiences, I knew that when solving a problem which involves a plenty of different terms, it is always helpful to find a few terms in advance.
Moreover, actually, all these questions involve the same key point: proving infinite terms through mathematical induction. Thanks to that experience, when I solve the similar question now, I almost immediately come up with the ideas.