COMPLEX NUMBERS

Spec-S1-Q7

7 Find the modulus and argument of $1+e^{2i\alpha}$ where $-\frac{1}{2}\pi < \alpha < \frac{1}{2}\pi$.

By using de Moivre's theorem, or otherwise, sum the series

$$\sum_{r=0}^{n} \frac{n!}{r!(n-r)!}\sin(2r+1)\alpha.$$

Spec-S2-Q3

3 Prove de Moivre's theorem, that

$$(\cos\theta + i\sin\theta)^n = \cos n\theta + i\sin n\theta,$$

where n is a positive integer.

Find all real numbers x and y which satisfy

$$x^3\cos 3y + 2x^2\cos 2y + 2x\cos y = -1,$$
$$x^3\sin 3y + 2x^2\sin 2y + 2x\sin y = 0.$$

Spec-S3-Q3

3 For the complex numbers z_1 and z_2 interpret geometrically the inequality

$$|z_1 + z_2| \leqslant |z_1| + |z_2|.$$

Prove that, if $|a_i| \leqslant 2$ for $i = 1, 2, \ldots, n$, then the equation

$$a_1 z + a_2 z^2 + \cdots + a_n z^n = 1$$

has no solutions with $|z| \leqslant \frac{1}{3}$.

87-S2-Q4

4 Explain the geometrical relationship between the points in the Argand diagram represented by the complex numbers z and $ze^{i\theta}$.

Write down necessary and sufficient conditions that the distinct complex numbers α, β and γ represent the vertices of an equilateral triangle taken in anticlockwise order.

Show that α, β and γ represent the vertices of an equilateral triangle (taken in any order) if and only if

$$\alpha^2 + \beta^2 + \gamma^2 - \beta\gamma - \gamma\alpha - \alpha\beta = 0.$$

Find necessary and sufficient conditions on the complex coefficients a, b and c for the roots of the equation

$$z^3 + az^2 + bz + c = 0$$

to lie at the vertices of an equilateral triangle in the Argand digram.

88-S2-Q4

4 The complex number w is such that $w^2 - 2x$ is real.

(i) Sketch the locus of w in the Argand diagram.

(ii) If $w^2 = x + iy$, describe fully and sketch the locus of points (x, y) in the x-y plane.

The complex number t is such that $t^2 - 2t$ is imaginary. If $t^2 = p + iq$, sketch the locus of points (p, q) in the p-q plane.

88-S3-Q3

3 Give a parametric form for the curve in the Argand diagram determined by $|z - \mathrm{i}| = 2$.

Let $w = (z + \mathrm{i})/(z - \mathrm{i})$. Find and sketch the locus, in the Argand diagram, of the point which represents the complex number w when

(i) $|z - \mathrm{i}| = 2$;

(ii) z is real;

(iii) z is imaginary.

89-S2-Q3

3 The real numbers x and y are related to the real numbers u and v by

$$2(u + \mathrm{i}v) = \mathrm{e}^{x+\mathrm{i}y} - \mathrm{e}^{-x-\mathrm{i}y}.$$

Show that the line in the x-y plane given by $x = a$, where a is a positive constant, corresponds to the ellipse

$$\left(\frac{u}{\sinh a}\right)^2 + \left(\frac{v}{\cosh a}\right)^2 = 1$$

in the u-v plane. Show also that the line given by $y = b$, where b is a constant and $0 < \sin b < 1$, corresponds to one branch of a hyperbola in the u-v plane. Write down the u and v coordinates of one point of intersection of the ellipse and hyperbola branch, and show that the curves intersect at right-angles at this point.

Make a sketch of the u-v plane showing the ellipse, the hyperbola branch and the line segments corresponding to:

(i) $x = 0$;

(ii) $y = \frac{1}{2}\pi, \quad 0 \leqslant x \leqslant a.$

90-S2-Q5

5 The distinct points L, M, P and Q of the Argand diagram lie on a circle S centred on the origin and the corresponding complex numbers are l, m, p and q. By considering the perpendicular bisectors of the chords, or otherwise, prove that the chord LM is perpendicular to the chord PQ if and only if $lm + pq = 0$.

Let A_1, A_2 and A_3 be three distinct points on S. For any given point A_1' on S, the points A_2', A_3' and A_1'' are chosen on S such that $A_1'A_2'$, $A_2'A_3'$ and $A_3'A_1''$ are perpendicular to A_1A_2, A_2A_3 and A_3A_1, respectively. Show that for exactly two positions of A_1', the points A_1' and A_1'' coincide.

If, instead, A_1, A_2, A_3 and A_4 are four given distinct points on S and, for any given point A_1', the points A_2', A_3', A_4' and A_1'' are chosen on S such that $A_1'A_2'$, $A_2'A_3'$, $A_3'A_4'$ and $A_4'A_1''$ are respectively perpendicular to A_1A_2, A_2A_3, A_3A_4 and A_4A_1, show that A_1' coincides with A_1''.

Give the corresponding result for n distinct points on S.

90-S3-Q1

1 Show, using de Moivre's theorem, or otherwise, that

$$\tan 9\theta = \frac{t(t^2-3)(t^6-33t^4+27t^2-3)}{(3t^2-1)(3t^6-27t^4+33t^2-1)}, \qquad \text{where } t = \tan\theta.$$

By considering the equation $\tan 9\theta = 0$, or otherwise, obtain a cubic equation with integer coefficients whose roots are

$$\tan^2\left(\frac{\pi}{9}\right), \qquad \tan^2\left(\frac{2\pi}{9}\right) \qquad \text{and} \qquad \tan^2\left(\frac{4\pi}{9}\right).$$

Deduce the value of

$$\tan\left(\frac{\pi}{9}\right)\tan\left(\frac{2\pi}{9}\right)\tan\left(\frac{4\pi}{9}\right).$$

Show that

$$\tan^6\left(\frac{\pi}{9}\right) + \tan^6\left(\frac{2\pi}{9}\right) + \tan^6\left(\frac{4\pi}{9}\right) = 33273.$$

91-S1-Q3

3 A path is made up in the Argand diagram of a series of straight line segments P_1P_2, P_2P_3, $P_3P_4, \ldots$ such that each segment is d times as long as the previous one, $(d \neq 1)$, and the angle between one segment and the next is always θ (where the segments are directed from P_j towards P_{j+1}, and all angles are measured in the anticlockwise direction). If P_j represents the complex number z_j, express

$$\frac{z_{n+1} - z_n}{z_n - z_{n-1}}$$

as a complex number (for each $n \geqslant 2$), briefly justifying your answer.

If $z_1 = 0$ and $z_2 = 1$, obtain an expression for z_{n+1} when $n \geqslant 2$. By considering its imaginary part, or otherwise, show that if $\theta = \frac{1}{3}\pi$ and $d = 2$, then the path crosses the real axis infinitely often.

91-S3-Q2

2 The distinct points P_1, P_2, P_3, Q_1, Q_2 and Q_3 in the Argand diagram are represented by the complex numbers z_1, z_2, z_3, w_1, w_2 and w_3 respectively. Show that the triangles $P_1P_2P_3$ and $Q_1Q_2Q_3$ are similar, with P_i corresponding to Q_i $(i = 1, 2, 3)$ and the rotation from 1 to 2 to 3 being in the same sense for both triangles, if and only if

$$\frac{z_1 - z_2}{z_2 - z_3} = \frac{w_1 - w_2}{w_1 - w_3}.$$

Verify that this condition may be written

$$\det\begin{pmatrix} z_1 & z_2 & z_3 \\ w_1 & w_2 & w_3 \\ 1 & 1 & 1 \end{pmatrix} = 0.$$

(i) Show that if $w_i = z_i^2$ $(i = 1, 2, 3)$ then triangle $P_1P_2P_3$ is not similar to triangle $Q_1Q_2Q_3$.

(ii) Show that if $w_i = z_i^3$ $(i = 1, 2, 3)$ then triangle $P_1P_2P_3$ is similar to triangle $Q_1Q_2Q_3$ if and only if the centroid of triangle $P_1P_2P_3$ is the origin. [The *centroid* of triangle $P_1P_2P_3$ is represented by the complex number $\frac{1}{3}(z_1 + z_2 + z_3)$.]

(iii) Show that the triangle $P_1P_2P_3$ is equilateral if and only if

$$z_2z_3 + z_3z_1 + z_1z_2 = z_1^2 + z_2^2 + z_3^2.$$

92-S1-Q4

4 Sketch the following subsets of the complex plane using Argand diagrams. Give reasons for your answers.

(i) $\{z : \mathrm{Re}((1+\mathrm{i})z) \geqslant 0\}$.

(ii) $\{z : |z^2| \leqslant 2, \mathrm{Re}(z^2) \geqslant 0\}$.

(iii) $\{z = z_1 + z_2 : |z_1| = 2, |z_2| = 1\}$.

92-S2-Q10

10 Let α be a fixed angle, $0 < x \leqslant \frac{1}{2}\pi$. In each of the following cases, sketch the locus of z in the Argand diagram (the complex plane):

(i) $\arg\left(\dfrac{z-1}{z}\right) = \alpha$,

(ii) $\arg\left(\dfrac{z-1}{z}\right) = \alpha - \pi$,

(iii) $\left|\dfrac{z-1}{z}\right| = 1$.

Let z_1, z_2, z_3 and z_4 be four points lying (in that order) on a circle in the Argand diagram. If

$$w = \frac{(z_1 - z_2)(z_3 - z_4)}{(z_4 - z_1)(z_2 - z_3)}$$

show, by considering $\arg w$, that w is real.

93-S1-Q5

5 If $z = x + \mathrm{i}y$ where x and y are real, define $|z|$ in terms of x and y. Show, using your definition, that if $z_1, z_2 \in \mathbb{C}$ then $|z_1 z_2| = |z_1|\,|z_2|$.

Explain, by means of a diagram, or otherwise, why $|z_1 + z_2| \leqslant |z_1| + |z_2|$.

Suppose that $a_j \in \mathbb{C}$ and $|a_j| \leqslant 1$ for $j = 1, 2, \ldots, n$. Show that, if $|z| \leqslant \frac{1}{2}$, then

$$\left|a_n z^n + a_{n-1} z^{n-1} + \cdots + a_1 z\right| < 1,$$

and deduce that any root w of the equation

$$a_n z^n + a_{n-1} z^{n-1} + \cdots + a_1 z + 1 = 0$$

must satisfy $|x| > \frac{1}{2}$.

93-S2-Q9

9 *In this question, the argument of a complex number is chosen to satisfy* $0 \leqslant \arg z < 2\pi$.

Let z be a complex number whose imaginary part is positive. What can you say about $\arg z$?

The complex numbers z_1, z_2 and z_3 all have positive imaginary part and $\arg z_1 < \arg z_2 < \arg z_3$. Draw a diagram that shows why

$$\arg z_1 < \arg(z_1 + z_2 + z_3) < \arg z_3.$$

Prove that $\arg(z_1 z_2 z_3)$ is never equal to $\arg(z_1 + z_2 + z_3)$.

93-S3-Q6

6 The point in the Argand diagram representing the complex number z lies on the circle with centre K and radius r, where K represents the complex number k. Show that

$$zz^* - kz^* - k^*z + kk^* - r^2 = 0.$$

The points P, Q_1 and Q_2 represent the complex numbers z, w_1 and w_2 respectively. The point P lies on the circle with OA as diameter, where O and A represent 0 and $2i$ respectively. Given that $w_1 = z/(z-1)$, find the equation of the locus L of Q_1 in terms of w_1 and describe the geometrical form of L.

Given that $w_2 = z^*$, show that the locus of Q_2 is also L. Determine the positions of P for which Q_1 coincides with Q_2.

94-S1-Q6

6 The function f is defined, for any complex number z, by

$$\mathrm{f}(z) = \frac{\mathrm{i}z - 1}{\mathrm{i}z + 1}.$$

Suppose throughout that x is a real number.

(i) Show that

$$\mathrm{Re}\,\mathrm{f}(x) = \frac{x^2 - 1}{x^2 + 1} \qquad \text{and} \qquad \mathrm{Im}\,\mathrm{f}(x) = \frac{2x}{x^2 + 1}.$$

(ii) Show that $\mathrm{f}(x)\mathrm{f}(x)^* = 1$, where $\mathrm{f}(x)^*$ is the complex conjugate of $\mathrm{f}(x)$.

(iii) Find expressions for $\mathrm{Re}\,\mathrm{f}(\mathrm{f}(x))$ and $\mathrm{Im}\,\mathrm{f}(\mathrm{f}(x))$.

(iv) Find $\mathrm{f}(\mathrm{f}(\mathrm{f}(x)))$.

94-S3-Q6

6 The four points A, B, C, D in the Argand diagram (complex plane) correspond to the complex numbers a, b, c, d respectively. The point P_1 is mapped to P_2 by rotating about A through $\pi/2$ radians. Then P_2 is mapped to P_3 by rotating about B through $\pi/2$ radians, P_3 is mapped to P_4 by rotating about C through $\pi/2$ radians and P_4 is mapped to P_5 by rotating about D through $\pi/2$ radians, each rotation being in the positive sense. If z_i is the complex number corresponding to P_i, find z_5 in terms of a, b, c, d and z_1.

Show that P_5 will coincide with P_1, irrespective of the choice of the latter if, and only if

$$a - c = \mathrm{i}(b - d)$$

and interpret this condition geometrically.

The points A, B and C are now chosen to be distinct points on the unit circle and the angle of rotation is changed to θ, where $\theta \neq 0$, on each occasion. Find the necessary and sufficient condition on θ and the points A, B and C for P_4 always to coincide with P_1.

95-S1-Q4

4 By applying de Moivre's theorem to $\cos 5\theta + \mathrm{i}\sin 5\theta$, expanding the result using the binomial theorem, and then equating imaginary parts, show that

$$\sin 5\theta = \sin\theta\left(16\cos^4\theta - 12\cos^2\theta + 1\right).$$

Use this identity to evaluate $\cos^2 \frac{1}{5}\pi$, and deduce that $\cos\frac{1}{5}\pi = \frac{1}{4}(1+\sqrt{5})$.

95-S2-Q6

6 If u and v are the two roots of $z^2 + az + b = 0$, show that $a = -u - v$ and $b = uv$.

Let $\alpha = \cos(2\pi/7) + \mathrm{i}\sin(2\pi/7)$. Show that α is a root of $z^6 - 1 = 0$ and express the roots in terms of α. The number $\alpha + \alpha^2 + \alpha^4$ is a root of a quadratic equation

$$z^2 + Az + B = 0$$

where A and B are real. By guessing the other root, or otherwise, find the numerical values of A and B.

Show that

$$\cos\frac{2\pi}{7} + \cos\frac{4\pi}{7} + \cos\frac{8\pi}{7} = -\frac{1}{2},$$

and evaluate

$$\sin\frac{2\pi}{7} + \sin\frac{4\pi}{7} + \sin\frac{8\pi}{7},$$

making it clear how you determine the sign of your answer.

95-S3-Q6

6 The variable non-zero complex number z is such that

$$|z - \mathrm{i}| = 1.$$

Find the modulus of z when its argument is θ. Find also the modulus and argument of $1/z$ in terms of θ and show in an Argand diagram the loci of points which represent z and $1/z$.

Find the locus C in the Argand diagram such that $w \in C$ if, and only if, the real part of $(1/w)$ is -1.

97-S2-Q5

5 The complex numbers $w = u + \mathrm{i}v$ and $z = x + \mathrm{i}y$ are related by the equation

$$z = (\cos v + \mathrm{i}\sin v)\mathrm{e}^u.$$

Find all w which correspond to $z = \mathrm{i}\,\mathrm{e}$.

Find the loci in the x–y plane corresponding to the lines $u =$ constant in the u–v plane. Find also the loci corresponding to the lines v = constant. Illustrate your answers with clearly labelled sketches.

Identify two subsets W_1 and W_2 of the u–v plane each of which is in one-to-one correspondence with the first quadrant $\{(x,\, y) :\, x > 0,\, y > 0\}$ of the x–y plane. Identify also two subsets W_3 and W_4 each of which is in one-to-one correspondence with the set $\{z\, : 0 < \,|z|\, < 1\}$.

[**NB** 'one-to-one' means here that to each value of w there is only one corresponding value of z, and vice-versa.]

97-S3-Q3

3 By considering the solutions of the equation $z^n - 1 = 0$, or otherwise, show that

$$(z - \omega)(z - \omega^2)\ldots(z - \omega^{n-1}) = 1 + z + z^2 + \cdots + z^{n-1},$$

where z is any complex number and $\omega = \mathrm{e}^{2\pi i/n}$.

Let $A_1, A_2, A_3, \ldots, A_n$ be points equally spaced around a circle of radius r centred at O (so that they are the vertices of a regular n-sided polygon).

Show that

$$\overrightarrow{OA_1} + \overrightarrow{OA_2} + \overrightarrow{OA_3} + \cdots + \overrightarrow{OA_n} = \mathbf{0}.$$

Deduce, or prove otherwise, that

$$\sum_{k=1}^{n} |A_1 A_k|^2 = 2r^2 n.$$

VECTORS

Spec-S2-Q9

9 **(i)** Let $\mathbf{a}$ and $\mathbf{b}$ be given vectors with $\mathbf{b} \neq \mathbf{0}$, and let $\mathbf{x}$ be a position vector. Find the condition for the sphere $|\mathbf{x}| = R$, where $R > 0$, and the plane $(\mathbf{x} - \mathbf{a}) \cdot \mathbf{b} = 0$ to intersect.

When this condition is satisfied, find the radius and the position vector of the centre of the circle in which the plane and sphere intersect.

(ii) Let $\mathbf{c}$ be a given vector, with $\mathbf{c} \neq \mathbf{0}$. The vector $\mathbf{x}'$ is related to the vector $\mathbf{x}$ by

$$\mathbf{x}' = \mathbf{x} - \frac{2(\mathbf{x} \cdot \mathbf{c})\mathbf{c}}{|\mathbf{c}|^2}.$$

Interpret this relation geometrically.

87-S1-Q9

9 ABC is a triangle whose vertices have position vectors $\mathbf{a}, \mathbf{b}, \mathbf{c}$ respectively, relative to an origin in the plane ABC. Show that an arbitrary point P on the segment AB has position vector

$$\rho\mathbf{a} + \sigma\mathbf{b},$$

where $\rho \geqslant 0$, $\sigma \geqslant 0$ and $\rho + \sigma = 1$.

Give a similar expression for an arbitrary point on the segment PC, and deduce that any point inside ABC has position vector

$$\lambda\mathbf{a} + \mu\mathbf{b} + \nu\mathbf{c},$$

where $\lambda \geqslant 0$, $\mu \geqslant 0$, $\nu \geqslant 0$ and $\lambda + \mu + \nu = 1$.

Sketch the region of the plane in which the point $\lambda\mathbf{a} + \mu\mathbf{b} + \nu\mathbf{c}$ lies in each of the following cases:

(i) $\lambda + \mu + \nu = -1$, $\lambda \leqslant 0$, $\mu \leqslant 0$, $\nu \leqslant 0$;

(ii) $\lambda + \mu + \nu = 1$, $\mu \leqslant 0$, $\nu \leqslant 0$.

87-S2-Q8

8 Let $\mathbf{r}$ be the position vector of a point in three-dimensional space. Describe fully the locus of the point whose position vector is $\mathbf{r}$ in each of the following four cases:

(i) $(\mathbf{a} - \mathbf{b}) \cdot \mathbf{r} = \frac{1}{2}(|\mathbf{a}|^2 - |\mathbf{b}|^2)$;

(ii) $(\mathbf{a} - \mathbf{r}) \cdot (\mathbf{b} - \mathbf{r}) = 0$;

(iii) $|\mathbf{r} - \mathbf{a}|^2 = \frac{1}{2}|\mathbf{a} - \mathbf{b}|^2$;

(iv) $|\mathbf{r} - \mathbf{b}|^2 = \frac{1}{2}|\mathbf{a} - \mathbf{b}|^2$.

Prove algebraically that the equations **(i)** and **(ii)** together are equivalent to **(iii)** and **(iv)** together. Explain carefully the geometrical meaning of this equivalence.

88-S2-Q10

10 The surface S in 3-dimensional space is described by the equation

$$\mathbf{a}\cdot\mathbf{r}+ar=a^2,$$

where $\mathbf{r}$ is the position vector with respect to the origin O, $\mathbf{a}(\neq\mathbf{0})$ is the position vector of a fixed point, $r=|\mathbf{r}|$ and $a=|\mathbf{a}|$. Show, with the aid of a diagram, that S is the locus of points which are equidistant from the origin O and the plane $\mathbf{r}\cdot\mathbf{a}=a^2$.

The point P, with position vector $\mathbf{p}$, lies in S, and the line joining P to O meets S again at Q. Find the position vector of Q.

The line through O orthogonal to $\mathbf{p}$ and $\mathbf{a}$ meets S at T and T'. Show that the position vectors of T and T' are

$$\pm\frac{1}{\sqrt{2ap-a^2}}\mathbf{a}\times\mathbf{p},$$

where $p=|\mathbf{p}|$.

Show that the area of the triangle PQT is

$$\frac{ap^2}{2p-a}.$$

91-S1-Q5

5 A set of n distinct vectors $\mathbf{a}_1,\mathbf{a}_2,\dots,\mathbf{a}_n$, where $n\geqslant 2$, is called *regular* if it satisfies the following two conditions:

(i) there are constants α and β, with $\alpha>0$, such that for any i and j,

$$\mathbf{a}_i\cdot\mathbf{a}_j=\begin{cases}\alpha^2 & \text{when } i=j\\ \beta & \text{when } i\neq j,\end{cases}$$

(ii) the centroid of $\mathbf{a}_1,\mathbf{a}_2,\dots,\mathbf{a}_n$ is the origin $\mathbf{0}$. [The centroid of vectors $\mathbf{b}_1,\mathbf{b}_2,\dots,\mathbf{b}_m$ is the vector $\frac{1}{m}(\mathbf{b}_1+\mathbf{b}_2+\dots+\mathbf{b}_m)$.]

Prove that **(i)** and **(ii)** imply that $(n-1)\beta=-\alpha^2$.

If $\mathbf{a}_1=\begin{pmatrix}1\\0\end{pmatrix}$, where $\mathbf{a}_1,\mathbf{a}_2,\dots,\mathbf{a}_n$ is a regular set of vectors in 2-dimensional space, show that either $n=2$ or $n=3$, and in each case find the other vectors in the set.

Hence, or otherwise, find all regular sets of vectors in 3-dimensional space for which $\mathbf{a}_1=\begin{pmatrix}1\\0\\0\end{pmatrix}$ and $\mathbf{a}_2$ lies in the x-y plane.

92-S2-Q9

9 Let $\mathbf{a},\mathbf{b}$ and $\mathbf{c}$ be the position vectors of points A,B and C in three-dimensional space. Suppose that A,B,C and the origin O are not all in the same plane. Describe the locus of the point whose position vector $\mathbf{r}$ is given by

$$\mathbf{r}=(1-\lambda-\mu)\mathbf{a}+\lambda\mathbf{b}+\mu\mathbf{c},$$

where λ and μ are scalar parameters. By writing this equation in the form $\mathbf{r}\cdot\mathbf{n}=p$ for a suitable vector $\mathbf{n}$ and scalar p, show that

$$-(\lambda+\mu)\mathbf{a}\cdot(\mathbf{b}\times\mathbf{c})+\lambda\mathbf{b}\cdot(\mathbf{c}\times\mathbf{a})+\mu\mathbf{c}\cdot(\mathbf{a}\times\mathbf{b})=0$$

for all scalars λ,μ.

Deduce that

$$\mathbf{a}\cdot(\mathbf{b}\times\mathbf{c})=\mathbf{b}\cdot(\mathbf{c}\times\mathbf{a})=\mathbf{c}\cdot(\mathbf{a}\times\mathbf{b}).$$

Say briefly what happens if A,B,C and O are all in the same plane.

93-S2-Q4

4 Two non-parallel lines in 3-dimensional space are given by $\mathbf{r} = \mathbf{p}_1 + t_1\mathbf{m}_1$ and $\mathbf{r} = \mathbf{p}_2 + t_2\mathbf{m}_2$ respectively, where $\mathbf{m}_1$ and $\mathbf{m}_2$ are unit vectors. Explain by means of a sketch why the shortest distance between the two lines is

$$\frac{|(\mathbf{p}_1 - \mathbf{p}_2)\cdot(\mathbf{m}_1 \times \mathbf{m}_2)|}{|(\mathbf{m}_1 \times \mathbf{m}_2)|}.$$

(i) Find the shortest distance between the lines in the case

$$\mathbf{p}_1 = (2,1,-1) \qquad \mathbf{p}_2 = (1,0,-2) \qquad \mathbf{m}_1 = \tfrac{1}{5}(4,3,0) \qquad \mathbf{m}_2 = \tfrac{1}{\sqrt{10}}(0,-3,1).$$

(ii) Two aircraft, A_1 and A_2, are flying in the directions given by the unit vectors $\mathbf{m}_1$ and $\mathbf{m}_2$ at constant speeds v_1 and v_2. At time $t = 0$ they pass the points $\mathbf{p}_1$ and $\mathbf{p}_2$, respectively. If d is the shortest distance between the two aircraft during the flight, show that

$$d^2 = \frac{|\mathbf{p}_1 - \mathbf{p}_2|^2\,|v_1\mathbf{m}_1 - v_2\mathbf{m}_2|^2 - [(\mathbf{p}_1 - \mathbf{p}_2)\cdot(v_1\mathbf{m}_1 - v_2\mathbf{m}_2)]^2}{|v_1\mathbf{m}_1 - v_2\mathbf{m}_2|^2}.$$

(iii) Suppose that v_1 is fixed. The pilot of A_2 has chosen v_2 so that A_2 comes as close as possible to A_1. How close is that, if $\mathbf{p}_1, \mathbf{p}_2, \mathbf{m}_1$ and $\mathbf{m}_2$ are as in **(i)**?

95-S3-Q8

8 A plane π in 3-dimensional space is given by the vector equation $\mathbf{r}\cdot\mathbf{n} = p$, where $\mathbf{n}$ is a unit vector and p is a non-negative real number. If $\mathbf{x}$ is the position vector of a general point X, find the equation of the normal to π through X and the perpendicular distance of X from π.

The unit circles C_i, $i = 1, 2$, with centres $\mathbf{r}_i$, lie in the planes π_i given by $\mathbf{r}\cdot\mathbf{n}_i = p_i$, where the $\mathbf{n}_i$ are unit vectors, and p_i are non-negative real numbers. Prove that there is a sphere whose surface contains both circles only if there is a real number λ such that

$$\mathbf{r}_1 + \lambda\mathbf{n}_1 = \mathbf{r}_2 \pm \lambda\mathbf{n}_2.$$

Hence, or otherwise, deduce the necessary conditions that

$$(\mathbf{r}_1 - \mathbf{r}_2)\cdot(\mathbf{n}_1 \times \mathbf{n}_2) = 0$$

and that

$$(p_1 - \mathbf{n}_1\cdot\mathbf{r}_2)^2 = (p_2 - \mathbf{n}_2\cdot\mathbf{r}_1)^2.$$

Interpret each of these two conditions geometrically.

13-S1-Q3

3 For any two points X and Y, with position vectors $\mathbf{x}$ and $\mathbf{y}$ respectively, $X * Y$ is defined to be the point with position vector $\lambda\mathbf{x} + (1-\lambda)\mathbf{y}$, where λ is a fixed number.

(i) If X and Y are distinct, show that $X * Y$ and $Y * X$ are distinct unless λ takes a certain value (which you should state).

(ii) Under what conditions are $(X * Y) * Z$ and $X * (Y * Z)$ distinct?

(iii) Show that, for any points X, Y and Z,

$$(X * Y) * Z = (X * Z) * (Y * Z)$$

and obtain the corresponding result for $X * (Y * Z)$.

(iv) The points $P_1, P_2, \ldots$ are defined by $P_1 = X * Y$ and, for $n \geqslant 2$, $P_n = P_{n-1} * Y$. Given that X and Y are distinct and that $0 < \lambda < 1$, find the ratio in which P_n divides the line segment XY.

STEP2 MATRICES QUESTIONS

87-S2-Q9

9 For any square matrix $\mathbf{A}$ such that $\mathbf{I} - \mathbf{A}$ is non-singular (where $\mathbf{I}$ is the unit matrix), the matrix $\mathbf{B}$ is defined by

$$\mathbf{B} = (\mathbf{I} + \mathbf{A})(\mathbf{I} - \mathbf{A})^{-1}.$$

Prove that $\mathbf{B}^{\mathrm{T}}\mathbf{B} = \mathbf{I}$ if and only if $\mathbf{A} + \mathbf{A}^{\mathrm{T}} = \mathbf{O}$ (where $\mathbf{O}$ is the zero matrix), explaining clearly each step of your proof.

[*You may quote standard results about matrices without proof.*]

93-S2-Q10

10 Verify that if

$$\mathbf{P} = \begin{pmatrix} 1 & 2 \\ 2 & -1 \end{pmatrix} \qquad \text{and} \qquad \mathbf{A} = \begin{pmatrix} -1 & 8 \\ 8 & 11 \end{pmatrix}$$

then $\mathbf{PAP}$ is a diagonal matrix.

Put $\mathbf{x} = \begin{pmatrix} x \\ y \end{pmatrix}$ and $\mathbf{x}_1 = \begin{pmatrix} x_1 \\ y_1 \end{pmatrix}$. By writing

$$\mathbf{x} = \mathbf{P}\mathbf{x}_1 + \mathbf{a}$$

for a suitable vector $\mathbf{a}$, show that the equation

$$\mathbf{x}^{\mathrm{T}}\mathbf{A}\mathbf{x} + \mathbf{b}^{\mathrm{T}}\mathbf{x} - 11 = 0,$$

where $\mathbf{b} = \begin{pmatrix} 18 \\ 6 \end{pmatrix}$ and $\mathbf{x}^{\mathrm{T}}$ is the transpose of $\mathbf{x}$, becomes

$$3x_1^2 - y_1^2 = c$$

for some constant c (which you should find).

92-S2-Q5

5 Explain what is meant by the order of an element g of a group G.

The set S consists of all 2×2 matrices whose determinant is 1. Find the inverse of the element $\mathbf{A}$ of S, where

$$\mathbf{A} = \begin{pmatrix} w & x \\ y & x \end{pmatrix}.$$

Show that S is a group under matrix multiplication (you may assume that matrix multiplication is associative). For which elements $\mathbf{A}$ is $\mathbf{A}^{-1} = \mathbf{A}$? Which element or elements have order 2? Show that the element $\mathbf{A}$ of S has order 3 if, and only if, $w + z + 1 = 0$. Write down one such element.

91-S2-Q9

9 Let G be the set of all matrices of the form

$$\begin{pmatrix} a & b \\ 0 & c \end{pmatrix},$$

where a, b and c are integeres modulo 5, and $a \neq 0 \neq c$. Show that G forms a group under matrix multiplication (which may be assumed to be associative). What is the order of G? Determine whether or not G is commutative.

Determine whether or not the set consisting of all elements in G of order 1 or 2 is a subgroup of G.

88-S2-Q8

8 In a crude model of population dynamics of a community of aardvarks and buffaloes, it is assumed that, if the numbers of aardvarks and buffaloes in any year are A and B respectively, then the numbers in the following year at $\frac{1}{4}A + \frac{3}{4}B$ and $\frac{3}{2}B - \frac{1}{2}A$ respectively. It does not matter if the model predicts fractions of animals, but a non-positive number of buffaloes means that the species has become extinct, and the model ceases to apply. Using matrices or otherwise, show that the ratio of the number of aardvarks to the number of buffaloes can remain the same each year, provided it takes one of two possible values.

Let these two possible values be x and y, and let the numbers of aardvarks and buffaloes in a given year be a and b respectively. By writing the vector (a, b) as a linear combination of the vectors $(x, 1)$ and $(y, 1)$, or otherwise, show how the numbers of aardvarks and buffaloes in subsequent years may be found. On a sketch of the a-b plane, mark the regions which correspond to the following situations

(i) an equilibrium population is reached as time $t \to \infty$;

(ii) buffaloes become extinct after a finite time;

(iii) buffaloes approach extinction as $t \to \infty$.

93-S2-Q6

6 *In this question,* $\mathbf{A}, \mathbf{B}$ *and* $\mathbf{X}$ *are non-zero* 2×2 *real matrices.*

Are the following assertions true or false? You must provide a proof or a counterexample in each case.

(i) If $\mathbf{AB} = \mathbf{0}$ then $\mathbf{BA} = \mathbf{0}$.

(ii) $(\mathbf{A} - \mathbf{B})(\mathbf{A} + \mathbf{B}) = \mathbf{A}^2 - \mathbf{B}^2$.

(iii) The equation $\mathbf{AX} = \mathbf{0}$ has a non-zero solution $\mathbf{X}$ if and only if $\det \mathbf{A} = 0$.

(iv) For any $\mathbf{A}$ and $\mathbf{B}$ there are at most two matrices $\mathbf{X}$ such that $\mathbf{X}^2 + \mathbf{AX} + \mathbf{B} = \mathbf{0}$.

89-S2-Q9

9 The matrix $\mathbf{F}$ is defined by

$$\mathbf{F} = \mathbf{I} + \sum_{n=1}^{\infty} \frac{1}{n!} t^n \mathbf{A}^n,$$

where $\mathbf{A} = \begin{pmatrix} -3 & -1 \\ 8 & 3 \end{pmatrix}$, and t is a variable scalar. Evaluate $\mathbf{A}^2$, and show that

$$\mathbf{F} = \mathbf{I} \cosh t + \mathbf{A} \sinh t.$$

Show also that $\mathbf{F}^{-1} = \mathbf{I} \cosh t - \mathbf{A} \sinh t$, and that $\dfrac{\mathrm{d}\mathbf{F}}{\mathrm{d}t} = \mathbf{FA}$.

The vector $\mathbf{r} = \begin{pmatrix} x(t) \\ y(t) \end{pmatrix}$ satisfies the differential equation

$$\frac{\mathrm{d}\mathbf{r}}{\mathrm{d}t} + \mathbf{Ar} = \mathbf{0},$$

with $x = \alpha$ and $y = \beta$ at $t = 0$. Solve this equation by means of a suitable matrix integrating factor, and hence show that

$$x(t) = \alpha \cosh t + (3\alpha + \beta) \sinh t$$
$$y(t) = \beta \cosh t - (8\alpha + 3\beta) \sinh t.$$

STEP3 MATRICES QUESTIONS

97-S3-Q8

8 Let R_α be the 2×2 matrix that represents a rotation through the angle α and let

$$A = \begin{pmatrix} a & b \\ b & c \end{pmatrix}.$$

(i) Find in terms of a, b and c an angle α such that $R_{-\alpha}AR_\alpha$ is a diagonal matrix (i.e. has the value zero in top-right and bottom-left positions).

(ii) Find values of a, b and c such that the equation of the ellipse

$$x^2 + (y + 2x \cot 2\theta)^2 = 1 \qquad (0 < \theta < \tfrac{1}{4}\pi)$$

can be expressed in the form

$$(x \quad y)\, A \begin{pmatrix} x \\ y \end{pmatrix} = 1.$$

Show that, for this A, $R_{-\alpha}AR_\alpha$ is diagonal if $\alpha = \theta$. Express the non–zero elements of this matrix in terms of θ.

(iii) Deduce, or show otherwise, that the minimum and maximum distances from the centre to the circumference of this ellipse are $\tan\theta$ and $\cot\theta$.

92-S3-Q2

2 The matrices $\mathbf{I}$ and $\mathbf{J}$ are

$$\mathbf{I} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \quad \text{and} \quad \mathbf{J} = \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}$$

respectively and $\mathbf{A} = \mathbf{I} + a\mathbf{J}$, where a is a non-zero real constant. Prove that

$$\mathbf{A}^2 = \mathbf{I} + \tfrac{1}{2}[(1+2a)^2 - 1]\mathbf{J} \quad \text{and} \quad \mathbf{A}^3 = \mathbf{I} + \tfrac{1}{2}[(1+2a)^3 - 1]\mathbf{J}$$

and obtain a similar form for $\mathbf{A}^4$.

If $\mathbf{A}^k = \mathbf{I} + p_k\mathbf{J}$, suggest a suitable form for p_k and prove that it is correct by induction, or otherwise.

98-S3-Q5

5 The exponential of a square matrix $\mathbf{A}$ is defined to be

$$\exp(\mathbf{A}) = \sum_{r=0}^{\infty} \frac{1}{r!}\mathbf{A}^r ,$$

where $\mathbf{A}^0 = \mathbf{I}$ and $\mathbf{I}$ is the identity matrix.

Let

$$\mathbf{M} = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} .$$

Show that $\mathbf{M}^2 = -\mathbf{I}$ and hence express $\exp(\theta\mathbf{M})$ as a single 2×2 matrix, where θ is a real number. Explain the geometrical significance of $\exp(\theta\mathbf{M})$.

Let

$$\mathbf{N} = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} .$$

Express similarly $\exp(s\mathbf{N})$, where s is a real number, and explain the geometrical significance of $\exp(s\mathbf{N})$.

For which values of θ does

$$\exp(s\mathbf{N})\ \exp(\theta\mathbf{M}) = \exp(\theta\mathbf{M})\ \exp(s\mathbf{N})$$

for all s? Interpret this fact geometrically.

93-S3-Q10

10 The transformation T of the point P in the x,y plane to the point P' is constructed as follows: Lines are drawn through P parallel to the lines $y = mx$ and $y = -mx$ to cut the line $y = kx$ at Q and R respectively, m and k being given constants. P' is the fourth vertex of the parallelogram $PQP'R$.

Show that if P is (x_1, y_1) then Q is

$$\left(\frac{mx_1 - y_1}{m - k}, \frac{k(mx_1 - y_1)}{m - k} \right) .$$

Obtain the coordinates of P' in terms of x_1, y_1, m and k, and express T as a matrix transformation. Show that areas are transformed under T into areas of the same magnitude.

96-S3-Q6

6 **(i)** Let S be the set of matrices of the form

$$\begin{pmatrix} a & a \\ a & a \end{pmatrix},$$

where a is any real non-zero number. Show that S is closed under matrix multiplication and, further, that S is a group under matrix multiplication.

(ii) Let G be a set of $n \times n$ matrices which is a group under matrix multiplication, with identity element $\mathbf{E}$. By considering equations of the form $\mathbf{BC} = \mathbf{D}$ for suitable elements $\mathbf{B}$, $\mathbf{C}$ and $\mathbf{D}$ of G, show that if a given element $\mathbf{A}$ of G is a singular matrix (i.e. $\det \mathbf{A} = 0$), then all elements of G are singular. Give, with justification, an example of such a group of singular matrices in the case $n = 3$.

Spec-S3-Q9

9 Prove that the set of all matrices of the form

$$\begin{pmatrix} 1 & 0 & 0 \\ x & 1 & 0 \\ y & z & 1 \end{pmatrix},$$

where x, y and z are real numbers, is a group G under matrix multiplication. (You may assume that matrix multiplication is associative.)

Does the subset consisting of these matrices where x, y, z are restricted to the integers form a subgroup of G? Is there an element $\mathbf{A}$ in G, with $\mathbf{A}$ not equal to the identity matrix, such that $\mathbf{AB} = \mathbf{BA}$ for all $\mathbf{B}$ belonging to G? Justify your answer.

91-S3-Q2

2 The distinct points P_1, P_2, P_3, Q_1, Q_2 and Q_3 in the Argand diagram are represented by the complex numbers z_1, z_2, z_3, w_1, w_2 and w_3 respectively. Show that the triangles $P_1P_2P_3$ and $Q_1Q_2Q_3$ are similar, with P_i corresponding to Q_i $(i = 1, 2, 3)$ and the rotation from 1 to 2 to 3 being in the same sense for both triangles, if and only if

$$\frac{z_1 - z_2}{z_2 - z_3} = \frac{w_1 - w_2}{w_1 - w_3}.$$

Verify that this condition may be written

$$\det \begin{pmatrix} z_1 & z_2 & z_3 \\ w_1 & w_2 & w_3 \\ 1 & 1 & 1 \end{pmatrix} = 0.$$

(i) Show that if $w_i = z_i^2$ $(i = 1, 2, 3)$ then triangle $P_1P_2P_3$ is not similar to triangle $Q_1Q_2Q_3$.

(ii) Show that if $w_i = z_i^3$ $(i = 1, 2, 3)$ then triangle $P_1P_2P_3$ is similar to triangle $Q_1Q_2Q_3$ if and only if the centroid of triangle $P_1P_2P_3$ is the origin. [The *centroid* of triangle $P_1P_2P_3$ is represented by the complex number $\frac{1}{3}(z_1 + z_2 + z_3)$.]

(iii) Show that the triangle $P_1P_2P_3$ is equilateral if and only if

$$z_2z_3 + z_3z_1 + z_1z_2 = z_1^2 + z_2^2 + z_3^2.$$

89-S3-Q7

7 The linear transformation T is a shear which transforms a point P to the point P' defined by

(i) $\overrightarrow{PP'}$ makes an acute angle α (anticlockwise) with the x-axis,

(ii) $\angle POP'$ is clockwise (i.e. the rotation from OP to OP' clockwise is less than π),

(iii) $PP' = k \times PN$, where PN is the perpendicular onto the line $y = x \tan \alpha$, where k is a given non-zero constant.

If T is represented in matrix form by $\begin{pmatrix} x' \\ y' \end{pmatrix} = \mathbf{M} \begin{pmatrix} x \\ y \end{pmatrix}$, show that

$$\mathbf{M} = \begin{pmatrix} 1 - k \sin \alpha \cos \alpha & k \cos^2 \alpha \\ -k \sin^2 \alpha & 1 + k \sin \alpha \cos \alpha \end{pmatrix}.$$

Show that the necessary and sufficient condition for $\begin{pmatrix} p & q \\ r & t \end{pmatrix}$ to commute with $\mathbf{M}$ is

$$t - p = 2q \tan \alpha = -2r \cot \alpha.$$

94-S3-Q8

8 Let a, b, c, d, p, q, r and s be real numbers. By considering the determinant of the matrix product

$$\begin{pmatrix} z_1 & z_2 \\ -z_2^* & z_1^* \end{pmatrix} \begin{pmatrix} z_3 & z_4 \\ -z_4^* & z_3^* \end{pmatrix},$$

where z_1, z_2, z_3 and z_4 are suitably chosen complex numbers, find expressions L_1, L_2, L_3 and L_4, each of which is linear in a, b, c and d and also linear in p, q, r and s, such that

$$(a^2 + b^2 + c^2 + d^2)(p^2 + q^2 + r^2 + s^2) = L_1^2 + L_2^2 + L_3^2 + L_4^2.$$

89-S3-Q3

3 The matrix $\mathbf{M}$ is given by

$$\mathbf{M} = \begin{pmatrix} \cos(2\pi/m) & -\sin(2\pi/m) \\ \sin(2\pi/m) & \cos(2\pi/m) \end{pmatrix},$$

where m is an integer greater than 1. Prove that

$$\mathbf{M}^{m-1} + \mathbf{M}^{m-2} + \cdots + \mathbf{M}^2 + \mathbf{M} + \mathbf{I} = \mathbf{O},$$

where $\mathbf{I} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$ and $\mathbf{O} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$.

The sequence $\mathbf{X}_0, \mathbf{X}_1, \mathbf{X}_2, \ldots$ is defined by

$$\mathbf{X}_{k+1} = \mathbf{P}\mathbf{X}_k + \mathbf{Q},$$

where $\mathbf{P}, \mathbf{Q}$ and $\mathbf{X}_0$ are given 2×2 matrices. Suggest a suitable expression for $\mathbf{X}_k$ in terms of $\mathbf{P}$, $\mathbf{Q}$ and $\mathbf{X}_0$, and justify it by induction.

The binary operation $*$ is defined as follows:

$\mathbf{X}_i * \mathbf{X}_j$ is the result of substituting $\mathbf{X}_j$for $\mathbf{X}_0$in the expression for $\mathbf{X}_i$.

Show that if $\mathbf{P} = \mathbf{M}$, the set $\{\mathbf{X}_1, \mathbf{X}_2, \mathbf{X}_3, \ldots\}$ forms a finite group under the operation $*$.

93-S3-Q5

5 The set S consists of ordered pairs of complex numbers (z_1, z_2) and a binary operation $\circ$ on S is defined by

$$(z_1, z_2) \circ (w_1, w_2) = (z_1 w_1 - z_2 w_2^*,\ z_1 w_2 + z_2 w_1^*).$$

Show that the operation $\circ$ is associative and determine whether it is commutative. Evaluate $(z, 0) \circ (w, 0)$, $(z, 0) \circ (0, w)$, $(0, z) \circ (w, 0)$ and $(0, z) \circ (0, w)$.

The set S_1 is the subset of S consisting of A, B, $\ldots$, H, where $A = (1, 0)$, $B = (0, 1)$, $C = (i, 0)$, $D = (0, i)$, $E = (-1, 0)$, $F = (0, -1)$, $G = (-i, 0)$ and $H = (0, -i)$. Show that S_1 is closed under $\circ$ and that it has an identity element. Determine the inverse and order of each element of S_1. Show that S_1 is a group under $\circ$.
[You are not required to compute the multiplication table in full.]

Show that $\{A, B, E, F\}$ is a subgroup of S_1 and determine whether it is isomorphic to the group generated by the 2×2 matrix $\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$ under matrix multiplication.

PROOF

Spec-S3-Q1

1 **(i)** Guess an expression for

$$\left(1-\frac{1}{4}\right)\left(1-\frac{1}{9}\right)\cdots\left(1-\frac{1}{n^2}\right),$$

valud for $n \geqslant 2$, and prove by mathematical induction the correctness of your guess.

(ii) Show that, if n is a positive integer,

$$\sum_{r=0}^{k}(-1)^r\binom{n}{r} = (-1)^k\binom{n-1}{k}, \qquad \text{for } k = 0, 1, \dots, n-1.$$

Spec-S3-Q2

2 Show by using the binomial expansion, or otherwise, that $(1+x)^n > nx$ whenever $x \geqslant 0$ and n is a positive integer. Deduce that if $y > 1$ then, given any number K, an integer N can be found such that $y^n > K$ for all integers $n \geqslant N$.

Show further that if $y > 1$ then, given any K, an integer N can be found such that $\dfrac{y^n}{n} > K$ for all integers $n \geqslant N$.

Spec-S3-Q10

10 **(i)** Show that every odd square leaves remainder 1 when divided by 8, and that every even square leaves remainder 0 or 4. Deduce that a number of the form $8n + 7$, where n is a positive integer, cannot be expressed as a sum of three squares.

(ii) Prove that 17 divides $2^{3n+1} + 3(5^{2n+1})$ for all integers $n \geqslant 0$.

87-S3-Q1

1 Find the set of positive integers n for which n does not divide $(n-1)!$. Justify your answer. [Note that small values of n may require special consideration.]

90-S3-Q5

5 Prove that, for any integers n and r, with $1 \leqslant r \leqslant n$,

$$\binom{n}{r} + \binom{n}{r-1} = \binom{n+1}{r}.$$

Hence or otherwise, prove that

$$(uv)^{(n)} = u^{(n)}v + \binom{n}{1}u^{(n-1)}v^{(1)} + \binom{n}{2}u^{(n-2)}v^{(2)} + \cdots + uv^{(n)},$$

where u and v are functions of x and $z^{(r)}$ means $\dfrac{\mathrm{d}^r z}{\mathrm{d}x^r}$.

Prove that, if $y = \sin^{-1} y$, then $(1-x^2)y^{(n+2)} - (2n+1)xy^{(n+1)} - n^2y^{(n)} = 0$.

94-S2-Q1

1 In this question we consider only positive, non-zero integers written out in the usual (decimal) way. We say, for example, that 207 ends in 7 and that 5310 ends in 1 followed by 0. Show that, if n does not end in 5 or an even number, then there exists m such that $n \times m$ ends in 1.

Show that, given any n, we can find m such that $n \times m$ ends either in 1 or in 1 followed by one or more zeros.

Show that, given any n which ends in 1 or in 1 followed by one or more zeros, we can find m such that $n \times m$ contains all the digits $0, 1, 2, \dots, 9$.

94-S2-Q6

6 Prove by induction, or otherwise, that, if $0 < \theta < \pi$,

$$\frac{1}{2}\tan\frac{\theta}{2} + \frac{1}{2^2}\tan\frac{\theta}{2^2} + \cdots + \frac{1}{2^n}\tan\frac{\theta}{2^n} = \frac{1}{2^n}\cot\frac{\theta}{2^n} - \cot\theta.$$

Deduce that

$$\sum_{r=1}^{\infty}\frac{1}{2^r}\tan\frac{\theta}{2^r} = \frac{1}{\theta} - \cot\theta.$$

96-S2-Q6

6 A *proper factor* of a positive integer N is an integer M, with $M \neq 1$ and $M \neq N$, which divides N without remainder. Show that 12 has 4 proper factors and 16 has 3.

Suppose that N has the prime factorisation

$$N = p_1^{m_1}p_2^{m_2}\dots p_r^{m_r},$$

where $p_1, p_2, \dots, p_r$ are distinct primes and $m_1, m_2, \dots, m_r$ are positive integers. How many proper factors does N have and why?

Find:

(i) the smallest positive integer which has precisely 12 proper factors;

(ii) the smallest positive integer which has at least 12 proper factors.

96-S3-Q4

4 Find the integers k satisfying the inequality $k \leqslant 2(k-2)$.

Given that N is a strictly positive integer consider the problem of finding strictly positive integers whose sum is N and whose product is as large as possible. Call this largest possible product $P(N)$. Show that $P(5) = 2\times 3$, $P(6) = 3^2$, $P(7) = 2^2\times 3$, $P(8) = 2\times 3^2$ and $P(9) = 3^3$.

Find $P(1000)$ explaining your reasoning carefully.

97-S2-Q1

1 Find the sum of those numbers between 1000 and 6000 every one of whose digits is one of the numbers 0, 2, 5 or 7, giving your answer as a product of primes.

97-S2-Q2

2 Suppose that

$$3 = \frac{2}{x_1} = x_1 + \frac{2}{x_2} = x_2 + \frac{2}{x_3} = x_3 + \frac{2}{x_4} = \cdots.$$

Guess an expression, in terms of n, for x_n. Then, by induction or otherwise, prove the correctness of your guess.

98-S1-Q3

3 Which of the following statements are true and which are false? Justify your answers.

(i) $a^{\ln b} = b^{\ln a}$ for all $a, b > 0$.

(ii) $\cos(\sin\theta) = \sin(\cos\theta)$ for all real θ.

(iii) There exists a polynomial P such that $|\mathrm{P}(\theta) - \cos\theta| \leqslant 10^{-6}$ for all real θ.

(iv) $x^4 + 3 + x^{-4} \geqslant 5$ for all $x > 0$.

99-S1-Q1

1 How many integers greater than or equal to zero and less than a million are not divisible by 2 or 5? What is the average value of these integers?

How many integers greater than or equal to zero and less than 4179 are not divisible by 3 or 7? What is the average value of these integers?

04-S1-Q5

5 The positive integers can be split into five distinct arithmetic progressions, as shown:

$$\begin{aligned} A&: \ 1,\ 6,\ 11,\ 16,\ \ldots \\ B&: \ 2,\ 7,\ 12,\ 17,\ \ldots \\ C&: \ 3,\ 8,\ 13,\ 18,\ \ldots \\ D&: \ 4,\ 9,\ 14,\ 19,\ \ldots \\ E&: \ 5, 10,\ 15,\ 20,\ \ldots \end{aligned}$$

Write down an expression for the value of the general term in each of the five progressions. Hence prove that the sum of any term in B and any term in C is a term in E.

Prove also that the square of every term in B is a term in D. State and prove a similar claim about the square of every term in C.

(i) Prove that there are no positive integers x and y such that

$$x^2 + 5y = 243\,723\,.$$

(ii) Prove also that there are no positive integers x and y such that

$$x^4 + 2y^4 = 26\,081\,974\,.$$

06-S1-Q1

1 Find the integer, n, that satisfies $n^2 < 33\,127 < (n+1)^2$. Find also a small integer m such that $(n+m)^2 - 33\,127$ is a perfect square. Hence express $33\,127$ in the form pq, where p and q are integers greater than 1.

By considering the possible factorisations of $33\,127$, show that there are exactly two values of m for which $(n+m)^2 - 33\,127$ is a perfect square, and find the other value.

07-S1-Q1

1 A positive integer with $2n$ digits (the first of which must not be 0) is called a *balanced number* if the sum of the first n digits equals the sum of the last n digits. For example, 1634 is a 4-digit balanced number, but 123401 is not a balanced number.

(i) Show that seventy 4-digit balanced numbers can be made using the digits $0, 1, 2, 3$ and 4.

(ii) Show that $\frac{1}{6}k\,(k+1)\,(4k+5)$ 4-digit balanced numbers can be made using the digits 0 to k.

You may use the identity $\displaystyle\sum_{r=0}^{n} r^2 \equiv \tfrac{1}{6}n\,(n+1)\,(2n+1)\,.$

08-S1-Q1

1 What does it mean to say that a number x is *irrational*?

Prove by contradiction statements A and B below, where p and q are real numbers.

A: If pq is irrational, then at least one of p and q is irrational.

B: If $p+q$ is irrational, then at least one of p and q is irrational.

Disprove by means of a counterexample statement C below, where p and q are real numbers.

C: If p and q are irrational, then $p+q$ is irrational.

If the numbers e, π, π^2, e^2 and $\mathrm{e}\pi$ are irrational, prove that at most one of the numbers $\pi+\mathrm{e}$, $\pi-\mathrm{e}$, $\pi^2-\mathrm{e}^2$, $\pi^2+\mathrm{e}^2$ is rational.

10-S1-Q8

8 **(i)** Suppose that a, b and c are integers that satisfy the equation

$$a^3+3b^3=9c^3.$$

Explain why a must be divisible by 3, and show further that both b and c must also be divisible by 3. Hence show that the only integer solution is $a=b=c=0$.

(ii) Suppose that p, q and r are integers that satisfy the equation

$$p^4+2q^4=5r^4.$$

By considering the possible final digit of each term, or otherwise, show that p and q are divisible by 5. Hence show that the only integer solution is $p=q=r=0$.

11-S1-Q8

8 **(i)** The numbers m and n satisfy

$$m^3=n^3+n^2+1. \qquad (*)$$

(a) Show that $m>n$. Show also that $m<n+1$ if and only if $2n^2+3n>0$. Deduce that $n<m<n+1$ unless $-\frac{3}{2}\leqslant n\leqslant 0$.

(b) Hence show that the only solutions of $(*)$ for which both m and n are integers are $(m,n)=(1,0)$ and $(m,n)=(1,-1)$.

(ii) Find all integer solutions of the equation

$$p^3=q^3+2q^2-1.$$

14-S1-Q1

1 *All numbers referred to in this question are non-negative integers.*

(i) Express each of the numbers 3, 5, 8, 12 and 16 as the difference of two non-zero squares.

(ii) Prove that any odd number can be written as the difference of two squares.

(iii) Prove that all numbers of the form $4k$, where k is a non-negative integer, can be written as the difference of two squares.

(iv) Prove that no number of the form $4k + 2$, where k is a non-negative integer, can be written as the difference of two squares.

(v) Prove that any number of the form pq, where p and q are prime numbers greater than 2, can be written as the difference of two squares in exactly two distinct ways. Does this result hold if p is a prime greater than 2 and $q = 2$?

(vi) Determine the number of distinct ways in which 675 can be written as the difference of two squares.

15-S1-Q8

8 Show that:

(i) $1 + 2 + 3 + \cdots + n = \frac{1}{2}n(n + 1)$;

(ii) if N is a positive integer, m is a non-negative integer and k is a positive odd integer, then $(N - m)^k + m^k$ is divisible by N.

Let $S = 1^k + 2^k + 3^k + \cdots + n^k$, where k is a positive odd integer. Show that if n is odd then S is divisible by n and that if n is even then S is divisible by $\frac{1}{2}n$.

Show further that S is divisible by $1 + 2 + 3 + \cdots + n$.

GRAPHING

Spec-S2-Q4

4 **(i)** Show that

$$\sin^{-1}(\tanh x) = \tan^{-1}(\sinh x),$$

when principal values only are considered.

(ii) Show that

$$\sinh^{-1}(\tan y) = \tanh^{-1}(\sin y),$$

when $-\frac{1}{2}\pi < y < \frac{1}{2}\pi$.

Sketch the graphs of $\sinh^{-1}(\tan y)$ and $\tanh^{-1}(\sin y)$ in the interval $-\pi < y < \pi$ and find the relationship between the two expressions when $\frac{1}{2}\pi < y < \pi$.

88-S2-Q6

6 Show that the following functions are positive when x is positive:

(i) $x - \tanh x$

(ii) $x \sinh x - 2\cosh x + 2$

(iii) $2x\cosh 2x - 3\sinh 2x + 4x$.

The function f is defined for $x > 0$ by

$$\mathrm{f}(x) = \frac{x(\cosh x)^{\frac{1}{3}}}{\sinh x}.$$

Show that $\mathrm{f}(x)$ has no turning points when $x > 0$, and sketch $\mathrm{f}(x)$ for $x > 0$.

88-S3-Q1

1 Sketch the graph of

$$y = \frac{x^2 \mathrm{e}^{-x}}{1+x},$$

for $-\infty < x < \infty$.

Show that the value of

$$\int_0^\infty \frac{x^2\mathrm{e}^{-x}}{1+x}\,\mathrm{d}x$$

lies between 0 and 1.

89-S2-Q4

4 The function f is defined by

$$\mathrm{f}(x) = \frac{(x-a)\,(x-b)}{(x-c)\,(x-d)} \qquad (x \neq c,\ x \neq d)\,,$$

where a, b, c and d are real and distinct, and $a + d \neq c + d$. Show that

$$\frac{x\mathrm{f}'(x)}{\mathrm{f}(x)} = \left(1 - \frac{a}{x}\right)^{-1} + \left(1 - \frac{b}{x}\right)^{-1} - \left(1 - \frac{c}{x}\right)^{-1} - \left(1 - \frac{d}{x}\right)^{-1},$$

$(x \neq 0, x \neq a, x \neq b)$ and deduce that when $|x|$ is much larger than each of $|a|\,, |b|\,, |c|$ and $|d|$, the gradient of $\mathrm{f}(x)$ has the same sign as $(a + b - c - d)$.

It is given that there is a real value of real value of x for which $\mathrm{f}(x)$ takes the real value z if and only if

$$\left[(c-d)^2\, z + (a-c)\,(b-d) + (a-d)\,(b-c)\right]^2 \geqslant 4\,(a-c)\,(b-d)\,(a-d)\,(b-c)\,.$$

Describe briefly a method by which this result could be proved, but do not attempt to prove it.

Given that $a < b$ and $a < c < d$, make sketches of the graph of f in the four distinct cases which arise, indicating the cases for which the range of f is not the whole of $\mathbb{R}$.

89-S3-Q4

4 Sketch the curve whose cartesian equation is

$$y = \frac{2x(x^2-5)}{x^2-4},$$

and give the equations of the asymptotes and of the tangent to the curve at the origin.

Hence, or otherwise, determine (giving reasons) the number of real roots of the following equations:

(i) $4x^2(x^2-5) = (5x-2)(x^2-4)$;

(ii) $4x^2(x^2-5)^2 = (x^2-4)^2(x^2+1)$;

(iii) $4z^2(z-5)^2 = (z-4)^2(z+1)$.

90-S2-Q3

3 Sketch the curves given by

$$y = x^3 - 2bx^2 + c^2x,$$

where b and c are non-negative, in the cases:

$$\textbf{(i)}\ \ 2b < c\sqrt{3}, \quad \textbf{(ii)}\ \ 2b = c\sqrt{3} \neq 0, \quad \textbf{(iii)}\ \ c\sqrt{3} < 2b < 2c, \quad \textbf{(iv)}\ \ b = c \neq 0,$$
$$\textbf{(v)}\ \ b > c > 0, \quad \textbf{(vi)}\ \ c = 0, b \neq 0, \quad \textbf{(vii)}\ \ c = b = 0.$$

Sketch also the curves given by $y^2 = x^3 - 2bx^2 + c^2x$ in the cases **(i)**, **(v)** and **(vii)**.

90-S3-Q7

7 The points $P\,(0,a)$, $Q\,(a,0)$ and $R\,(a,-a)$ lie on the curve C with cartesian equation

$$xy^2 + x^3 + a^2y - a^3 = 0, \qquad \text{where } a > 0.$$

At each of P, Q and R, express y as a Taylor series in h, where h is a small increment in x, as far as the term in h^2. Hence, or otherwise, sketch the shape of C near each of these points.

Show that, if (x,y) lies on C, then

$$4x^4 - 4a^3x - a^4 \leqslant 0.$$

Sketch the graph of $y = 4x^4 - 4a^3 - a^4$.

Given that the y-axis is an asymptote to C, sketch the curve C.

92-S2-Q6

6 Sketch the graphs of $y = \sec x$ and $y = \ln(2\sec x)$ for $0 \leqslant x \leqslant \frac{1}{2}\pi$. Show graphically that the equation

$$kx = \ln(2\sec x)$$

has no solution with $0 \leqslant x < \frac{1}{2}\pi$ if k is a small positive number but two solutions if k is large. Explain why there is a number k_0 such that

$$k_0 x = \ln(2\sec x)$$

has exactly one solution with $0 \leqslant x < \frac{1}{2}\pi$. Let x_0 be this solution, so that $0 \leqslant x_0 < \frac{1}{2}\pi$ and $k_0 x_0 = \ln(2\sec x)$. Show that

$$x_0 = \cot x_0 \ln(2\sec x_0).$$

Use any appropriate method to find x_0 correct to two decimal places. Hence find an approximate value for k_0.

92-S3-Q4

4 A set of curves S_1 is defined by the equation

$$y = \frac{x}{x-a},$$

where a is a constant which is different for different members of S_1. Sketch on the same axes the curves for which $a = -2, -1, 1$ and 2.

A second of curves S_2 is such that at each intersection between a member of S_2 and a member of S_1 the tangents of the intersecting curves are perpendicular. On the same axes as the already sketched members of S_1, sketch the member of S_2 that passes through the point $(1, -1)$.

Obtain the first order differential equation for y satisfied at all points on all members of S_1 (i.e. an equation connecting x, y and $\mathrm{d}y/\mathrm{d}x$ which does not involve a).

State the relationship between the values of $\mathrm{d}y/\mathrm{d}x$ on two intersecting curves, one from S_1 and one from S_2, at their intersection. Hence show that the differential equation for the curves of S_2 is

$$x = y(y-1)\frac{\mathrm{d}y}{\mathrm{d}x}.$$

Find an equation for the member of S_2 that you have sketched.

97-S3-Q2

2 Let

$$\mathrm{f}(t) = \frac{\ln t}{t} \quad \text{for } t > 0.$$

Sketch the graph of $\mathrm{f}(t)$ and find its maximum value. How many positive values of t correspond to a given value of $\mathrm{f}(t)$?

Find how many positive values of y satisfy $x^y = y^x$ for a given positive value of x. Sketch the set of points (x, y) which satisfy $x^y = y^x$ with $x, y > 0$.

99-S1-Q4

4 Sketch the following subsets of the x-y plane:

(i) $|x| + |y| \leqslant 1$;

(ii) $|x-1| + |y-1| \leqslant 1$;

(iii) $|x-1| - |y+1| \leqslant 1$;

(iv) $|x|\,|y-2| \leqslant 1$.

00-S1-Q6

6 Show that

$$x^2 - y^2 + x + 3y - 2 = (x - y + 2)(x + y - 1)$$

and hence, or otherwise, indicate by means of a sketch the region of the x-y plane for which

$$x^2 - y^2 + x + 3y > 2.$$

Sketch also the region of the x-y plane for which

$$x^2 - 4y^2 + 3x - 2y < -2.$$

Give the coordinates of a point for which both inequalities are satisfied or explain why no such point exists.

13-S1-Q5

5 The point P has coordinates (x, y) which satisfy

$$x^2 + y^2 + kxy + 3x + y = 0\,.$$

(i) Sketch the locus of P in the case $k = 0$, giving the points of intersection with the coordinate axes.

(ii) By factorising $3x^2 + 3y^2 + 10xy$, or otherwise, sketch the locus of P in the case $k = \frac{10}{3}$, giving the points of intersection with the coordinate axes.

(iii) In the case $k = 2$, let Q be the point obtained by rotating P clockwise about the origin by an angle θ, so that the coordinates (X, Y) of Q are given by

$$X = x\cos\theta + y\sin\theta\,, \quad Y = -x\sin\theta + y\cos\theta\,.$$

Show that, for $\theta = 45°$, the locus of Q is $\sqrt{2}Y = (\sqrt{2}X + 1)^2 - 1$.

Hence, or otherwise, sketch the locus of P in the case $k = 2$, giving the equation of the line of symmetry.

15-S1-Q1

1 **(i)** Sketch the curve $y = e^x(2x^2 - 5x + 2)$.

Hence determine how many real values of x satisfy the equation $e^x(2x^2 - 5x + 2) = k$ in the different cases that arise according to the value of k.

You may assume that $x^n e^x \to 0$ as $x \to -\infty$ for any integer n.

(ii) Sketch the curve $y = e^{x^2}(2x^4 - 5x^2 + 2)$.

15-S1-Q5

5 **(i)** The function f is defined, for $x > 0$, by

$$f(x) = \int_1^3 (t-1)^{x-1}\,dt\,.$$

By evaluating the integral, sketch the curve $y = f(x)$.

(ii) The function g is defined, for $-\infty < x < \infty$, by

$$g(x) = \int_{-1}^{1} \frac{1}{\sqrt{1 - 2xt + x^2}}\,dt\,.$$

By evaluating the integral, sketch the curve $y = g(x)$.