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数学月刊

Mathematics Monthly

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Mathematics changes our lives

PREFACE

This month, we are going to talk about the following questions.

Q1: From Dr Greg Taujanskas

① For what values of a will $a^{a^{\cdot^{\cdot^{\cdot}}}}$ be converge?

② Which is bigger, $e^{e^{-1}}$ or $\sqrt{2}$?

Q2: From Dr James Davies

Given that (a,b) is larger than (c,d) if a > c and b > d, where a, b, c, d are integers.

① For n×n grids, what is the maximum number of points such that no one is larger than another?

② What is the maximum number of points such that we do not have 3 increasingly large points, such as (1,1), (2,2) and (3,3)?

Q3: From Dr Ron Reid-Edwards

$y_0 = x$, $y_1 = ln(x)$, $y_2 = ln(y_1)$, ..., $y_{n+1} = ln(y_n)$, *find out a general expression for* $\frac{dy_n}{dx}$.

Q4: From Ms Lili Fehertoi-Nagy

Suppose there is a person, with 2m high, standing on the surface of the Earth, with 6000km radius. Given that the beams of sunlight are parallel, what is the largest length of the shadow of that human?

Question 5: From Dr Ron Reid-Edwards

① What is the locus of the intersection of

$$\begin{cases} x^2 + y^2 + z^2 = 1 \\ (x - a)^2 + y^2 + z^2 = 1 \end{cases}$$

② What is the locus of the intersection of

$$\begin{cases} x^2 + y^2 + z^2 + \omega^2 = 1 \\ (x - a)^2 + y^2 + z^2 + \omega^2 = 1 \end{cases}$$

This is the admission question from 2024 Cambridge Trinity Hall interview questions. If you have other brilliant ideas, email to anmiciuangray@163.com for surprising rewards!

1.(a)

Some auxiliary results

During the interview, for the first part, I thought about:

① using induction: useless since it is hard to compare P_k and P_{k+1}.

② proving upper boundary is converge: useless since unknown upper boundary.

③ considering y = $a^{a^{a^{\cdot^{\cdot^{\cdot}}}}}$ and find an relationship between a and y: useful.

Solution

For ①, we get a^y = y, so ylna = lny, $\frac{\ln y}{y}$ = lna. Obviously, a > 0 due to definition of logarithm.

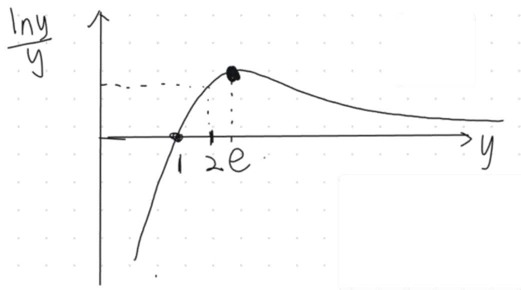
Then I sketch the graph of $\frac{\ln y}{y}$:

(1) domain: y > 0.

(2) approaching: when y → 0, $\frac{\ln y}{y}$ → -∞; when y → ∞, the rate of increase of denominator is dramatically larger than the rate of increase of numerator, so $\frac{\ln y}{y}$ → 0.

(3) derivative: $\frac{dy}{dx} = \frac{1-\ln x}{x^2}$, turning point occurs at (e,e^{-1}).

(4) zero point: (1,0).



If we consider RHS of $\frac{\ln y}{y}$ = lna as a horizontal test: if converge, lna ≤ e^{-1}, so 0 < a ≤ e^{-1}.

1.(b)

Some auxiliary results

For the last part, I thought about:

① compare $e^{e^{-1}}$ and $\sqrt{2}$ directly: useless and ridiculous.

② consider their physical meaning in the first part: useful.

For the last question, I now come up with a new idea of proving that a = $\sqrt{2}$ will make y converge.

Solution

For ②, we notice that: $\ln(\sqrt{2}) = \frac{\ln 2}{2}$, which is $\frac{\ln y}{y}$ for y = 2; $\ln(e^{e^{-1}}) = \frac{\ln e}{e}$, which is $\frac{\ln y}{y}$ for y = e. According to the sketch, $\ln(e^{e^{-1}}) > \ln(\sqrt{2}) \leftrightarrow e^{e^{-1}} > \sqrt{2}$.

I fact, another strategy is that: if I can prove that when a = $\sqrt{2}$, y is converge, then $e^{e^{-1}} > \sqrt{2}$ since $e^{e^{-1}}$ is the largest value of a such that y will be converge.

P_n: $\sqrt{2}^{\sqrt{2}^{\sqrt{2}^{\cdot^{\cdot^{\cdot^{\sqrt{2}}}}}}} < 2$, with n terms of $\sqrt{2}$, for all positive integers n.

P_1: $\sqrt{2}^{\sqrt{2}} < \sqrt{2}^{2} = 2$, so P_1 is true.

Assume P_k is true, so $\sqrt{2}^{\sqrt{2}^{\sqrt{2}^{\cdot^{\cdot^{\cdot^{\sqrt{2}}}}}}} < 2$, with k terms of $\sqrt{2}$.

Then for P_{k+1}: $\sqrt{2}^{[\sqrt{2}^{\sqrt{2}^{\sqrt{2}^{\cdot^{\cdot^{\cdot^{\sqrt{2}}}}}}}]} < \sqrt{2}^{2} = 2$, so $P_k \rightarrow P_{k+1}$.

Since P_1 is true and $P_k \rightarrow P_{k+1}$ is accurate. By induction, we can show that $\sqrt{2}^{\sqrt{2}^{\sqrt{2}^{\cdot^{\cdot^{\cdot^{\sqrt{2}}}}}}} < 2$, with n terms of $\sqrt{2}$, for all positive integers n. So when n → ∞, it will still have a upper boundary 2, meaning that it is converge.

That be said, $e^{e^{-1}} > \sqrt{2}$.

2.(a)

Some auxiliary results

During the interview, I thought about:

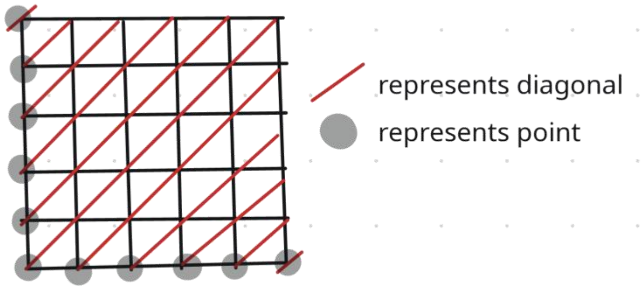
① using induction: useless since hard to consider the combination of $\mathrm{P_k}$ when $\mathrm{P_k} \to \mathrm{P_{k+1}}$.

② counting inverse situations: useless since too many cases without regularity.

③ using pigeon hole principle: useful since question relating to ‘at most’.

Solution

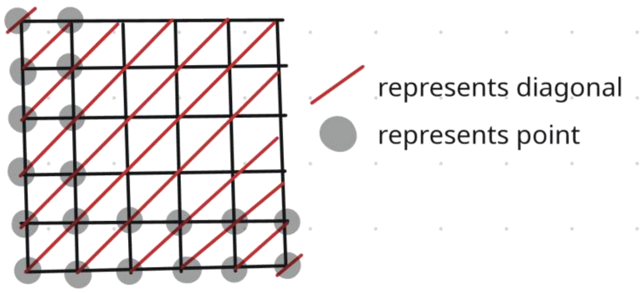
For ①, we notice that each upwards diagonal[because we want to focus on larger point] can at most have 1 point. So in the most optimal situation, for n×n grids, having (2n-1) diagonals[we can also consider there exists diagonals for vertices], we can have (2n-1) points. To prove my conjecture, we can distribute all the points on the first row and the first column.



2.(b)

Solution

For ②, similar strategy can be applied: each upwards diagonal can at most have 2 point. So in that way, we classify those upwards diagonals according to the number of points that each of them passing through: for those upwards diagonals passing through no more than 2 points, all those points can be chosen; for those upwards diagonals passing through more than 2 points, only 2 points can be chosen. Totally, we can have at most (4n-4) points. To prove my conjecture, we can distribute all the points on the first, second rows and the first, second columns.



3.

Solution

During the interview, I used induction to solve this question, because it is impossible for us to prove the accessibility of this conclusion by infinite trials.

P_n: $\frac{dy_n}{dx} = \prod_{i=0}^{n-1} \frac{1}{y_i}$ for all positive integers n.

P_1: $\frac{dy_1}{dx} = \frac{1}{x}$, so P_1 is true.

Assume P_k is true, so $\frac{dy_k}{dx} = \prod_{i=0}^{k-1} \frac{1}{y_i}$.

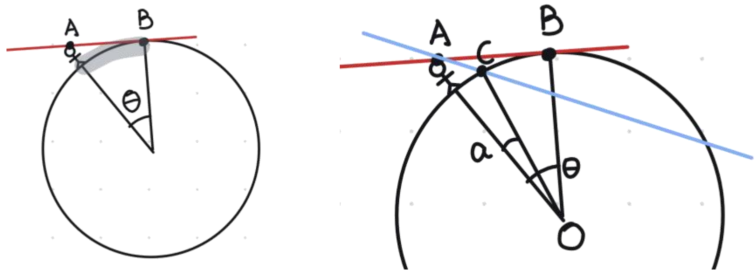
Then for P_{k+1}: $\frac{dy_{k+1}}{dx} = \frac{d}{dx}[\ln(y_k)] = \frac{1}{y_k}\frac{dy_k}{dx} = \frac{1}{y_k}\prod_{i=0}^{k-1}\frac{1}{y_i} = \prod_{i=0}^{k}\frac{1}{y_i}$, so $P_k \rightarrow P_{k+1}$.

Since P_1 is true and $P_k \rightarrow P_{k+1}$ is accurate. By induction, we can show that $\frac{dy_n}{dx} = \prod_{i=0}^{n-1}\frac{1}{y_i}$ for all positive integers n.

4.

Solution

We can interpret this question in a mathematical way: Given that a line must pass through A and another point on the circle, when will the corresponding arc be maximised?



In order to maximise the arc, we notice that we can maximise the arc by maximising the corresponding central angle.

That be said, we want to prove that $\theta > \alpha$.

Notice that OA is fixed, OC = OB. According to cosine rule, $AC^2 = OA^2 + OC^2$ - 2•OA•OC•cosα, $AB^2 = OA^2 + OB^2$ - 2•OA•OC•cosθ.

Given that AC < AB, $AC^2 < AB^2$, cosα > cosθ, $\theta > \alpha$. In that way, arc AB is the maximum shadow.

Notice that AB is tangent of the circle so AB is perpendicular to OB. θ = arccos$\frac{6000000}{6000000+2}$, length of shadow = 6000000arccos$\frac{6000000}{6000000+2}$.

For course, corresponding θ is really small, we can prove this by using either mathamtical value of cosθ or by considering physical meaning.

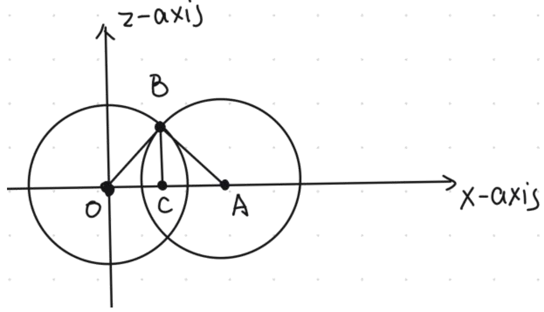
5.

Auxiliary Result

During the interview, I thought about:
① expressing x and y in terms of z: useless since hard to eliminate.
② considering their physical meaning.
For this question, I now come up with a new idea of considering the locus as rotation around x-axis.

Solution

For this question, all equations represent balls or some 4D graphs similar to balls, because they satisfy the definition of ball or something similar: the distance from the surface to the center is fixed.
For ①, $x^2 + y^2 + z^2 = 1$ represents a ball with center at origin and radius of 1; $(x-a)^2 + y^2 + z^2 = 1$ represents represents a ball with center at (a,0,0) and radius of 1.
By consider their physical meaning, if intersect, the points on the intersection, such as B, will have the equal distances to either O(0,0,0) or A(a,0,0) due to definition.
Given that BC is fixed, so OC = OA, so all the points of intersection will have x-coordinate of $\frac{a}{2}$.
Then, BC = $\sqrt{OB^2 - OC^2} = \sqrt{1 - \frac{a^2}{4}}$, so the locus is x = $\frac{a}{2}$, $y^2 + z^2 = 1 - \frac{a^2}{4}$.



Similarly, for ②, $x^2 + y^2 + z^2 + \omega^2 = 1$ represents a 4D graph similar to balls with center at origin and radius of 1; $(x-a)^2 + y^2 + z^2 + \omega^2 = 1$ represents represents a 4D graph similar to balls with center at (a,0,0,0) and radius of 1.
By consider their physical meaning, if intersect, the points on the intersection will have the equal distances to either O(0,0,0,0) or A(a,0,0,0) due to definition. So all the points of intersection will have x-coordinate of $\frac{a}{2}$.
Similarly, radius of locus can be found, so the locus is x = $\frac{a}{2}$, $y^2 + z^2 + \omega^2 = 1 - \frac{a^2}{4}$.

Afterwords

Another method is that consider the locus as rotation around x-axis.
For ①, we consider the graph of $x^2 + y^2 = 1$ and $(x-a)^2 + y^2 = 1$, and find out the intersection.
Then, we rotate the whole graph with x-axis, meaning that we turning y^2 into $y^2 + z^2$. In that way, we can still get the same result.
The same method can be applied to ②, as long as we consider the graph as the rotation of 3D space. In that way, we can get the same answer based on the result of ①.