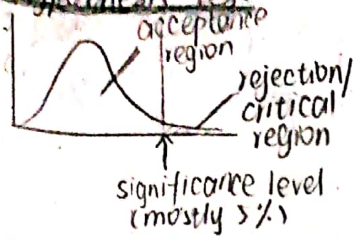
一 hypothesis test

① 

比方说 H_0、H_1 两种情况，其中 H_1 的 μ 在 H_0 的 critical region 内，那我可以认为 H_0 不被 accept

② in a hypothesis test, the claim is called the null hypothesis, H_0.
no to accept it, you must have an alternative hypothesis to accept, H_1.

③ 根据问题，选择 $\geq$、$\leq$、$>$、$<$、$\neq$
当选用 $\neq$ 时，significance level 要一分为二

④ type I error: H_0 对了你选 H_1
type II error: H_1 对了你选 H_0
* never say 'true' or 'right', only 'accept'

三 linear combination of random variables

① $\left.\begin{matrix} E(x+b) = E(x)+b \\ E(ax) = aE(x) \end{matrix}\right\} E(ax+b) = aE(x)+b$

$\left.\begin{matrix} Var(x+b) = Var(x) \\ Var(ax) = a^2 Var(x) \end{matrix}\right\} Var(ax+b) = a^2 Var(x)$

② $\left.\begin{matrix} E(X+Y) = E(X)+E(Y) \\ Var(X+Y) = Var(X)+Var(Y) \end{matrix}\right\} Var(2X) \neq Var(X)+Var(X)$

③ If continuous random variables X and Y respectively have a normal distribution, then $aX+b$ also has a normal distribution, so does $aX+bY$.
• $Z = \frac{X-\mu}{\sigma} \longrightarrow X = \sigma Z+\mu \longrightarrow$ linear combination

④ If $X \sim Po(\lambda)$, $Y \sim Po(\mu)$, then $X+Y \sim Po(\lambda+\mu)$

五 sampling

① $E(\bar{x}(n)) = \mu$, where $\mu = E(x)$
$Var(\bar{x}(n)) = \frac{\sigma^2}{n}$, where $\sigma^2 = Var(x)$

• $E(\bar{x}(n)) = E(\frac{1}{n}(x_1+x_2+\cdots+x_n))$
$= \frac{1}{n}(E(x_1)+E(x_2)+\cdots+E(x_n))$
$= E(x)\cdot\frac{1}{n}\cdot n = E(x)$

• $Var(\bar{x}(n)) = Var(\frac{1}{n}(x_1+x_2+\cdots+x_n))$
$= \frac{1}{n^2}(Var(x_1)+Var(x_2)+\cdots+Var(x_n))$
$= Var(x)\cdot\frac{1}{n^2}\cdot n = \frac{1}{n}Var(x)$

② provided $n > 30$ $\quad \bar{x}(n) \sim N(\mu, \frac{\sigma^2}{n})$
* 当 n 很小时，若图像像个 normal，则 n 可以适当取小；若不像，则 $n>30$ 时定像 normal

③ 解题：告诉一个 data 的 μ 与 σ^2，已知 n 个 data 为一个 sample，求 sample 概率
• 方法一 n 个 sample 一块的为 μ 与 $\frac{\sigma^2}{n}$
• 方法二 n 个 data 加一起，为 $n\mu$ 与 $n\sigma^2$

④ confidence interval (CI)：既 true 值在一定的可能性中会分布的区间
population mean: $(\bar{x} - k\frac{\sigma}{\sqrt{n}}, \bar{x}+k\frac{\sigma}{\sqrt{n}})$
population proportion: $(\hat{p} - k\sqrt{\frac{\hat{p}\hat{q}}{n}}, \hat{p}+k\sqrt{\frac{\hat{p}\hat{q}}{n}})$

二 poisson distribution

① when $\mu \approx \sigma^2$, we use poisson distribution
$X \sim Po(a)$ so $E(x) = a$ and $Var = a$
$P(X=r) = e^{-\lambda}\frac{\lambda^r}{r!}$
* $n \to \infty$, $p \to 0$, $np \to \lambda$
• used when events occur singly, at random and independently, in a given interval of space and time
• events occur at a constant rate; the mean average number of events in a given interval is proportional to that interval

② $X \sim B(n, \frac{\lambda}{n})$ for $n \to \infty$ $\approx X \sim Po(\lambda)$ for $n>50$, $\lambda<5$
• $P(X=x) = {}^nC_x \cdot (\frac{\lambda}{n})^x (1-\frac{\lambda}{n})^{n-x}$
$= \frac{n!}{x!(n-x)!}\cdot\frac{\lambda^x}{n^x}\cdot(1-\frac{\lambda}{n})^{n-x}$
$= \frac{n(n-1)(n-2)\cdots(n-x+1)}{n^x}\cdot\frac{\lambda^x}{x!}\cdot(1-\frac{\lambda}{n})^n\cdot(1-\frac{\lambda}{n})^{-x}$
$\because \frac{n(n-1)(n-2)\cdots(n-x+1)}{n^x}$ 分子 x 项，分母 x 项，为 1
$\because (1-\frac{\lambda}{n})^{-x}$, $n\to\infty$, $1-\frac{\lambda}{n} \to 1$，为 1
$\because (1-\frac{\lambda}{n})^n \to$ if $\frac{1}{y} = -\frac{\lambda}{n}$, $n = -\lambda y$
$(1+\frac{1}{y})^{-\lambda y} = e^{-\lambda}$ 为 $e^{-\lambda}$
$\therefore P(X=x) = e^{-\lambda}\cdot\frac{\lambda^x}{x!}$

③ $X \sim Po(\lambda) \approx X \sim N(\lambda, \lambda)$ for $\lambda > 15$
* discrete 要扩

四 continuous random variables

① if X is a continuous random variable with probability density function (PDF) $f(x)$:
$\int_{-\infty}^{\infty} f(x)dx = 1$
$f(x) \geq 0$ for all x
$P(a \leq x \leq b) = \int_a^b f(x)dx$
$P(x=a) = 0$
$E(x) = \int_{-\infty}^{\infty} x f(x)dx$
$Var(x) = \int_{-\infty}^{\infty} x^2 f(x)dx - [\int_{-\infty}^{\infty} x f(x)dx]^2$
* 不一定积分要取到极限，取到 boundary 就行

② 求 x percentile，则 $P(x \leq m) = x$，求 m 即可

六 estimation

① $\hat{v}$ is unbiased estimate for V if $E(\hat{v}) = V$
• the most effecient estimate is one that is unbiased and has smaller variance

② $E(V) = \frac{n-1}{n}\sigma^2 \rightarrow E(\frac{nV}{n-1}) = \sigma^2$
$\sigma^2 = \frac{1}{n-1}(\Sigma x^2 - n\bar{x}^2) = \frac{1}{n-1}(\Sigma x^2 - \frac{(\Sigma x)^2}{n})$
$= \frac{1}{n-1}(\Sigma x^2 - \frac{(\Sigma x)^2}{n})$
$= \frac{1}{n-1}(nE((x-\bar{x})^2))$
$= \frac{1}{n-1}(\Sigma(x-\bar{x})^2)$
* 注意是求 population 还是 set 的 Var
注意是 $\frac{1}{n}$ 还是 $\frac{1}{n-1}$

③ $\because$ 平均每一次的 sd 为 $\sqrt{\frac{\sigma^2}{n}} = \frac{\sigma}{\sqrt{n}}$ $[\sigma^2+\sigma^2+\cdots+\sigma^2 = n\sigma^2]$
$\therefore \bar{\sigma} = \frac{\sigma}{\sqrt{n}}$
$\therefore$ 用 normal 验证假说，$\bar{\mu} = \mu$，$\bar{\sigma} = \frac{\sigma}{\sqrt{n}}$
$z = \frac{\bar{x}-\mu}{\frac{\sigma}{\sqrt{n}}}$
• 要是不给 σ^2 则要先 estimate

二. polynomial equation

① 定义 $\sum\alpha = \alpha+\beta+\gamma+\cdots = -\frac{b}{a}$

$\sum\alpha\beta = \alpha\beta+\alpha\gamma+\cdots = \frac{c}{a}$

$\sum\alpha\beta\gamma = \alpha\beta\gamma+\cdots = -\frac{d}{a}$

$\vdots$

$S_n = \alpha^n+\beta^n+\gamma^n+\cdots$

$S_1 = \sum\alpha$

$S_2 = \sum\alpha^2 = (\sum\alpha)^2 - 2\sum\alpha\beta$

$S_{-1} = \sum\frac{1}{\alpha} = \frac{\sum\alpha\beta\gamma\cdots[n-1个]}{\sum\alpha\beta\gamma\cdots[n个]}$ [n为最高次次数]

$= -\frac{T_1系数}{T_0系数}$

e.g. $\sum\alpha^3 = (\sum\alpha)^3 - 3\sum\alpha\beta\sum\alpha + 3\sum\alpha\beta\gamma$

② $ax^n+bx^{n-1}+\cdots+cx+d=0$

$\downarrow$

$aS_n+bS_{n-1}+\cdots+cS_1+dS_0=0$

$\downarrow$

$aS_{n+w}+bS_{n-1+w}+\cdots+cS_{1+w}+dS_{0+w}=0$

③ if roots are $\alpha,\beta,\gamma\cdots$, find the equation whose roots are $f(\alpha), f(\beta), f(\gamma)\cdots$

- let $y=f(x)$, turn it into $x=f^{-1}(y)$
- use $f^{-1}(y)$ in the equation instead of x
- get the equation.

$\downarrow$

if want to find S_{na}, let $y=x^n$, turn into a new equation, then the S_a of new equation is equal to the S_{na} of original one

三. summation of series

① $\sum_{r=1}^{n} r = \frac{n(n+1)}{2}$

- $2^2=(1+1)^2=1^2+2\times1+1$

$3^2=(2+1)^2=2^2+2\times2+1$

$\vdots$

$(n+1)^2=n^2+2\times n+1$

$\downarrow \quad \downarrow$

$2^2+3^2+\cdots+(n+1)^2 = (1^2+2^2+\cdots+n^2)+2\times(1+2+\cdots+n)+n$

$1^2+2^2+3^2+\cdots+(n+1)^2 = (1^2+2^2+\cdots+n^2)+2\times(1+2+\cdots+n)+n+1$

$(n+1)^2+\sum_{r=1}^{n} r^2 = \sum_{r=1}^{n} r^2 + 2\sum_{r=1}^{n} r + n+1$

$2\sum_{r=1}^{n} r = (n+1)(n+1-1)$

$\sum_{r=1}^{n} r = \frac{n(n+1)}{2}$

同理推导:

$\sum_{r=1}^{n} r^2 = \frac{1}{6}n(n+1)(2n+1)$

$\sum_{r=1}^{n} r^3 = \frac{1}{4}n^2(n+1)^2$

② 计算 $\sum_{r=b}^{an} f(r)$, 把an当成n代入 $\sum_{k=1}^{n} f(r)$, finally

$\sum_{r=b}^{an} f(r) = \sum_{r=1}^{an} f(r) - \sum_{r=1}^{b-1} f(r)$

③ 计算 $\sum_{r=1}^{n} \frac{a}{(r+b)(r+c)}$, 化成 $\sum_{r=1}^{n}\left(\frac{A}{r+b}+\frac{B}{r+c}\right)$, 然后消

④ for convergence, $\sum_{r=a}^{n} f(n) = b + \frac{c}{dn+e}+\cdots$, allow $n \xrightarrow{\lim} \infty$

三. rational functions

①

function	intersection	asymptote
$y=\frac{ax+b}{cx+d}$	$(0,\frac{b}{d})$, $(-\frac{b}{a},0)$	$y=\frac{a}{c}$, $x=-\frac{d}{c}$
$y=\frac{ax+b}{(cx+d)(ex+f)}$	$(0,\frac{b}{df})$, $(-\frac{b}{a},0)$	$y=0$, $x=-\frac{d}{c}$, $x=-\frac{f}{e}$
$y=\frac{(ax+b)(cx+d)}{(ex+f)(gx+h)}$	$(0,\frac{bd}{fh})$, $(-\frac{b}{a},0)$ $(-\frac{d}{c},0)$	$y=\frac{ac}{eg}$, $x=-\frac{f}{e}$, $x=-\frac{h}{g}$
$y=\frac{(ax+b)(cx+d)}{ex^2+fx+g}$ where $ex^2+fx+g\neq0$	$(0,\frac{bd}{g})$, $(-\frac{b}{a},0)$ $(-\frac{d}{c},0)$	$y=\frac{ac}{e}$
$y=\frac{(ax+b)(cx+d)}{ex+f}$	$(0,\frac{bd}{f})$, $(-\frac{b}{a},0)$ $(-\frac{d}{c},0)$	$y=\frac{ac}{e}x+B$, $x=-\frac{f}{e}$

* to find turning point / range

- using $\frac{dy}{dx}=0$ to find the number of turning points
- turn $y=f(x)$ into $ayx^2+byx+cy=0$, then let $\Delta<0$ $((by)^2-4(ay)(cy)<0)$ (means y can't be in the set

** oblique asymptotes

$y=\frac{ax^2+bx+c}{ex+f} \rightarrow y=Ax+B+\frac{C}{ex+f}$, where $A=\frac{a}{e}$.

asymptotes: $y=Ax+B$

*** using when $x \xrightarrow{\lim}$ asymptotes, $y>0$ (or $y<0$) to stretch [also useful for oblique asymptotes]

**** $y=\frac{ax^2+bx+c}{dx+e}$ only has 0 or 2 turning points

when it has 2 turning points, the minimum for the top branch is greater than the maximum for the bottom branch

② stretch

asymptotes → intersection → $\frac{dy}{dx}$ 找拐点 →

$x \xrightarrow{\lim}$ asymptotes 找y.

③ 不等式

画图 → $f(x)=c$ 求关键值 → 取范围

④ range 如上

⑤ transformation:

$y=\frac{1}{f(x)}$: 有 $f(x)=0$ → 用 asymptotes 画,

无 $f(x)=0$ → 用 range 画,

$y=|f(x)|$: y轴下面的翻到上去,

$y=f(|x|)$: x轴左右对称, 先画右边

$y^2=f(x)$: turn into $y=\pm\sqrt{f(x)}$, 先画 $y=\sqrt{f(x)}$, 再以y轴对称.

四 matrices.

① $\begin{pmatrix}a&b\\c&d\end{pmatrix}\pm\begin{pmatrix}e&f\\g&h\end{pmatrix}=\begin{pmatrix}a\pm e&b\pm f\\c\pm g&d\pm h\end{pmatrix}$ → column, ↓ row

$k\begin{pmatrix}a&b\\c&d\end{pmatrix}=\begin{pmatrix}ka&kb\\kc&kd\end{pmatrix}$

$\begin{pmatrix}a&b\\c&d\end{pmatrix}\cdot\begin{pmatrix}e&f\\g&h\end{pmatrix}=\begin{pmatrix}ae+bg&af+bh\\ce+dg&cf+dh\end{pmatrix}$

* $(\bullet) = (\longrightarrow)\times(\downarrow)$. generally, $AB\neq BA$

② when one matrices is a zero matrix, $AB=BA$

$A^mA^n=A^nA^m=A^{m+n}$

the identity matrix, I, is $\begin{pmatrix}1&0&\cdots&0\\0&1&&0\\\vdots&&\ddots&\vdots\\0&0&\cdots&1\end{pmatrix}$, $IA=AI=A$

$(AB)CD = A(BC)D = AB(CD) = (AB)(CD)$

③ inverse matrix：试数，整行加/减整行
【一般多用含0行作为被加/减行】

formulation: $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ $A^{-1} = \frac{1}{ad-bc}\begin{pmatrix} d & -b \\ -c & a \end{pmatrix}$

* any $n \times n$ matrix with a repeated row cannot have an inverse $\longrightarrow$ singular matrix

$(AB)^{-1} = B^{-1}A^{-1}$

④ $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$, $\det(A) = \begin{vmatrix} a & b \\ c & d \end{vmatrix} = ad - bc$

* for any matrix $\begin{pmatrix} a & b \\ kc & kd \end{pmatrix}$ where $a, b \neq 0$. $\det = 0$, so there's no inverse matrix

for a 3×3 determinant, by multiplying each top element by a corresponding 2×2 determinant

$$\begin{vmatrix} a_1 & a_2 & a_3 \\ a_4 & a_5 & a_6 \\ a_7 & a_8 & a_9 \end{vmatrix} = a_1\begin{vmatrix} a_5 & a_6 \\ a_8 & a_9 \end{vmatrix} - a_2\begin{vmatrix} a_4 & a_6 \\ a_7 & a_9 \end{vmatrix} + a_3\begin{vmatrix} a_4 & a_5 \\ a_7 & a_8 \end{vmatrix}$$

* $\begin{vmatrix} + & - & + \\ - & + & - \\ + & - & + \end{vmatrix}$. $(-1)^{m+n}$

when you turn $r_{n_1} \to ar_{n_1} + br_{n_2}$. $a \cdot \det$ as well

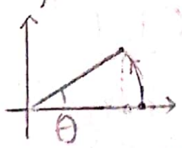
* this measure is used to simplify the way of finding out det. by turning a matrix into another matrix which has as many 0 as possible, then divide the new det by $a_1, a_2 \cdots\cdots$

$\det(AB) = \det(BA) = \det(A)\det(B)$

⑤ transformation in 2×2 matrix

- stretch by a scale factor k in x-axis $\begin{pmatrix} k & 0 \\ 0 & 1 \end{pmatrix}$, y-axis $\begin{pmatrix} 1 & 0 \\ 0 & k \end{pmatrix}$
- enlargement with center of enlargement the origin by a scale factor k: $\begin{pmatrix} k & 0 \\ 0 & k \end{pmatrix}$
- reflection in
  - x-axis $\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$
  - y-axis $\begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}$
  - $y = x$ $\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$
- rotation about the origin by θ in anticlockwise direction: $\begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}$

* proof:



from $\begin{pmatrix} x \\ 0 \end{pmatrix}$ to $\begin{pmatrix} x\cos\theta \\ x\sin\theta \end{pmatrix}$

$\begin{pmatrix} \cos\theta & b \\ \sin\theta & d \end{pmatrix}\begin{pmatrix} x \\ 0 \end{pmatrix} = \begin{pmatrix} x\cos\theta \\ x\sin\theta \end{pmatrix}$

$\because \det = 1$, $\cos^2\theta + \sin^2\theta = 1$

$\therefore \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}\begin{pmatrix} x \\ 0 \end{pmatrix} = \begin{pmatrix} x\cos\theta \\ x\sin\theta \end{pmatrix}$

PS:
det=0时
只有一条invariant line.

shearing in x-axis $\begin{pmatrix} 1 & k \\ 0 & 1 \end{pmatrix}$, y-axis $\begin{pmatrix} 1 & 0 \\ k & 1 \end{pmatrix}$

* if we want to find the invariant lines:（经过变化仍在该解析式上）

$\begin{pmatrix} a & b \\ c & d \end{pmatrix}\begin{pmatrix} t \\ mt \end{pmatrix} = \begin{pmatrix} T \\ mT \end{pmatrix}$

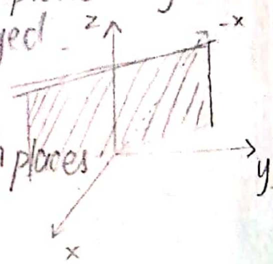
$\begin{cases} at + bmt = T \\ ct + dmt = mT \end{cases} \rightarrow \frac{a+bm}{c+dm} = \frac{1}{m}$

and the line $y = mx$ is the invariant line (after all, (0,0) never changes)

⑥ transformation in 3×3 matrix

- rotation by angle θ in the anticlockwise direction in the
  - x-axis $\begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos\theta & -\sin\theta \\ 0 & \sin\theta & \cos\theta \end{pmatrix}$
  - y-axis $\begin{pmatrix} \cos\theta & 0 & -\sin\theta \\ 0 & 1 & 0 \\ \sin\theta & 0 & \cos\theta \end{pmatrix}$
  - z-axis $\begin{pmatrix} \cos\theta & -\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{pmatrix}$
- enlargement with centre of enlargement the origin by scale factor k $\begin{pmatrix} k & 0 & 0 \\ 0 & k & 0 \\ 0 & 0 & k \end{pmatrix}$

* something interesting:
when reflection in plane $x + y = 0$



① z-axis unchanged
$\begin{pmatrix} a & b & 0 \\ c & d & 0 \\ 0 & 0 & 1 \end{pmatrix}$

② -x and y switch places
$\begin{pmatrix} 0 & -1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}$

总结：分解成一轴不变，简化为两轴变

~~1. 只有方阵才有inverse~~

if want to find inverse:

$\begin{pmatrix} a_{11} & a_{21} & a_{31} \\ a_{12} & a_{22} & a_{32} \\ a_{13} & a_{23} & a_{33} \end{pmatrix}$

turn $a_{11} \to (-1)^{1+1}\begin{vmatrix} a_{22} & a_{32} \\ a_{23} & a_{33} \end{vmatrix} \cdot \frac{1}{\det}$

⋮

turn $a_{22} \to (-1)^{2+2}\begin{vmatrix} a_{11} & a_{31} \\ a_{13} & a_{33} \end{vmatrix} \cdot \frac{1}{\det}$

⋮

2. $\begin{pmatrix} 0 & 1 & 0 \\ - & - & - \\ - & - & - \end{pmatrix}\begin{pmatrix} - & - & - \\ - & - & - \\ - & - & - \end{pmatrix}$ 取出第2个row

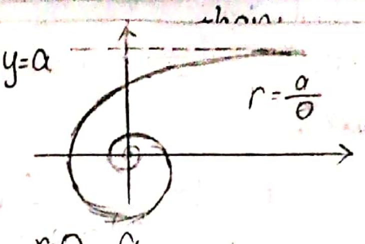
$\begin{pmatrix} - & - & - \\ - & - & - \\ - & - & - \end{pmatrix}\begin{pmatrix} 0 & - & - \\ 1 & - & - \\ 0 & - & - \end{pmatrix}$ 取出第2个column

五 polar coordinate (r,θ)

① $r=f(\theta)$ for $r>0$

$\begin{cases} x=r\cos\theta \\ y=r\sin\theta \\ x^2+y^2=r^2 \end{cases}$

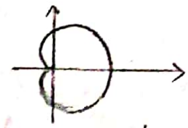
解equation: • $x,y \leftrightarrow r\square\theta$
• $r=\square\theta$
$r^2=r\square\theta \to x,y$
$x^2+y^2 \to x,y$



$r\theta=a$
when $\theta \xrightarrow{\lim} 0$, $\theta \doteq \sin\theta$
$y=r\sin\theta=\frac{a}{\theta}\sin\theta=\frac{a}{\sin\theta}\sin\theta=a$
渐近线 $y=a$

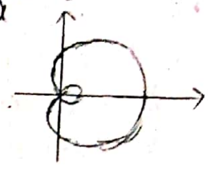
$r=a+b\cos\theta$

$a=b$



$\theta=\frac{\pi}{2}/\frac{3\pi}{2}$ 时不是极值

$a<b$

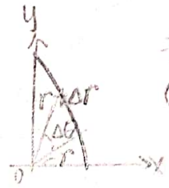


$a>b$

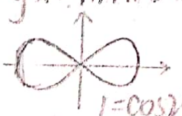
画图 · key value 列表
• symmetry: $\theta=0$ for $r=\cos\theta$
$\theta=\frac{\pi}{2}$ for $r=\sin\theta$
* similiar to Cartesian form
for $r=a+b\square c\theta$, $c\theta=\square$
• when $\theta \xrightarrow{\lim} 0$, $\sin\theta \approx \theta$

② application
• intersection
polar curves intersect
$\longrightarrow$ Cartesian equation too
[注意原点不一定相交]
步骤: form an equation & solve
↓
write down all possible values
↓
check $r\geq 0$

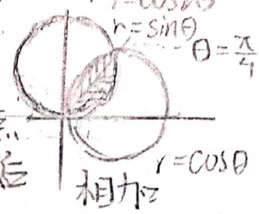
• Area



if $\Delta\theta \xrightarrow{\lim} 0$, $A \to \frac{1}{2}r^2\theta$
$A=\frac{1}{2}\int_a^b r^2\,d\theta$
[求面积,画图找symmetry和limits,
不要考虑 $r<0$ 时面积]

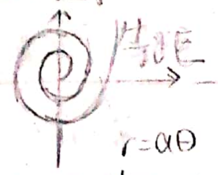


相差面积: $\frac{1}{2}\int_a^b R^2-r^2\,d\theta$
求交集面积: 找交点
↓
分别求交点
至limit面积
相加

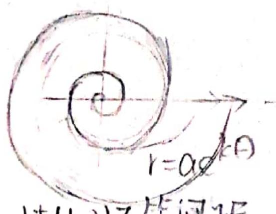


求 $r/x/y$ 极值 $\frac{dr}{d\theta}/\frac{dx}{d\theta}/\frac{dy}{d\theta}=0$

③ 典型例图



过(0,0),等间距

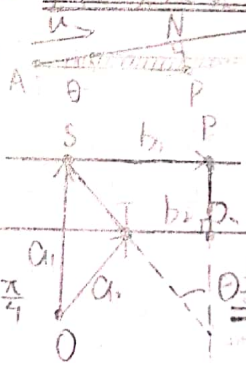


过(1,0)不等间距
if $\theta \xrightarrow{\lim} -\infty$, $r\to 0$

六 vector

① $a\times b=|a||b|\sin\theta\,\hat{n}$
where $\hat{n}$ is a unit vector that is a common perpendicular to a and b
投影在一个方向上,则长度为面积
vector determinant: $\begin{vmatrix} i & j & k \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix}=\begin{pmatrix} a_1 \\ a_2 \\ a_3 \end{pmatrix}\times\begin{pmatrix} b_1 \\ b_2 \\ b_3 \end{pmatrix}$
求两边四边形: $|a\times b|$
三角形: $\frac{1}{2}|a\times b|$
求三边体积: $\frac{1}{6}|(a\times b)\cdot c|$

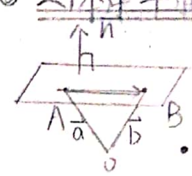
② 求点线长度: 用面积转化成长度



if $\vec{u}$ is a unit vector
NP为阴影四边形中以 $\vec{u}$ 为底时的高
$|\vec{AP}\times\vec{u}|=|\vec{u}|\times NP$
$NP=|\vec{AP}\times\vec{u}|$ as $|\vec{u}|=1$

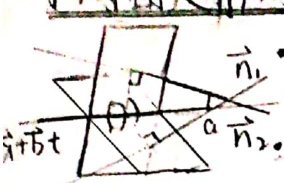
③ 求线线距离
当两条线所在面平行(否则为0)
• $\begin{cases} r_1=a_1+b_1t \\ r_2=a_2+b_2s \end{cases}$
$PQ=|ST\cos\theta|$
$=\left||\vec{ST}|\cdot\frac{\vec{ST}\cdot\vec{PQ}}{|\vec{ST}|\cdot|\vec{PQ}|}\right|$
$=\left|\frac{(a_2-a_1)\cdot(b_1\times b_2)}{|b_1\times b_2|}\right|$ (保证为正)
• 设两点P and Q在 r_1,r_2 上
可求 $\vec{PQ}$, $\vec{PQ}\cdot r_1=0$, $\vec{PQ}\cdot r_2=0$, 求解s与t

④ 立体中平面表示:



• $(\vec{b}-\vec{a})\cdot\vec{n}=0$ ($\vec{n}$ is a normal vector)
$\begin{pmatrix} x \\ y \\ z \end{pmatrix}\vec{n}=\vec{b}\cdot\vec{n}$ (if $\vec{b}$ is known, $\vec{a}=\begin{pmatrix} x \\ y \\ z \end{pmatrix}$)
• if $\vec{r}=\vec{a}+\vec{b}s+\vec{c}t$, 求another form
$\vec{n}=\vec{b}\times\vec{c}$
$\vec{a}\cdot\vec{n}=\begin{pmatrix} x \\ y \\ z \end{pmatrix}\vec{n}$

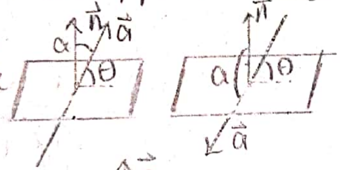
求两面夹角/两面交线



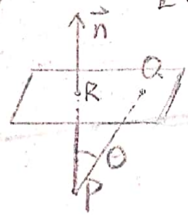
- $\theta=\pi-\alpha$ where $\cos\alpha=\frac{\vec{n}_1\cdot\vec{n}_2}{|\vec{n}_1||\vec{n}_2|}$
- for the line of intersection , details
  $\vec{b}=\vec{n}_1\times\vec{n}_2$
  for $\vec{a}$: 设 x/y/z 任一为 0
- 消元, 将 x、y、z 用任一一个唯一 x、y、z 表示, 再将此值標为 t
  $\left.\begin{matrix}x=a_1z+b_1\\ y=a_2z+b_2\end{matrix}\right\}\rightarrow r=\begin{pmatrix}b_1\\b_2\\0\end{pmatrix}+\begin{pmatrix}a_1\\a_2\\1\end{pmatrix}t$

求线面交点/线面夹角

- turn the equation of line into parametric form
  $r=\begin{pmatrix}a_1\\a_2\\a_3\end{pmatrix}+\begin{pmatrix}b_1\\b_2\\b_3\end{pmatrix}t\rightarrow r=(a_1+b_1t)i+(a_2+b_2t)j+(a_3+b_3t)k$
  then combine it with the equation of plane
  $\because Ax+By+Cz=D\rightarrow(a_1+b_1t)A+(a_2+b_2t)B+(a_3+b_3t)C=D$
  solve t
- 平行 → 法向量点乘 = 0
- parametric form 带入 equation of plane 后发现 D = D' (t 相消) → 包含
  parametric form 带入 equation of plane 后发现 E ≠ D (t 相消) → 平行但不包含
- 求角度 → acute: $\theta=\frac{\pi}{2}-\alpha$
  obtuse $\theta=\alpha-\frac{\pi}{2}$



求点面距离



- 法向量为方向,
  if $\vec{n}=ai+bj+ck$
  $\therefore \vec{PR}=\begin{pmatrix}A\\B\\C\end{pmatrix}+\begin{pmatrix}a\\b\\c\end{pmatrix}t$
  带入求 R 坐标后算距离 or 带入平面求 t 后
  距离为 $t\sqrt{a^2+b^2+c^2}$
- $|\vec{PR}|=\left||\vec{PQ}|\cdot\cos\theta\right|=\left|\frac{\vec{PQ}\cdot n}{|n|}\right|$
  where Q can be any points on plane

osborne's rule
to move from a trigonometric identity to a hyperbolic identity:
change a cos to a cosh
change the sign of a products of sinhs

C. proof by induction

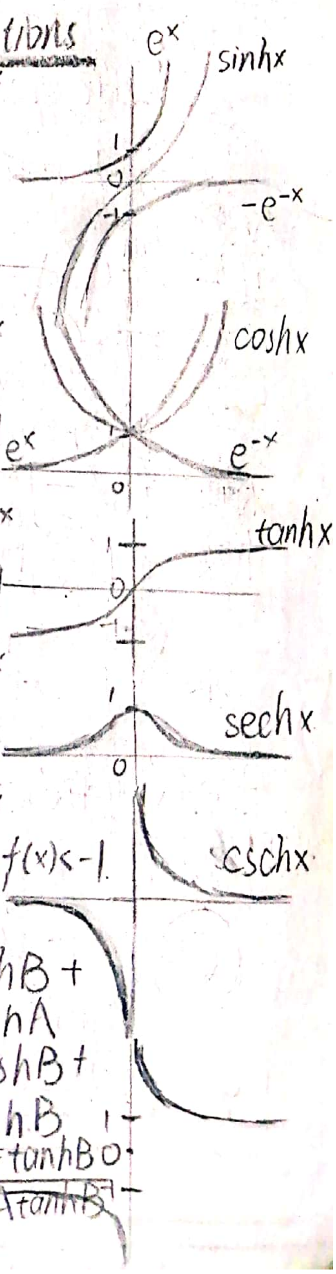
① 基本思想: P_k 代表 $r=k$ 时 statement
$\begin{cases}P_n\text{ (}n\text{ for the minimum requirement.) is true}\\ P_k\Rightarrow P_{k+1}\text{ (assum }P_k\text{ is true ...)}\end{cases}$
details ↓

② summations: 带入 (k+1) 即可
deviation: 带入 k, $\frac{d}{dx}\left(\frac{d}{dx}\left(\cdots\left(\frac{d}{dx}f(k)\right)\cdots\right)\right)$ 直到 k+1 成立.
recurrence relations: if $u_n >/</\geq/\leq m$.
$u_{n+1}-m=f(u_n)-m$
when $u_n >/</\geq/\leq m$
$f(u_n)-m >/</\geq/\leq 0$
so $u_{n+1} >/</\geq/\leq m$
satisfied.
matrix: 带入 (k+1) 即可
divisibility: $f(k+1)-f(k)$ is divisible

③ since P_n (n for the minimum requirement) is true and $P_k\Rightarrow P_{k+1}$ by mathematical induction P_n is true for all $n\geq n$.

入. hyperbolic functions

① • $\sinh x=\frac{e^x-e^{-x}}{2}$
$x\in R,\ f(x)\in R$
• $\cosh x=\frac{e^x+e^{-x}}{2}$
$x\in R,\ f(x)\geq 1$
• $\tanh x=\frac{e^x-e^{-x}}{e^x+e^{x}}$
$x\in R,\ -1<f(x)<1$
• $\operatorname{sech}x=\frac{2}{e^x+e^{-x}}$
$x\in R,\ 0<f(x)\leq 1$
• $\operatorname{csch}x=\frac{2}{e^x-e^{-x}}$
$x\neq 0,\ f(x)\neq 0$
• $\coth x=\frac{e^x+e^{-x}}{e^x-e^{-x}}$
$x=0,\ f(x)>1$ and $f(x)<-1$



② $\cosh^2x-\sinh^2x\equiv 1$
$\sinh(A+B)=\sinh A\cosh B+\sinh B\cosh A$
$\cosh(A+B)=\cosh A\cosh B+\sinh A\sinh B$
$\tanh(A+B)=\frac{\tanh A+\tanh B}{1+\tanh A\tanh B}$

③ $\sinh^{-1}x = \ln(x+\sqrt{x^2+1})$

$\sinh^{-1}x$

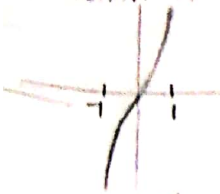
$x \in R, f(x) \in R$

• $\cosh^{-1}x = \ln(x+\sqrt{x^2-1})$

$\cosh^{-1}x$

$x \geq 1, f(x) \geq 0$

• $\tanh^{-1}x = \frac{1}{2}\ln\left(\frac{1+x}{1-x}\right)$



$\tanh^{-1}x$

$-1 < x < 1, f(x) \in R$

• $\coth^{-1}x = \frac{1}{2}\ln\left(\frac{x+1}{x-1}\right)$

• $\operatorname{sech}^{-1}x = \ln\left(\frac{1+\sqrt{1-x^2}}{x}\right)$

• $\operatorname{csch}^{-1}x = \ln\left(\frac{1\pm\sqrt{1+x^2}}{x}\right)$

④ $\sinh(\cosh^{-1}x) = \sqrt{x^2-1}$

$\ln x = \tanh^{-1}\left(\frac{x^2-1}{x^2+1}\right)$

九 advanced matrice

① sometimes we tranform a vector into another, it's direction doesn't change

$M\begin{pmatrix}a_{11}\\a_{12}\end{pmatrix} = \lambda\begin{pmatrix}a_{11}\\a_{12}\end{pmatrix}$ where M is a matrice, λ is a constant

so $(M-\lambda I)\begin{pmatrix}a_{11}\\a_{12}\end{pmatrix} = 0$

$\det(M-\lambda I)=0$

$M-\lambda I = \begin{pmatrix}b_{11}-\lambda & b_{21}\\ b_{12} & b_{22}-\lambda\end{pmatrix}$

so, the λ is called eigenvalues

the vector $\begin{pmatrix}a_{11}\\a_{12}\end{pmatrix}$ we found is known as eigenvectors

* it seems like invariant line,

pos

** given eigenvalue and eigenvector expressed by k, we could find k.

② if A has eigenvalue λ_i and corresponding eigenvector e_i, and B: μ_i and e_i

then $(A+B)e_i = (\lambda_i+\mu_i)e_i$

$\downarrow$

if $B = aA + bI$, given A has eigenvector e_i, then B has eigenvector e_i

(their eigenvalues: $\mu_i = a\lambda_i + b$)

cuz $Be_i = aAe_i + bIe_i$, direction doesn't change

then

$ABe_i = \lambda_i\mu_ie_i$

so, understandably

$A^ne_i = \lambda^ne_i$

③ 因此，如果A有eigenvalue及eigenvector $\begin{pmatrix}a_{11}\\a_{12}\end{pmatrix}$ & c_1 与 $\begin{pmatrix}a_{21}\\a_{22}\end{pmatrix}$ & c_2 (we'll start with 2x2 matrice)

then, 当应用A a transformation时，可将它视为作一个在基为$\begin{pmatrix}a_{11}\\a_{12}\end{pmatrix}$与$\begin{pmatrix}a_{21}\\a_{22}\end{pmatrix}$下的呈方向拉伸

(即在此基底下进行对角矩阵の变换)

$\begin{pmatrix}a_{11} & a_{21}\\a_{12} & a_{22}\end{pmatrix}^{-1}A\begin{pmatrix}a_{11} & a_{21}\\a_{12} & a_{22}\end{pmatrix} = \begin{pmatrix}c_1 & 0\\0 & c_2\end{pmatrix}$

$A = \begin{pmatrix}a_{11} & a_{21}\\a_{12} & a_{22}\end{pmatrix}\begin{pmatrix}c_1 & 0\\0 & c_2\end{pmatrix}\begin{pmatrix}a_{11} & a_{21}\\a_{12} & a_{22}\end{pmatrix}^{-1}$

recall $P_A(\lambda)$ (our characteristic equation)

$\because P_A(\lambda) = |A-\lambda I| = 0$

$\therefore P_A(\lambda) = |PBP^{-1} - P(\lambda I)P^{-1}|$

where $P = \begin{pmatrix}a_{11} & a_{21}\\a_{12} & a_{22}\end{pmatrix}$ $B = \begin{pmatrix}c_1 & 0\\0 & c_2\end{pmatrix}$

$= |P||B-\lambda I||P^{-1}|$

$= |B-\lambda I| = f_B(\lambda)$

$= (c_1-\lambda)(c_2-\lambda) = 0$

$\therefore f_A(A) = (c_1I-A)(c_2I-A)$

$= (P(c_1I)P^{-1} - PBP^{-1})(P(c_2I)P^{-1} - PBP^{-1})$

$= P(c_1I-B)(c_2I-B)P^{-1} = 0$

$= (c_1I-B)(c_2I-B) = 0$

just remember:

$\lambda^n + a_{n-1}\lambda^{n-1} + \cdots + a_1\lambda + a_0 = 0$

can be turned into

$A^n + a_{n-1}A^{n-1} + \cdots + a_1A + a_0I = \vec{0}$

use this to find inverse!

$A^n + a_{n-1}A^{n-1} + \cdots + a_1A = -a_0I$

$\downarrow$

$\frac{A^{n-1} + a_{n-1}A^{n-2} + \cdots + a_1}{-a_0} = A^{-1}$

as $a_0 \neq 0$, $\det(A) \neq 0$.

* the above-mentioned $A = PBP^{-1}$ is called diagonalisable, which is a good way to figure out

$A^n = PB^nP^{-1}$

注意这里的P与B要扩充A的所有dimensions，否则无法diagonalisable.

⑩ use matrice to solve linear equation!

$$\begin{pmatrix} a_{11} & a_{21} & a_{31} \\ a_{12} & a_{22} & a_{32} \\ a_{13} & a_{23} & a_{33} \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} c_1 \\ c_2 \\ c_3 \end{pmatrix}$$

用高斯消元、依次求出

* 当 $a_{13}, a_{23}, a_{33}, c_3$ 均为0，则用

$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} b_1 \\ b_2 \\ b_3 \end{pmatrix} + t\begin{pmatrix} b_4 \\ b_5 \\ b_6 \end{pmatrix}$ 表示答案

当 $a_{13}, a_{23}, a_{33} = 0$ 而 $c_3 \neq 0$，无解

十. advanced differentiation

① $\frac{d}{dx}\left(\frac{dy}{dx}\right) = \frac{d^2y}{dx^2}$
$\frac{dy}{dx}\cdot\frac{dy}{dx} = \left(\frac{dy}{dx}\right)^2$

it's better to start with $\frac{d}{dx}(\cdots) = \frac{d}{dx}(\cdots)$ then differentiate them

② $(uxw)' = u'xw + ux'w + uxw'$

③ $\frac{dy}{dx} = \frac{d}{dt}(y)\cdot\frac{dt}{dx}$

$\frac{d^2y}{dx^2} = \frac{d}{dt}\left(\frac{dy}{dx}\right)\cdot\frac{dt}{dx}$

$\frac{d}{dt}\left(\frac{dy}{dx}\right) = \frac{d}{dt}\left(\frac{dy}{dt}\cdot\frac{dt}{dx}\right)$

$\frac{d}{dx}\left(\frac{dy}{dx}\right)\cdot\frac{dx}{dt} = \frac{d}{dt}\left(\frac{dy}{dx}\right)$

④ $\frac{d}{dx}(\cosh x) = \sinh x$

$\frac{d}{dx}(\sinh x) = \cosh x$

$\frac{d}{dx}(\tanh x) = \operatorname{sech}^2 x$

$\frac{d}{dx}(\operatorname{sech} x) = -\operatorname{sech} x \tanh x$

$\frac{d}{dx}(\operatorname{csch} x) = -\operatorname{csch} x \coth x$

$\frac{d}{dx}(\coth x) = -\operatorname{csch}^2 x$

recall
$\cos x = \frac{1}{\sec x}$
$\sin x = \frac{1}{\csc x}$

⑤ $\frac{d}{dx}(\sin^{-1}x) = \frac{1}{\sqrt{1-x^2}}$

$\frac{d}{dx}(\cos^{-1}x) = -\frac{1}{\sqrt{1-x^2}}$

$\frac{d}{dx}(\tan^{-1}x) = \frac{1}{1+x^2}$

⑥ $\frac{d}{dx}(\sinh^{-1}x) = \frac{1}{\sqrt{1+x^2}}$

$\frac{d}{dx}(\cosh^{-1}x) = \frac{1}{\sqrt{x^2-1}}$

$\frac{d}{dx}(\tanh^{-1}x) = \frac{1}{1-x^2}$

⑦ $f(x) = \frac{f(0)}{0!} + \frac{xf'(0)}{1!} + \frac{x^2f''(x)}{2!} + \cdots\cdots$

$= \sum_{n=1}^{\infty}\frac{x^n f^{(n)}(0)}{n!}$ (证明见附)

⑧ shorthand

$y = f(x)$，构建 $g(y) = f'(x)$

$\therefore y' = g(y)$，$y'' = g'(y)\cdot y' = g'(y)\cdot g(y)$

十一. advanced integration

① $\int\frac{1}{\sqrt{a^2-x^2}}dx = \sin^{-1}\left(\frac{x}{a}\right)$ by $x = a\sin\theta$

$\int\frac{1}{a^2+x^2}dx = \frac{1}{a}\tan^{-1}\left(\frac{x}{a}\right)$ by $x = a\tan\theta$

$\int\frac{1}{\sqrt{a^2+x^2}}dx = \sinh^{-1}\left(\frac{x}{a}\right)$ by $x = a\sinh\theta$

$\int\frac{1}{\sqrt{x^2-a^2}}dx = \cosh^{-1}\left(\frac{x}{a}\right)$ by $x = a\cosh\theta$

$\int\frac{1}{a^2-x^2}dx = \frac{1}{a}\tanh^{-1}\left(\frac{x}{a}\right)$ by $x = a\tanh\theta$

* $\int\sec x\,dx = \int\frac{\sec^2x + \sec x\tan x}{\sec x + \tan x}dx$

$= \ln|\sec x + \tan x| = \ln\left|\tan\left(\frac{1}{2}x + \frac{1}{4}\pi\right)\right|$

$\int\csc x\,dx = \int\frac{\csc^2x + \csc x\cot x}{\csc x + \cot x}dx$

$= -\ln|\csc x + \cot x| = \ln\left|\tan\left(\frac{1}{2}x\right)\right|$

** sometimes，当分母为二次多项式，化为完全平方加减一常数

② $\int\sinh x\,dx = \cosh x$

$\int\cosh x\,dx = \sinh x$

$\int\tanh x\,dx = \ln|\cosh x|$

$\int\coth x\,dx = \ln|\sinh x|$

* 这些东西解不开可化成用e表示的形式，然后换元 e^x

或者尝试用u、v求解（v'代入1，则 $v = x$）

③ reduction rules:

- if $I_n = \int\cos^n x\,dx$ for $x \geq 2$.

$u = \cos^{n-1}x$, $\frac{du}{dx} = (n-1)\cos^{n-2}x\sin x$

$\frac{dv}{dx} = \cos x$, $v = \sin x$

$\therefore I_n = \sin x\cos^{n-1}x - (n-1)\int\cos^{n-2}x\sin^2x\,dx$

$= \sin x\cos^{n-1}x - (n-1)\int\cos^{n-2}x(1-\cos^2x)dx$

$= \sin x\cos^{n-1}x - (n-1)I_{n-2} + (n-1)I_n$

$\therefore I_n = \frac{1}{n}\left(\sin x\cos^{n-1}x - (n-1)I_{n-2}\right)$

- similarly:

if $I_n = \int\sin^n x\,dx$

$I_n = \frac{1}{n}\left(-\cos x\sin^{n-1}x + (n-1)I_{n-2}\right)$

- if $I_n = \int\cosh^n 2x\,dx$

$u = \cosh^{n-1}2x$, $\frac{du}{dx} = 2(n-1)\cosh^{n-2}2x\cdot\sinh 2x$

$\frac{dv}{dx} = \cosh 2x$ $v = \frac{1}{2}\sinh 2x$

$I_n = \frac{1}{2}\cosh^{n-1}2x\sinh 2x - (n-1)\int\cosh^{n-2}2x\sinh^2 2x\,dx$

$= \frac{1}{2}\cosh^{n-1}2x\sin 2x - (n-1)\int \cosh^{n-2}2x(\cosh^2 2x - 1)dx$

$= \frac{1}{2}\cosh^{n-1}2x\sin 2x - (n-1)I_n + (n-1)I_{n-2}$

$\therefore I_n = \frac{1}{2n}(\sinh 2x\cosh^{n-1}2x + (n-1)I_{n-2})$

④ for arc in Cartesian form

if $\Delta x \to 0$

$(\Delta x)^2 + (\Delta y)^2 = (\Delta s)^2$

$\frac{(\Delta x)^2}{(\Delta x)^2} + \frac{(\Delta y)^2}{(\Delta x)^2} = \frac{(\Delta s)^2}{(\Delta x)^2}$ $\Big\}\ (\frac{dy}{dx})^2 \neq \frac{d^2y}{dx^2}$!

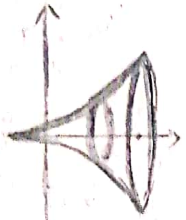
$1 + (\frac{dy}{dx})^2 = (\frac{ds}{dx})^2$

$S = \int_{x_1}^{x_2}\sqrt{1 + (\frac{dy}{dx})^2}\,dx$ (加上下限即可)

for arc in parametric form

$S = \int_{t_1}^{t_2}\sqrt{(\frac{dx}{dt})^2 + (\frac{dy}{dt})^2}\,dt$

for arc in polar form



$(\Delta r)^2 + (r\Delta\theta)^2 = (\Delta s)^2$

$\frac{(\Delta r)^2}{(\Delta\theta)^2} + \frac{(r\Delta\theta)^2}{(\Delta\theta)^2} = \frac{(\Delta s)^2}{(\Delta\theta)^2}$

$(\frac{dr}{d\theta})^2 + r^2 = (\frac{\Delta s}{\Delta\theta})^2$

$\Delta S = \int_{\theta_1}^{\theta_2}\sqrt{r^2 + (\frac{dr}{d\theta})^2}\,d\theta$

for surface generated when area of a curve is rotated about x-axis

$\Delta S = 2\pi y \Delta s$

$S = \int_{x_1}^{x_2} 2\pi y\sqrt{1 + (\frac{dy}{dx})^2}\,dx$

for Cartesian curves

$S = \int_{t_1}^{t_2} 2\pi y\sqrt{(\frac{dx}{dt})^2 + (\frac{dy}{dt})^2}\,dt$

for parametric curves.

$\ast\cdot \int \sqrt{a^2 + b^2x^2}\,dx = \int \frac{a^2}{b}\cosh^2\theta\, d\theta$

when $bx = a\sinh\theta$.

or $LHS = \int \frac{a^2}{b}\sec^3\theta\, d\theta$

when $bx = a\tan\theta$

about y-axis:

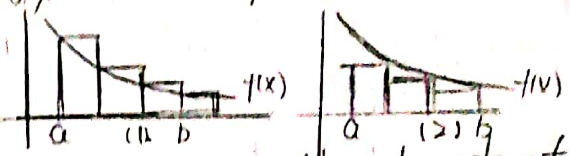
$S = \int_{x_1}^{x_2} 2\pi x\sqrt{1 + (\frac{dy}{dx})^2}\,dx$

for Cartesian curves

$S = \int_{t_1}^{t_2} 2\pi x\sqrt{(\frac{dx}{dt})^2 + (\frac{dy}{dt})^2}\,dt$

for parametric curves

⑤ limits of area



find the upper and lower bounds of $\sum_{i=a}^{b} f(i)$

for (1): $\int_a^{b+1} f(x)\,dx < \sum_{i=a}^{b} f(i)$

for (2): $\int_a^b f(x)\,dx > \sum_{i=a+1}^{b} f(i)$

$\int_a^b f(x)\,dx + f(a) > \sum_{i=a}^{b} f(i)$

判断converge

- upper bound converge
- 一个比它大的值converge

判断diverge

- lower bound diverge
- 一个比它小的值diverge

十一 advanced complex number

① $z = r(\cos\theta + i\sin\theta)$

$z^n = r^n(\cos n\theta + i\sin n\theta)$

$\ast$ 将polynomial换成" a 分母 = b 分子"形式

则 $\frac{a}{b} = \frac{分子}{分母}$(换成三角函数)求解

$z^n + \frac{1}{z^n} = 2\cos n\theta$ $\quad e^{i\theta} + e^{-i\theta} = 2\cos\theta$

$z^n - \frac{1}{z^n} = 2i\sin n\theta$ $\quad e^{i\theta} - e^{-i\theta} = 2i\sin\theta$.

$\ast\ \because z + \frac{1}{z} = 2\cos\theta$, 则 $(z + \frac{1}{z})^n = 2^n\cos^n\theta$

将 $(z + \frac{1}{z})^n$ 展开后用 $z^n + \frac{1}{z^n} = 2\cos n\theta$

换成一次三角函数

② $z^n = 1$, the n^{th} roots of unity are

$1, e^{\frac{2\pi i}{n}}, e^{\frac{4\pi i}{n}}, \ldots, e^{\frac{2(n-1)\pi i}{n}}$

$(\omega_1, \omega_2, \ldots, \omega_n)$

if you have already solved one root a for polynomial $\alpha^n = b$, all the roots are:

$a\omega_1, a\omega_2, \ldots, a\omega_n$

③ if we want to find out $\sum_{n=0}^{N-1}\cos n\theta$

turn it into $\sum_{n=0}^{N-1} z^n$, which is a geometric sum

$\sum_{n=0}^{N-1} z^n = \frac{z^N - 1}{z - 1} = \frac{e^{Ni\theta} - 1}{e^{i\theta} - 1}$

$= \frac{e^{(N-\frac{1}{2})i\theta} - e^{-\frac{1}{2}i\theta}}{e^{\frac{1}{2}i\theta} - e^{-\frac{1}{2}i\theta}}$ 上下同乘 $e^{-\frac{1}{2}i\theta}$

$= \frac{\cos(N-\frac{1}{2})\theta + i\sin(N-\frac{1}{2})\theta - \cos\frac{1}{2}\theta + i\sin\frac{1}{2}\theta}{2i\sin\theta}$

we just need real part for $\sum_{n=0}^{N-1}\cos n\theta$

※ • 有时上下同乘$(az\pm b)$，反正分母不带e
• 有时让求值，找清楚θ用什么替换

$$\sum_{n=0}^{\infty} a^{-n}\sin\left(\frac{n\pi}{5}\right) \Rightarrow \sum_{n}^{\infty} a^{-n}\sin n\theta \Rightarrow \sum_{n}^{\infty}\left(\frac{z}{a}\right)^n$$

十二. superb-advanced differential

① $\frac{dy}{dx} + Fy = G$ where F, G are functions about x

let $v = y$, $v' = \frac{dy}{dx}$, $u = I$, $u' = IF$, so:

$$I\frac{dy}{dx} + IFy = IG$$

$$\frac{du}{dx} = \frac{d}{dx}I = IF \Rightarrow \int \frac{1}{I}\,dI = \int F\,dx$$

$$\ln I = \int F\,dx$$

$$I = e^{\int F dx}$$

after find I, $IG = \frac{d}{dx}(Iy)$

② • if $a\frac{dy}{dx} + by = 0$

try $y = Ae^{\lambda x}$ where $A \neq 0$

$\therefore a\lambda(Ae^{\lambda x}) + b(Ae^{\lambda x}) = 0$

this is called homogeneous differential equation

$a\lambda + b = 0$

this is called auxiliary equation

• if $a\frac{d^2y}{dx^2} + b\frac{dy}{dx} + cy = 0$

try $y = Ae^{\lambda x}$

$a\lambda^2(Ae^{\lambda x}) + b\lambda(Ae^{\lambda x}) + c(Ae^{\lambda x}) = 0$

$a\lambda^2 + b\lambda + c = 0$

if $\lambda_1 \neq \lambda_2$

complementary function $y = Ae^{\lambda_1 x} + Be^{\lambda_2 x}$

if $\lambda_1 = \lambda_2$

complementary function $y = (Ax+B)e^{\lambda x}$

if λ_1, λ_2 are complex number $m \pm ni$

complementary function: $y = e^{mx}(A\cos nx + B\sin nx)$

③ • if $a\frac{d^2y}{dx^2} + b\frac{dy}{dx} + cy = f(x)$ where $f(x)$ is polynomial function of degree n.

use $a\frac{d^2y}{dx^2} + b\frac{dy}{dx} + cy = 0$ to find CF

apply $y = a_n x^n + a_{n-1}x^{n-1} + \cdots$ into equation to find PI (particular integral)

then GS = CF + PI

• if $f(x)$ is an exponential function $ke^{\lambda x}$

then try $y = ae^{\lambda x}$.

• 如果两者compound那就integrate就成

• if $f(x) = k_1\cos\lambda x + k_2\sin\lambda x$

then try $y = a\cos\lambda x + b\sin\lambda x$

• 如果你求PI的形式与CF重复了，用$y = axf(x)$找PI

如果$y = axf(x)$也重复，用$y = ax^2f(x)$

• 有些时候要换元以满足格式

$a\frac{d^2y}{dx^2} + b\frac{dy}{dx} + cy = f(x)$

主要是照着cy变

④ • if $\frac{dy}{dx} = f(x,y)$, turn $F(x,y) = u$

then $\frac{du}{dx}$ 可以将 $\frac{dy}{dx}$ 替换

so $A\frac{du}{dx} = g(u) + B$

求积分即可，尤其$u = \frac{y}{x}$ / $u = ax + by$

• if $f(x)\frac{d^2y}{dx^2} + g(x)\frac{dy}{dx} + h(x)k(y) = F(x)$

一般是让$u = k(y)$

则可求出 $\frac{du}{dy}$, $\frac{du}{dx}$ ($= k'(y)\cdot\frac{dy}{dx}$，保留$\frac{dy}{dx}$即可)，$\frac{d^2u}{dx^2}$

则可以将式子化为

$a\frac{d^2u}{dx^2} + b\frac{du}{dx} + cu = G(x)$

FM-M

一. projectiles

① $y = x\tan\theta \pm \frac{gx^2}{2u^2}\sec^2\theta$（起始同一高度）

其中 x/y 代表 x/y 轴上距离（矢量）

u 代表初速率. θ 代表一开始手抛与水平夹角

当往上抛, 用 +, 往下抛用 −

用 $y = ax + bx^2$ 代换求 θ/u

① 是否存在一瞬间使抛体与地面θ

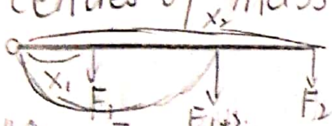
通过 θ 列出 $V_x = kV_y$

$u\cos\theta = k(u\sin\theta + \frac{1}{2}gt^2)$

求 t, 然后验证 t 是否可能

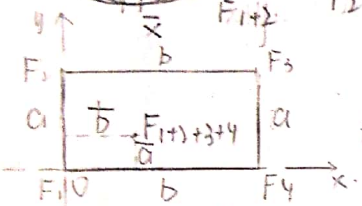
二. equilibrium of a rigid body

① centres of mass rod.



用 F_{1+2} 等效代换 $F_1 + F_2$

那 $F_1x_1 + F_2x_2 = F_{1+2}\bar{x}$

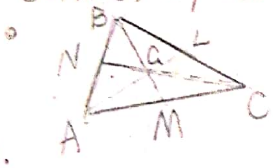


用 $F_{1+2+3+4}$ 代换 $F_1+F_2+F_3+F_4$

那 $F_2a + F_3a = F_{1+2+3+4}\bar{a}$

$F_3b + F_4b = F_{1+2+3+4}\bar{b}$

② centres of lamina



$\vec{OG} = \vec{OA} + \alpha\vec{AL}$

$= \vec{OC} + \beta\vec{CN}$

$= \vec{OB} + \gamma\vec{BM}$

where $\vec{AL} = (\vec{OC} - \vec{OA}) + \frac{1}{2}(\vec{OB} - \vec{OC})$

$\vec{CN} = (\vec{OB} - \vec{OC}) + \frac{1}{2}(\vec{OB} - \vec{OA})$

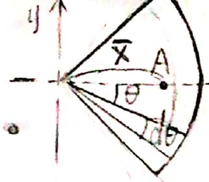
$\vec{BM} = (\vec{OA} - \vec{OB}) + \frac{1}{2}(\vec{OC} - \vec{OA})$

已知以上三式系数一致

则 $\frac{GL}{AL} = \frac{GM}{BM} = \frac{GN}{CN} = \frac{1}{3}$

if three vertices of triangle $(x_1, y_1)(x_2, y_2)(x_3, y_3)$

the centre $\left(\frac{x_1+x_2+x_3}{3}, \frac{y_1+y_2+y_3}{3}\right)$



r 与 2α

• 对于一个小的 dθ, 扇形看作三角形

则 centre 的 x 为 $\frac{2}{3}r\cos\theta$

设 mass per unit area ρ

则 total mass $\frac{1}{2}r^2 \cdot 2\alpha \cdot \rho = r^2\alpha\rho$

total moment:

$\int_{-\alpha}^{\alpha} \frac{1}{2}r^2 \cdot \rho \cdot \frac{2}{3}r\cos\theta\, d\theta$

$\therefore r^2\alpha\rho\bar{x} = \int_{-\alpha}^{\alpha} \frac{1}{3}r^3\rho\cos\theta\, d\theta$

$\bar{x} = \frac{2r\sin\alpha}{3\alpha}$

• 对于 conposite shape, 用 ρ 与每个图形的中心画点, 像 mass rods 一样找

用 area × ρ = mass 代表 force

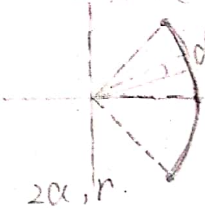
用 force × 重心对应边长代表 moment

注: 1. 用总 mass 代表总 force

2. x/y 轴分居两侧的 moment 符号要不同

3. 当大图中去除小图时, 相对应的去除小图的 moment 是负的!

③ centres of wire



2α, r

for dθ, wire has mass rρdθ

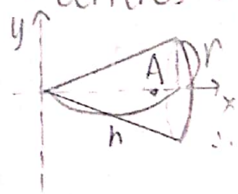
and corresponding x: $r\cos\theta$

total mass $2r\alpha\rho$

$2r\alpha\rho\bar{x} = \int_{-\alpha}^{\alpha} r^2\rho\cos\theta\, d\theta$

$\bar{x} = \frac{r\sin\alpha}{\alpha}$

④ centres of solid

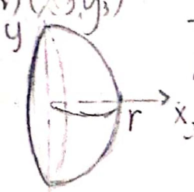


total mass $\frac{1}{3}\pi r^2 h \cdot \rho$

mass for dx $\rho\pi y^2 dx$

$\therefore \frac{1}{3}\rho\pi r^2 h\bar{x} = \int_0^h \rho\pi(\frac{r}{h}x)dx$

$\bar{x} = \frac{3}{4}h$



total mass $\frac{2}{3}\pi r^3 \cdot \rho$

mass for dx $\rho\pi y^2 dx$

$\therefore \frac{2}{3}\pi r^3\rho\bar{x} = \int_0^r \rho\pi(r^2 - x^2)dx$

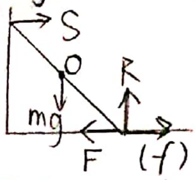
$\bar{x} = \frac{3}{8}r$

• 左右两边 balance

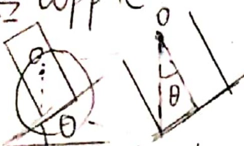
$F_1\bar{x}_1 = F_2\bar{x}_2$

$r = kh$

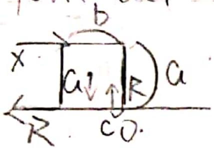
① object in equilibrium

- 

$S=F$, $mg=R$
stable → $f \geq F$
$S=kR$ → moment about O

- 是否 topple


当重心在最低点正上方
就是topple的 limit θ

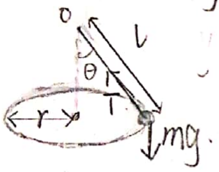
- suspend wire与gravitational force 共线
- force act to slide/topple


slide 当 $X>R$
topple 当R的作用点为O或O的右方时
$aX+cR=\frac{1}{2}ba$

三 circular motion

① 

圆盘上滚动 f 指里

② 

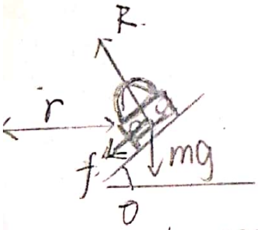
$$\begin{cases} x: T\sin\theta = mr\omega^2 \\ y: T\cos\theta = mg \end{cases}$$

$\because \frac{r}{l}=\sin\theta$

$T\sin\theta = ml\sin\theta\omega^2$

$T=ml\omega^2=\frac{mg}{\cos\theta}$

$\omega=\sqrt{\frac{g}{l\cos\theta}}$



$$\begin{cases} x: f\cos\theta + R\sin\theta = m\frac{v^2}{r} \\ y: R\cos\theta = mg + f\sin\theta \end{cases}$$

$\because f=\mu R$

$R\cos\theta-\mu R\sin\theta=mg$

$R=\frac{mg}{\cos\theta-\mu\sin\theta}$

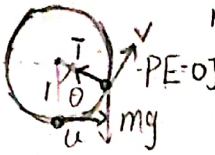
$R(\mu\cos\theta+\sin\theta)=m\frac{v^2}{r}$

$\frac{\mu\cos\theta+\sin\theta}{\cos\theta-\mu\sin\theta}mg=m\frac{v^2}{r}$

$\frac{v^2}{r}=g\frac{\mu+\tan\theta}{1-\mu\tan\theta}$

f 向下因为 exceed velocity 时 f 向上

三 l 住の平面方定转考虑 a

③ 

$[PE=0]$

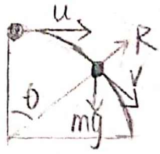
initially
$PE=-mgr$ $KE=\frac{1}{2}mu^2$
generally
$PE=-mgr\cos\theta$ $KE=\frac{1}{2}mv^2$

in order to complete full circles.
at top. $[KE=0]$
assume no f

$$\begin{cases} \frac{1}{2}mu^2-mgr=\frac{1}{2}mv^2-mgr\cos\theta \\ T-mg\cos\theta=\frac{mv^2}{r} \end{cases}$$

$T-mg\cos\theta=\frac{mu^2}{r}-2mg+2mg\cos\theta$

注意 1. T为向心力，要让小球complete full circle/不离开轨道 $T\geq 0$.
$u\geq\sqrt{5gr}$
2. 如果要求max/min tension，再考虑 a (cuz T 仅为向心力).
3. 一般用能量守恒与向心力公式列
4. 找运动最高点 → 脱轨后仍会上升



initially
PE mgr KE $\frac{1}{2}mu^2$
generally
PE $mgr\cos\theta$ KE $\frac{1}{2}mv^2$

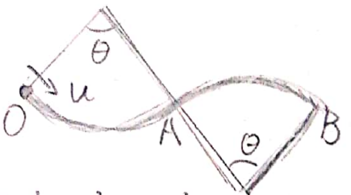
$\therefore \frac{1}{2}mu^2+mgr=\frac{1}{2}mv^2+mgr\cos\theta$

$\frac{mv^2}{r}=\frac{mu^2}{r}+2mg-2mg\cos\theta$

$\because mg\cos\theta-R=\frac{mv^2}{r}$

$\therefore -R=\frac{mu^2}{r}+2mg-3mg\cos\theta$

注意：如果小球飞出，$R=0$
飞出所需要的u的range $0<u<\sqrt{gr}$
θ的range $0<\theta<\cos^{-1}\frac{2}{3}$



O、A、B点 v相同
要想满足运动：
1. 到达top
2. 不飞出去.

四 hooke's law

① $T=\frac{\lambda x}{l}$ where l is original length
λ is the modulus of elasticity

一定要考虑 a！

② $EPE=\int_0^x\frac{\lambda x}{l}dx=\frac{\lambda x^2}{2l}$

考虑弹力本 EPE

五. linear motion under a variable force

① $F = ma$，a用$\frac{dv}{dt}$或$v\frac{dv}{dx}$表示

② $a = f(x)$.

$\frac{dv}{dx} \cdot \frac{dx}{dt} = f(x)$

$v\frac{dv}{dx} = f(x)$

$\int v\,dv = \int f(x)\,dx$

③ 求min/max用二阶导

六 momentum

① newton's experimental law

$e = \frac{v_2 - v_1}{u_1 - u_2}$

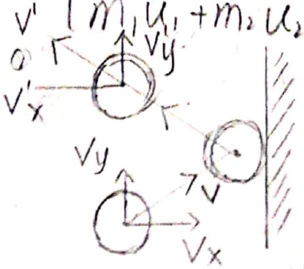
where e is coefficient of restitution. $0 \leq e \leq 1$

完全elastic. $e = 1$

完全inelastic (粘一块) $e = 0$

注: 求v_1 v_2时,用

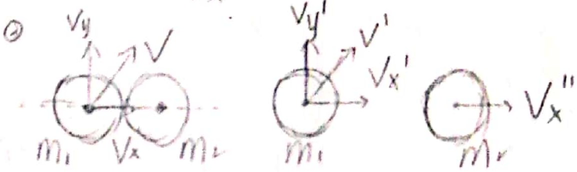
$$\begin{cases} e = \frac{v_2 - v_1}{u_1 - u_2} \\ m_1 u_1 + m_2 u_2 = m_1 v_1 + m_2 v_2 \end{cases}$$



$V_y = V_y'$

$eV_x = V_x'$

②



$V_y = V_y'$.

$m_1 V_x = m_1 V_x' + m_2 V_x''$

注意 求angle还是angle deflection

FM-S

一 continuous random variables

① $f(x)$ 是PDF(函数本身)

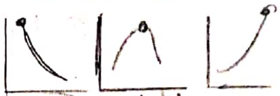
$F(x)$ 是CDF(probability解析式)

$F(x)=\int f(x)dx \quad f(x)=\frac{d}{dx}F(x)$

注意：是题目中给/问的是PDF还是CDF？求mean/variance用PDF.

② 求mode:

PDF求导找stationary，然后起、始、stationary point 比大小



就三种情况

③ $E(g(X))=\int g(x)f(x)dx$

④ 已知$F(X)$, $Y=h(x)$, 求$G(y)$

$$G(y)=P(Y\le y)=\begin{cases} G(X\le h^{-1}(y))=F(h^{-1}(y)) & ① \\ G(X>h^{-1}(y))=1-F(h^{-1}(y)) & ② \end{cases}$$

- 当$f(x)$的critical value由小至大排序与$g(x)$的相同时，用①；反之则②
- $f(x) \to F(x) \to G(Y) \to g(y)$

上述计算仅满足$F(x)$ CDF → $G(Y)$ CDF!

看准是PDF or CDF!

二 inferential statistics

① t-distribution

- 当 assume underlying normal distribution
  variance is unknown
  sample size is small ($n<30$)

用t-distribution去做hypothesis test

- $s^2=\frac{1}{n-1}\sum(x-\bar{x})^2=\frac{1}{n-1}(\sum x^2-n\bar{x}^2)$
- test statistic $=\frac{\bar{x}-\mu}{\frac{s}{\sqrt{n}}}$
- t的查表

$t_{a,b}$, 其中a是one-trial的significant value (如果two-trial除以2)

b是自由度$(n-1)$

根据上述查出一个value c, 已算出一个test statistic d

if d negative, $-c>d$, 才能reject H_0

if d positive, $c<d$, 才能reject H_0

② 现在我们想比较两个distribution, n is large

为了方便比较，我们控制变量

则假设variance相同，比较mean

$\because X\sim N(\mu_x,\sigma_x^2)$

$Y\sim N(\mu_y,\sigma_y^2)$

$\therefore \bar{X}-\bar{Y}\sim(\mu_x-\mu_y, \frac{\sigma_x^2}{n_x}+\frac{\sigma_y^2}{n_y})$

$$\therefore Z=\frac{(\bar{x}-\bar{y})-(\mu_x-\mu_y)}{\sqrt{\frac{\sigma_x^2}{n_x}+\frac{\sigma_y^2}{n_y}}}\sim N(0,1)$$

我们用这个来hypothesis test

③ 但当$n<15$, 太小了, 不能反映population, 反而更像samples

$$\therefore S_p^2=\frac{\sum(x-\bar{x})^2+\sum(y-\bar{y})^2}{n_x+n_y-2}$$

$$也即=\frac{\sum x^2-\frac{(\sum x)^2}{n_x}+\sum y^2-\frac{(\sum y)^2}{n_y}}{n_x+n_y-2}$$

如果unbiased, $\sigma_x^2=\sigma_y^2=S_p^2$

$$\therefore T=\frac{(\bar{x}-\bar{y})-(\mu_x-\mu_y)}{\sqrt{S_p^2(\frac{1}{n_x}+\frac{1}{n_y})}}\sim t_{n_x+n_y-2}$$

注意这个查t的表

④ 如果我们要有两个small sample之间的mean difference:

test statistic $=\frac{\bar{d}-k}{\frac{S_d}{\sqrt{n}}}\sim t_{n-1}$

where $d_i=x_i-y_i$

$S_d^2=\frac{\sum d^2-\frac{(\sum d)^2}{n}}{n-1}$

因为① 差值也是个underlying normal distribution

② 数据太少

⑤ a $100(a-1)\%$ confidence interval for μ:

$$\bar{x}\pm t_{\frac{\alpha}{2},n-1}\frac{s}{\sqrt{n}}$$

注意 $S^2=\frac{\sum x^2-\frac{(\sum x)^2}{n}}{n-1}$

⑥ a $100(a-1)\%$ confidence interval for difference in mean

large samples:

$$\bar{x}-\bar{y}\pm Z_{\frac{\alpha}{2}}\sqrt{\frac{S_x^2}{n_x}+\frac{S_y^2}{n_y}}$$

small samples

$$\bar{x}-\bar{y}\pm t_{\frac{\alpha}{2},n_1+n_2-2}\cdot S_p\sqrt{\frac{1}{n_x}+\frac{1}{n_y}}$$

mean difference

$\bar{d} \pm t(\frac{\alpha}{2}, n-1) \cdot \frac{s_d}{\sqrt{n}}$

where $s^2 = \frac{\sum x^2 - \frac{(\sum x)^2}{n}}{n-1}$

三 chi-squared test

① 自由度

ν = number of expected value − 1 − number of parameters estimated

② given the condition that all $E_i \geq 5$.

$\chi^2 = \sum\left(\frac{(O_i - E_i)^2}{E_i}\right) \sim \chi^2_\nu(1-\alpha\%)$ (α% significant)

where O_i: observed frequency

E_i: expected frequency

for binomial distribution:

$\hat{p} = \frac{\bar{x}}{n}$

$\bar{x} = \frac{\sum(r_i \cdot O_i)}{N}$

where n: the number of trials in binomial distribution

N: the number of times the experiment is repeated

$r_i = 0, 1, 2, \dots, n$

for poisson distribution

$\hat{\lambda} = \frac{\sum(r_i \cdot O_i)}{N}$

注意: 估计 $\hat{p}$, $\hat{\lambda}$ 自由度减1

③ $X \sim U[a,b]$: rectangular distribution

then $f(x) = \begin{cases} \frac{1}{b-a} & a \leq x \leq b \\ 0 & \text{otherwise} \end{cases}$

④ test association through contingency table

observed	A	B	
A	AA	AB	R_1
B	BA	BB	R_2
	C_1	C_2	T

expected	A	B	
A	$\frac{R_1 C_1}{T}$	$\frac{R_1 C_2}{T}$	R_1
B	$\frac{R_2 C_1}{T}$	$\frac{R_2 C_2}{T}$	R_2
	C_1	C_2	T

此时, $\nu = (r-1)(c-1)$, $\chi^2 = \sum\left(\frac{(O_{ij} - E_{ij})^2}{E_{ij}}\right)$

H_0 independent (no association)

H_1 dependent (association)

注意 $E_{ij} = \frac{R_i \times C_j}{T} \geq 5$

四 non-parametric test

① single-sample sign test

- when underlying data are continuous
  data are independent
- n个数据, 验证 population的median 是不是 α

做法: 大于α的标+, 小于的标−

$X \sim B(n, 0.5)$

$P(X > k$ 标+的个数$) \sim Z$.

如果 $n > 10$, 可估计成 $T \sim N(\frac{n}{2}, \frac{n}{4})$

注意修正!

要全部区间都在reject域中才能reject!

② single-sample Wilcoxon signed-rank test

- when underlying data are symmetric
  underlying data are continuous
  data are independent

做法: 将每一个数据与给定的median作差后取绝对值, 按绝对值大小赋值, 从1, 2, 3到n

其中, 数据大于median的差的绝对值的赋值为P, 反之为N

P, N 各自求和, $T = \min(P, N)$

根据n与α查表

[注意: ● test statistic < critical value to reject H_0 here!

● a, b绝对值一样, 赋 $\frac{a+b}{2}$]

- for large n

$T \sim N\left(\frac{n(n+1)}{4}, \frac{n(n+1)(2n+1)}{24}\right) \sim Z$

注意修正!

以上两个是给定median与一组数据求正误. 下面三个是两组数据判断是否相同

③ paired-sample sign test

做法：$L_i - R_i$ 求差，正的标+ 负的标−

· 取数量小的一组，$X \sim Bin(n, 0.5) \sim Z$

④ wilcoxon matched-pairs signed-rank test

做法：求差后，1包对值排序，分组算求小

· 查表［注意 ·α

· test statistic < critical value to reject H_0 here］

⑤ Wilcoxon rank-sum test (not match)

当两组 size 不一样但要求有无 difference 时

做法从小到大（或从大到小）将两组数据混合后赋值

C_1 与 C_2 然后分别求和，如果 C_1 有 m 个数据，C_2 有 n 个且 $m \le n$，C_1 和为 R_m，C_2 和为 R_n

$W = \min(R_m, m(n+m+1) - R_m)$

随后查表［注意 ·α

· test statistic < critical value to reject H_0 here］

· for large n

$W \sim N\left(\frac{m(n+m+1)}{2}, \frac{mn(n+m+1)}{12}\right) \sim Z$

注意修正！

［所有的检验用 α 还是要看是 one-trial 还是 two-trials］

五 probability generating function

① $G_X(t) = \sum_x t^{x_i} P(X = x_i) = E(t^X)$

（目前把它当作 table 的一种表达方式就行）

for a uniform distribution: $P(X = x_i) = \begin{cases} \frac{1}{n} & i = 1, 2, \cdots, n \\ 0 & \text{otherwise} \end{cases}$

$G_X(t) = \frac{t(1-t^n)}{n(1-t)}$

for $X \sim B(n, p)$: $G_X(t) = (q + pt)^n$

for $X \sim G(p)$: $G_X(t) = \frac{pt}{1-qt}$

for $X \sim Po(\lambda)$: $G_X(t) = e^{\lambda(t-1)}$

② $E(X) = G_X'(1)$

$Var(X) = G_X''(1) + G_X'(1) - [G_X'(1)]^2$

③ 用泰勒展开 PGF 看各项概率

$P(X = r) = \frac{G_X^{(r)}(0)}{r!}$

④ for independent X and Y

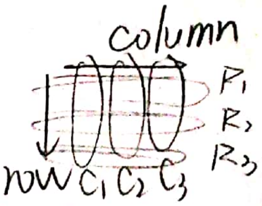
$G_{X+Y}(t) = G_X(t) \times G_Y(t)$

$G_{aX+b}(t) = t^b G_X(t^a)$

*此处的 t^a 用来代替式中所有 t

$G_{\underbrace{X+X+X+\cdots+X}_{n个}}(t) = [G(t)]^n$

读题!



P1.

① $\Sigma\alpha$、$\Sigma\alpha\beta\gamma$ 别忘了符号，考试时点出 $\Sigma\alpha = -\frac{b}{a}$ etc

② symptotes 包括 x 与 y，别忘了 x 的 symptotes，画图要记得交点！

③ matrices 从右往左，横column竖row，$\begin{pmatrix}0&1&0\\-&-&-\end{pmatrix}\begin{pmatrix}-&-&-\\-&-&-\end{pmatrix}$ 取第2row，$\begin{pmatrix}-&-&-\end{pmatrix}\begin{pmatrix}0&-&-\\1&-&-\\0&-&-\end{pmatrix}$ 第2 column etc

④ for $A = \begin{pmatrix}a&b\\c&d\end{pmatrix}$, $A^{-1} = \frac{1}{ad-bc}\begin{pmatrix}d&-b\\-c&a\end{pmatrix}$

for $B = \begin{pmatrix}a_{11}&a_{12}&a_{13}\\a_{21}&a_{22}&a_{23}\\a_{31}&a_{32}&a_{33}\end{pmatrix}$ to find B^{-1}, $a_{11} \to \frac{(-1)^{1+1}}{\det B}\begin{vmatrix}a_{22}&a_{23}\\a_{32}&a_{33}\end{vmatrix}$, $a_{32} \to \frac{(-1)^{3+2}}{\det B}\begin{vmatrix}a_{11}&a_{13}\\a_{21}&a_{23}\end{vmatrix}$ etc

$(AB)^{-1} = B^{-1}A^{-1}$ $\quad \det A^{-1} = \frac{1}{\det A}$

只有方阵有逆，当找invariant line所可行性范围时注意 $\det \neq 0$

⑤ matrices types: reflect, enlarge, rotate, shear, stretch $\quad \begin{pmatrix}\cos\theta&-\sin\theta\\\sin\theta&\cos\theta\end{pmatrix}$

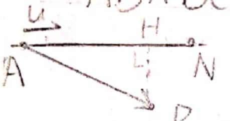
⑥ polar coordinate (r, θ) $\quad$ B1 for initial line drawn (切到x轴)

when $\theta \to 0$, $\frac{\sin\theta}{\theta} \to 1$

⑦ vector 的公式整理

• 面的表示: $r = \vec{OA} + t\vec{AB} + s\vec{BC}$ (已知, 求 $ax+by+cz=d$) 注意 $\vec{OA}$ 是 position!

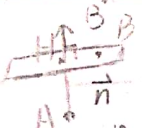
$\vec{n} = \vec{AB} \times \vec{BC}$ $\quad \begin{pmatrix}x\\y\\z\end{pmatrix}\vec{n} = \vec{OA}\cdot\vec{n}$

• 点线距离 

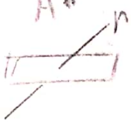
$PH = |\vec{AP} \times \vec{u}|$ 原理是平行四边形面积

• 线线距离 

$a_1 + tb_1$, $a_2 + sb_2$ $\quad AB = \left|\frac{(a_2-a_1)\cdot(b_1 \times b_2)}{|b_1 \times b_2|}\right|$ 切记下面要开根!

• 点面距离 

$\vec{OH} = \vec{OA} + s\vec{n}$, $AH = s|\vec{n}|$ 或 $AH = \left|\frac{\vec{AB}\cdot\vec{n}}{|\vec{n}|}\right|$ (B在面上)

• 线面距离 

$r = \begin{pmatrix}\alpha_1 + t\beta_1\\\alpha_2 + t\beta_2\\\alpha_3 + t\beta_3\end{pmatrix}$ $\quad a(\alpha_1+t\beta_1) + b(\alpha_2+t\beta_2) + c(\alpha_3+t\beta_3) = d$

化简: $d = d$ 重合

$f(t) = d$ 交点

$e \neq d$ 平行但不重合

夹角用 $\vec{n}$ 算，注意题目求 acute angle?

• 面面距离 夹角用 $\vec{n}$ 算，注意题目 acute angle?

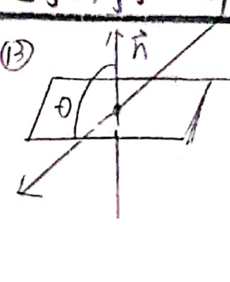
共线: ① 消元，$x = a_1z + b_1$, $y = a_2z + b_2$, 则 $r = \begin{pmatrix}b_1\\b_2\\0\end{pmatrix} + t\begin{pmatrix}a_1\\a_2\\1\end{pmatrix}$

② if $r = \vec{a} + t\vec{b}$, 则 $\vec{b} = \vec{n_1} \times \vec{n_2}$, $\vec{a}$ 带数求

⑩ for $P_k \Rightarrow P_{k+1}$, firstly: assume P_k is true for some positive integer k. 列出 P_k，用K而不是n写!

⑪ 分数求和考虑所有消不完的项，别忘了消了一部分的项！

⑫ $l = \underbrace{A_1i + B_1j + C_1k}_{\text{不能扩大}} + t\underbrace{(A_2i + B_2j + C_2k)}_{\text{可以等比例扩大}}$

⑬ 

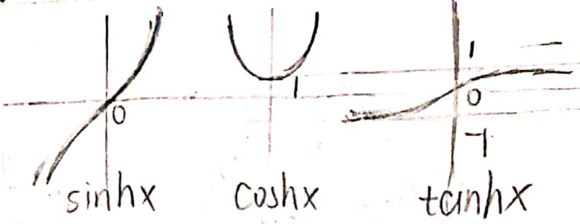
这种情况下仍是线面夹角

⑬ rotate (anti)clockwise $n°$ about the centre of origin
- reflect about line ---
- shear in x/y direction
- stretch, parallel to x/y-axis scale factor k

⑭ vector equation $ai+bj+ck$ coordinate (a,b,c)

P2.

① 双曲函数与三角函数一样，为了one-one计算结果只有一解
但答案可能多解，画图！



② for $A\vec{v} = I\lambda\vec{v}$，$\lambda$为eigen value
用此找对应eigenvector即可
如果问范围，注意 $\det \neq 0$

③ characteristic equation 一般化为 $a_nA^n + a_{n-1}A^{n-1} + \cdots a_1A = -a_0I$

④ $\frac{d}{dx}\left(\frac{dy}{dx}\right) = \frac{d^2y}{dx^2}$，$\frac{d}{dt}\left(\frac{dy}{dx}\right) \cdot \frac{dt}{dx} = \frac{d^2y}{dx^2}$

⑤ shorthand：$y = f(x)$，构建 $g(y) = f'(x)$，则 $y' = g(y)$，$y'' = g'(y) \cdot y' = g'(y) \cdot g(y)$

⑥ reduction rule 1.题目说啥选啥 2.逆推，从题目推过程 3.拆分X
4.远离离谱求导积分错误（$\sin^3 x$，$(1+x^2)^{\frac{1}{2}n}$ etc.）

⑦ 分母是多项式求积分：化成平方±常数项 或用 u, v ($v' = 1$)

⑧ $s = \int\sqrt{1+\left(\frac{dy}{dx}\right)^2}\,dx = \int\sqrt{\left(\frac{dx}{dt}\right)^2+\left(\frac{dy}{dt}\right)^2}\,dt = \int\sqrt{r^2+\left(\frac{dr}{d\theta}\right)^2}\,d\theta$

$S = 2\pi x\Delta s \,/\, 2\pi y \Delta s$

* $\int\sqrt{a^2+b^2x^2}\,dx = \int \frac{a^2}{b}\cosh^2\theta\, d\theta$ if $bx = a\sinh\theta$
$= \int \frac{a^2}{b}\sec^3\theta\, d\theta$ if $bx = a\tan\theta$

⑨ limit of area：判断converge：upper converge
判断diverge：lower diverge

⑩ $\begin{cases} z^n + \frac{1}{z^n} = 2\cos n\theta & e^{i\theta} + e^{-i\theta} = 2\cos\theta \\ z^n - \frac{1}{z^n} = 2i\sin n\theta & e^{i\theta} - e^{-i\theta} = 2i\sin\theta \end{cases}$ 用这个换成三角函数

if 求 $\sum_{n=0}^{N-1} \cos n\theta \,/\, \sin n\theta$，换成 $\sum_{n=0}^{N-1} z^n = \frac{z^N - 1}{z - 1} = \frac{e^{Ni\theta} - 1}{e^{i\theta} - 1} = \frac{e^{(N-\frac{1}{2})i\theta} - e^{-\frac{1}{2}i\theta}}{e^{\frac{1}{2}i\theta} - e^{-\frac{1}{2}i\theta}}$ 例
让分母以1对称划换元，最后看要z的real/imaginary part 就行

* 有时上下同乘 $(az^{-1} - 1)$ 或 $(a - \cos\theta + i\sin\theta)$
注意什么要替换 $a^{-n}\sin\frac{n\pi}{b} \rightarrow a^{-n}\sin n\theta \rightarrow$ 换成 $\left(\frac{z}{a}\right)^n$

⑪ for $\frac{dy}{dx} + Fy = G$，$I = e^{\int F dx}$，$IG = \frac{d}{dx}(Iy)$ 别忘了积IG！
since $GS = CF + PI$，for CF：1. $\lambda_1 \neq \lambda_2$：$y = Ae^{\lambda_1 x} + Be^{\lambda_2 x}$
2. $\lambda_1 = \lambda_2$：$y = (Ax+B)e^{\lambda x}$
3. complex numbers $m \pm ni$ $y = e^{mx}(A\cos nx + B\sin nx)$

for PI：if $f(x) = ke^{\lambda x}$，use $y = ae^{\lambda x}$
if $f(x) = k_1\cos\lambda x + k_2\sin\lambda x$，$y = a\cos\lambda x + b\sin\lambda x$
如果与CF重复格式，乘 x、x^2、……

for $\frac{dy}{dx} = f(x,y)$，turn $F(x,y) = u$，so $\frac{du}{dx}$ 可得，$\frac{dy}{dx}$ 可换为 $\frac{du}{dx}$ so $A\frac{du}{dx} = g(u) + B$
注意问谁与谁的equation？用结论找过程！

⑫ boundary limit 题型一定要写几环 A

⑬ Moivre's theorem: equation $=0$ 时，只满足分子 $=0$ 　　多个 $100t$

同时考虑周期性（多个解可以合并吗？）

⑭ $\tanh^{-1}x = \frac{1}{2}\ln\left(\frac{1+x}{1-x}\right)$ 推导：

$$\tanh x = \frac{e^{2x}-1}{e^{2x}+1}$$

$$te^{2x}+t = e^{2x}-1$$

$$e^{2x}(1-t) = 1+t$$

$$x = \frac{1}{2}\ln\left|\frac{1+t}{1-t}\right|$$

$$\tan^{-1}x = \frac{1}{2}\ln\left|\frac{1+x}{1-x}\right|$$

⑮ geometric meaning（要写出 $x=\cdots, y=\cdots, z=\cdots$）

- two parallel planes, not identical, not parallel to third plane
- three planes intersect at a single point
- two planes identical, a line of intersection with other plane
- three planes form a triangular prism

⑯ $z^n=1,\ z\neq 1 \Longrightarrow z+z^2+z^3+\cdots+z^n=0 \Rightarrow 1+z+z^2+\cdots+z^{n-1}=0$

⑰ $\lim_{x\to\infty}$ 题点出 $e^{-x}\to 0$

⑱ $P^{-1}AP=D \Longrightarrow A=PDP^{-1}$ 　A is original matrix

⑲ $Ae=\lambda e \Rightarrow e=\lambda A^{-1}e \Rightarrow \lambda^{-1}e=A^{-1}e$

① hypothesis test 别忘了写 there's sufficient/insufficient evidents to prove
要reject 必须全部在 critical value 外！(注意Binomial)
type I error H_0对 拒H_1, type II error H_1对 拒H_0.

② $Var(X) = \int_{-\infty}^{\infty} x^2 f(x)dx - [\int_{-\infty}^{\infty} x f(x)dx]^2$ $\quad E(X) = \int_{-\infty}^{\infty} x f(x)dx$

$= G_t''(1) + G_t'(1) - (G_t'(1))^2 \qquad = G_t'(1)$

③ 搞清population与sample! $S_p^2 = \dfrac{\Sigma x^2 - \frac{(\Sigma x)^2}{n}}{n-1}$!

④ PDF 加 otherwise, CDF 加 $F(x)=0$, when $x<a$; $F(x)=1$ when $x>b$

$f(x) \rightarrow F(x) \rightarrow G() \rightarrow g()$

$G(y) = P(Y \le y) = \begin{cases} G(X \le h^{-1}(y)) = F(h^{-1}(y)) \\ G(X \ge h^{-1}(y)) = 1 - F(h^{-1}(y)) \end{cases}$ 考虑大小排序 $y=h(x)$ } 本质为X换

⑤ t distribution 的 ν 永远 $n-1$. $t = \frac{\bar{x}-\mu}{\sqrt{\frac{s^2}{n}}}$
χ^2 distribution 的 ν 跟着变, 如果E<5要合并(合并的话看成一个自由)
single-sample Wilcoxon/matched-pair signed-rank/rank sum test
要 critical < statistic 来 accept H_0
preference 等不知 distribution 的比较 → H_0: the difference of population median = 0

在用N估计时, 修正discrete!
考虑one-trial还是two-trials
是否无偏取决于是不是sample; 用t还是Z查取决于n的大小

⑥
test	assumption
sign test	continuous, independent
Wilcoxon signed-rank test	symmetric, continuous, independent
paired sign test	in matched pairs, independent differences are continuous
Wilcoxon matched-pairs signed-rank test	in matched pairs, independent differences symmetric & continuous
Wilcoxon rank-sum test	independent, symmetric, continuous
t distribution	normal

⑦
distribution	PGF
$P(X=x_i) = \begin{cases} \frac{1}{n} & i=1,2,\dots n \\ 0 & \text{otherwise} \end{cases}$	$G_X(t) = \frac{t(1-t^n)}{n(1-t)}$
$X \sim B(n,p)$	$G_X(t) = (q+pt)^n$
$X \sim G(p)$	$G_X(t) = \frac{pt}{1-qt}$
$X \sim Po(\lambda)$	$G_X(t) = e^{\lambda(t-1)}$

考虑有没有包含n=0! 没有的话重新推导!

修正
书写格式 CDF PDF
(χ^2) two/one-sides
无偏时 n or (n-1)
搞明白秩 等 rank
ν 还是 n? $\nu=n-1$? $n-2$?

⑧ the circumstances under which a non-parametric test of significance should be used rather than a parametric test

- when population cannot be assumed to be normally distributed

⑨ population variance 时 $\sqrt{\frac{S^2}{n_x} + \frac{S^2}{n_y}}$

注意 $S^2 = \frac{\Sigma(x-\bar{x})^2 + \Sigma(y-\bar{y})^2}{n_x+n_y-2}$

$= \frac{(n_x-1)S_x^2 + (n_y-1)S_y^2}{n_x+n_y-2}$

⑩ result support $\mu > k$ → 说明 $\bar{x} >> k$ →

即 $\bar{x}-\mu = \bar{x}-k > 0$，越大越极限

⑪ $G_x(t)$ 展开：用 $(1+x)^n = 1+nx+\frac{n(n-1)}{2}x^2+\cdots$

⑫ χ^2 估计时看有没有 estimate element：$Po(\lambda)$ 算一个

⑬ sign test 用 $B(n, 0.5)$ 算　　$(n-1-N)$

⑭ 用 Z 估计 large n 的 rank 时，修正 discrete

⑮ significant level 边上

confidence interval 中间

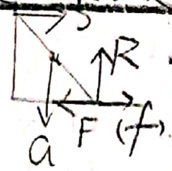
⑯ central limit theorem applies

⑰ more group / details / degree of freedom

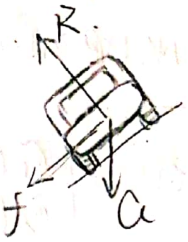
Date

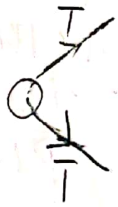
No.

画图！读题！(express A in B? B in A?)
看清楚字母含义！
(a是全长?腰长?是不是km?
$r=x+a$?
V_y or y
$r=x\sin\theta$?)



① 搞清楚 V_x V_y X Y 公式
$y=x\tan\theta+\frac{gx^2}{2u^2}\sec^2\theta$ 各字母含义
如果用顶端 $\frac{1}{2}$ 的做法需说明
② 重心 F.X. 注意 对称性 以及物体重心位置
不要用moment找力(只会找到分力,无用)用x,y找力
判断rotate 依心不会移动,用moment
考虑所有力(a)用moment不考虑部分力

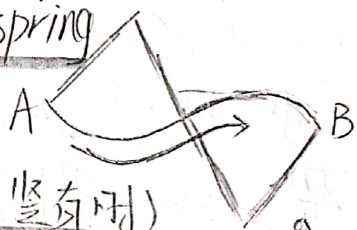




③ 绳中间放个球,两边不管方向如何均有T
三维的平面旋转考虑a,給ω转换成V
要完成一圈环形轨道,R>0
如果考虑 T_{MAX} $_{MIN}$ 考虑a
如果考 h_{MAX},球脱离轨道后还会向上

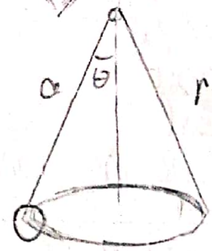
$PE=-mgr\cos\theta$
$F_{向}=T-G\cos\theta$
V在变,则F向变

④ gain/loss in KE/GPE/EPE
注意转化关系!EPE原本就有一些!string/spring
$T=\frac{\lambda x}{l}$, $EPE=\frac{\lambda x^2}{2l}$
注意题目, $v=0$ 与 $a=0$ 不同



A→B
过顶点
不飞出

⑤ resistive force的acceleration为负!考虑G(竖直时)
$a=v\frac{dv}{dx}$



r是 $a\sin\theta$.

⑥ $e=\frac{v_2-v_1}{u_1-u_2}\Rightarrow v_2-v_1=e(u_1-u_2)$ 用这个关系求范围
⑦ hollow/shell △ 差 from vertex, 右 from base
⑧ $\alpha+\beta=90°$
$\tan\alpha\cdot\tan\beta=1$
$l_1\perp l_2$
$\tan\alpha\cdot\tan\beta=-1$
实际考虑
⑨ 平抛时 $KE\neq0$!求平抛最高点不要用conservation of energy!
⑩ collision求e时考虑方向相反!

补: ⑪ angle deflection是偏离原轨
⑫ 平抛问题: ①是否存在θ角: $V_x=kV_y$
$u\cos\theta=k(u\sin\theta+\frac{1}{2}gt^2)$
验证t
· $\frac{dy}{dx}=\pm k$ 区间内less than θ

⑭ 会不会slide,范围:
用"="先求,然后确定范围,注意 $0\leq x$
$\tan\theta>\frac{a}{x}$ 时倒
$\therefore\frac{a}{x}>\tan\theta$

⑬ 不要老考虑公式!h_{max} 直接套 $\frac{u^2\sin^2\theta}{2g}$
问 $\frac{n}{m}h_{MAX}$ 时公式, $V=\frac{u\cos\theta}{\cos\alpha}$ ↑ $(v\sin\alpha)^2=(u\sin\theta)^2-2g\cdot\frac{n}{m}\frac{u^2\sin^2\theta}{2g}$
⑬ a/v/s/t 开平方根/ln绝对值都要保留正负, take negative sign to meet initial condition → not satisfy

⑮受力分析是整体平衡不是个点平衡

x、y轴从众即可