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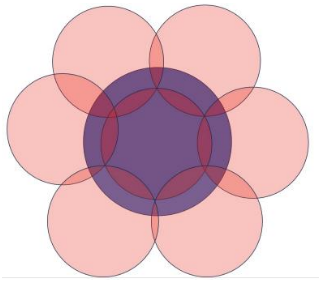
Mathematics changes our lives

PREFACE

This month, we are going to talk about the following questions.

Q1

Consider a disk D of radius 1. We say a collection of shapes hides D if we can place the shapes from that collection on top of the disk in such a way that each point of the disk is under some part of at least one of the shapes. For example, D can be hidden by a single disk of radius greater-than-or-equal-to 1. However, D cannot be hidden by a single disk of smaller radius than 1. As another example, D can be hidden by 7 disks of radius 3/4 by placing one radius 3/4 disk at the center and the remaining radius 3/4 disks around the edge of D as follows:



(a) Find the smallest radius r so that two disks of radius r hide D (and any disk of radius less than r cannot hide it).

(b) What is the smallest radius r you can find such that four disks of radius r can hide D?

(c) What is the smallest radius r that you can find such that five disks of radius r can hide D? You don't need to prove this radius is optimal; you only need to show that you found a solution with your radius r. With the radius you found, can you hide D in such a way that the centers of the disks are not vertices of a regular pentagon?

Q2

Consider a game where you have five distinct coins in a row, and you are allowed to move them in pairs and insert them between other coins in the row (without changing the order of the pair). For example, here is an allowable move:

As above you can slide the coins over horizontally to create space to insert the pair, or to remove extra space, as long as you do not change the order of the coins in the row.

(a) Suppose the coins are placed in order of increasing value; is there a sequence of moves as described above that reverses the order (so they are in order of decreasing value)?

(b) Again, suppose the coins are placed in order of increasing value; is there a sequence of

moves that switches the first coin with the last, and where the middle three end up in their starting positions?

This is the admission question from 2024 SUMaC Program. If you have other brilliant ideas, email to anmiciuangray@163.com **for surprising rewards!**

1.(a)

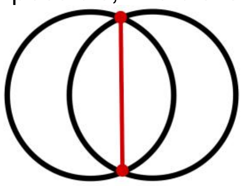
Some auxiliary results

Theory 5.1 Two circles are placed together, then they are possibly intersecting, tangent, or apart.

Theory 5.2 The length of any chord of a circle is not greater than the length of its diameter.

Solution

Understandably, in these situation, if possible, two disks can only be intersecting.

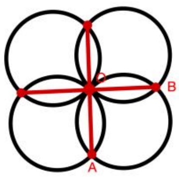


We merely pay attention to the the lengths of lines in direction of the red line.
We notice that this red line is a chord. According to theory 5.2, its length cannot be greater than the length of diameter.
If the radius of disks is r, which is less than 1, then the length of the red line, the chord, is less than or equal to 2r, which is less than 2.
But the width of D in the direction of the red line is 2, meaning no matter how two disks are put, they fail to hide D.
So, the minimum r must be 1. At that time, two disks can just totally overlap and hide D easily.

1.(b)

Solution

Understandably, in these situation, we'll put four disks as below.
Because single chord of the disk is always not longer than the diameter, based on Theory 5.2, so we want to connect one chord with another chord in order to maximize the width in the direction of the chord



we can find that the parallel red long line and the vertical red long line are the shortest lines among all the lines passing through the center(O), meaning that their length are both 2, the diameter of D.
So

$$AO = AB = 1.$$

Since $\angle AOB = 90°$,

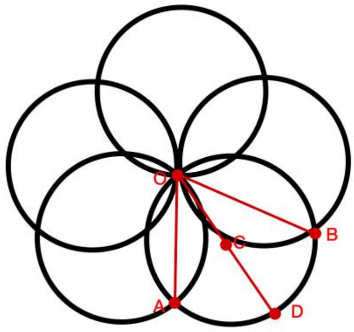
$$AB = \sqrt{2},$$

$$r = \frac{\sqrt{2}}{2}.$$

1.(c)

Solution

Understandably, the most obvious way to achieve this is to put dishes according to the graph.



Here, five disks intersect at one common point O, C is the center of one disk, OD passes through C.
So, OA = 1 and ∠AOB = 72°, so ∠AOD = 36°.
Since OD is the diameter of the disk, so ∠OAD = 90°.
Now, we just need to know the value of cos36° in order to find the value of r.
Since

$$\cos 36° = \sin 54°.$$
$$\cos 36° = \sin(36° + 18°).$$
$$1 - 2\sin^2 18° = \sin 36° \cos 18° + \cos 36° \sin 18°.$$
$$1 - 2\sin^2 18° = 2\sin 18° \cos^2 18° + \sin 18° - 2\sin^3 18°.$$
$$1 - 2\sin^2 18° = 2\sin 18° - 2\sin^3 18° + \sin 18° - 2\sin^3 18°.$$
$$4\sin^3 18° - 2\sin^2 18° - 3\sin 18° + 1 = 0.$$
$$(\sin 18° - 1)(4\sin^2 18° + 2\sin 18° - 1) = 0.$$
$$4\sin^2 18° + 2\sin 18° - 1 = 0.$$
$$\sin 18°_1 = \frac{-\sqrt{5} - 1}{4} \text{ (ignore).}$$
$$\sin 18°_2 = \frac{\sqrt{5} - 1}{4}.$$
$$\cos 36° = 1 - 2\sin^2 18° = \frac{\sqrt{5} + 1}{4}.$$

So

$$OD = \frac{OA}{\cos 36°} = \frac{4}{\sqrt{5} + 1}.$$
$$r = \frac{2}{\sqrt{5} + 1}.$$

I have to say, I don’t think that these five disks can be put in another way.
Obviously, none disk can be put further from the center, otherwise they will fail to hide the center of D.
Besides, none disk can be rotated with the center of O, otherwise there must be one chord with length less than 1.
All in all, I think this is the only way using 5 disks with the radius of $\frac{2}{\sqrt{5} + 1}$ to hide D.

2.(a)

Introduction

We have five coins, namely ①②③④⑤.
We want to change their order from ①②③④⑤ to ⑤④③②①.

Solution

At the beginning, ⑤ can only be grouped with ④, and we will find a place to insert ④⑤.
Since we eventually need to let ⑤ be the first one, meaning ④ must be grouped with another coin.
So, understandably, ④⑤ will be inserted between ① and ②.

①②③④⑤ → ①④⑤②③

Then, we have to consider ①④ as a pair and clearly, ①④ will be inserted between ② and ③.

①④⑤②③ → ⑤②①④③ → ⑤④③②①

Eventually, we can reverse the order easily.

2.(b)

Introduction

We want to change their order from ①②③④⑤ to ⑤②③④①.

Some auxiliary results

Lemma 6.2.1 All the movements can be consider as single time or several times of the change in the positions of three adjacent coins, where the coins chosen every time may not be the same.
Proof.

ABCD → CDAB = ABCD → CABD → CDAB

Lemma 6.2.2 If we consider the exchange of positions between two coins as one operation, then every movement can be consider as even number of operations.
Proof.
For ABC → CAB:
we firstly exchange of positions of B and C, then exchange of positions of A and C
Similarly for ABC → BCA.
As we can see, every movement relating to three coins can be consider as two operations.
According to Lemma 6.2.1, all the movements can be consider as single time or several times of the change in the positions of three adjacent coins, meaning that all the movements can be consider as single time or several times of two operations.

Lemma 6.2.3 If the movement can be consider as odd(or even) number of operations, then it cannot be consider as even(or odd) number of operations.
Proof.
Assuming that, after even(or odd) number of operations, we get ABCDE.
We now do some further operations based on this result, but still ending up with ABCDE.
Notice that, during the process of further operations, there always exists a so-called 'furthest stage', indicating that the minimum number of operations we further need to take to return to ABCDE is the maximum, compared with the minimum number of operations we further need to take to return to ABCDE in other stages.

For example:
(1)During the process of ABCDE → ACBDE → ACEDB, ACEDB is 'furthest stage'.
(2)During the process of ABCDE → ACBDE → ACEDB → ACBDE, ACEDB is 'furthest stage'.

At the 'furthest stage', the method of returning to ABCDE using the minimum number of operations is to return by the original route. Because every operation is independent and unique: after all, operations relating to exchange of positions of A and C will not influence the order of B in the sequence; if we want to change the order of B in the sequence, then returning by the original route is the most effective way.
Notice that, 'furthest stage' is either the final stage(like example(1)) or the middle stage(like example(2)). For the latter, the stages behind 'furthest stage' are on the way of returning by the original route.
Notice that, since we return by the original route, meaning that the number of further operations is always even.
Thus, if the movement can be consider as odd(or even) number of operations, then it cannot be consider as even(or odd) number of operations.

Solution

⑤②③④① experiences one operation.
Whereas, according to Lemma 6.2.2 and 6.2.3, every movement can be consider as even number of operations, and if the movement can be consider as odd(or even) number of operations, then it cannot be consider as even(or odd) number of operations.
So, ⑤②③④①, which is consider as odd number of operations, could not be achieve by several movements, which are consider as even number of operations.

Afterword

I really want to mention some of my other ideas here, because this problem has been bothering me for a long time.

Method 1
The most straight-forward way is to list all the results---although this is impossible because there are 120 possibilities for us to deal with.

Method 2
Another way to make a quick conclusion is that, since in the first part of the question, we ended up with ⑤④③②①, we just need to prove that it is impossible for us to turn ④③② into ②③④, fixing the positions of ① and ⑤.
However, this method is not rigorous enough: what if, in the process, we change the positions of ① and ⑤, and eventually make ① and ⑤ turn to their original places?

Method 3
Understandably, we need to convert this question into another form, but how?
For each movement, I once performed some changes on the affected coins in order to mark them.
But after several trials, disappointingly, I found that if I want to performed some changes on the affected coins, I needed to track the specific position of each coin, making the whole method useless.