代数部分

1.不等式

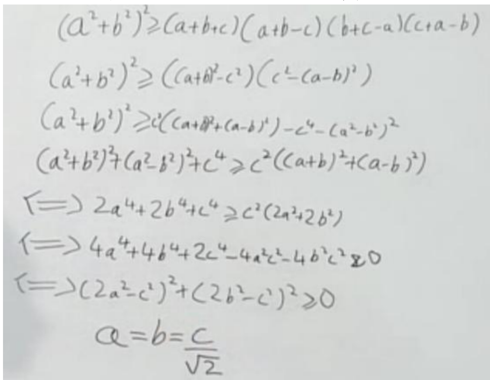
(1)BMO1-2006-5

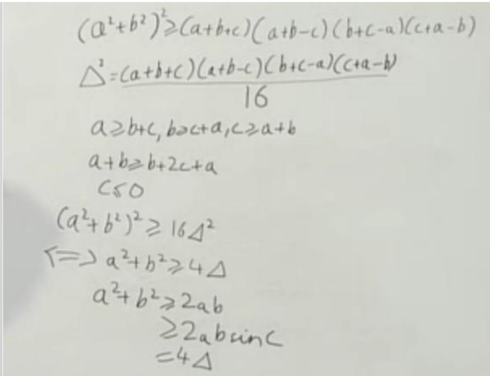
For positive real numbers a, b, c, prove that

$$(a^2+b^2)^2 \geq (a+b+c)(a+b-c)(b+c-a)(c+a-b).$$

思路

因为我们可以看到单纯的不等式不是很好解,所以说第一种方法就是用平方不为负数的性质解
我们可以知道: 1. 平方差公式的项可能是多项式 2. 不要害怕开平方 3. 最后要化成平方形式
第二种方法涉及均值不等式和面积公式的灵活运用
因为 RHS 真的看上去很想海伦公式,我们能不能用面积公式来推导?
我们可以知道: 1. 相似形式的转化 2. 寻找相同意义然后转化形式





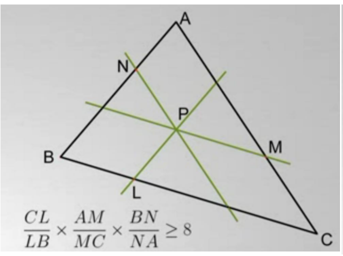
(2)BMO1-2007-5

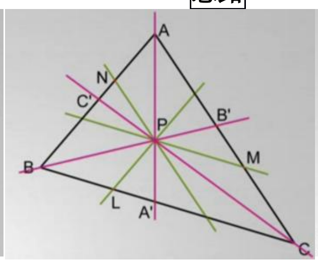
Let P be an internal point of triangle ABC. The line through P parallel to AB meets BC at L, the line through P parallel to BC meets CA at M, and the line through P parallel to CA meets AB at N. Prove that

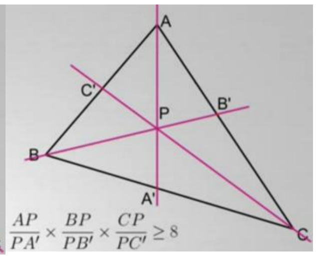
$$\frac{BL}{LC} \times \frac{CM}{MA} \times \frac{AN}{NB} \leq \frac{1}{8}$$

and locate the position of P in triangle ABC when equality holds.

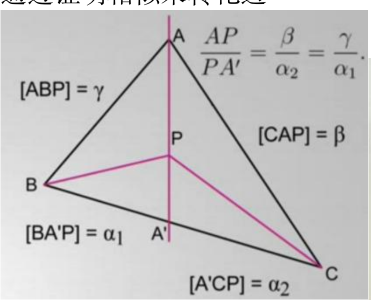
思路

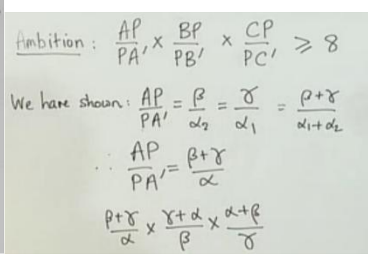


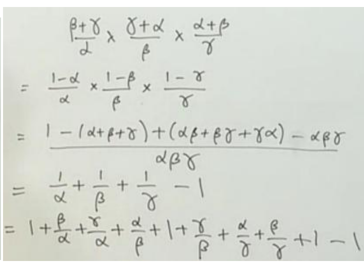




通过证明相似来转化边







一开始设面积为 1,三个三角形面积分别为 a,b,c,1=a+b+c

经过面积法转化我们了比例

$$= 2+\left(\frac{\alpha}{\beta}+\frac{\beta}{\alpha}\right)+\left(\frac{\beta}{\gamma}+\frac{\gamma}{\beta}\right)+\left(\frac{\gamma}{\alpha}+\frac{\alpha}{\gamma}\right)$$

$$\left[\ \left(\sqrt{x}-\frac{1}{\sqrt{x}}\right)^2 \geqslant 0\right.$$

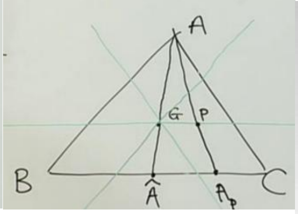
$$\therefore\ x+\frac{1}{x}-2 \geqslant 0$$

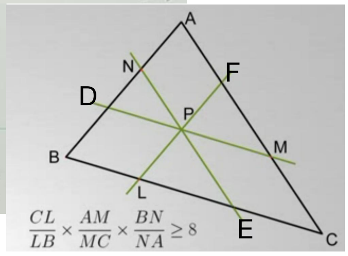
$$\left.\therefore\ x+\frac{1}{x} \geqslant 2\ \right]$$

$$\geqslant 2+2+2+2=8.$$

$\alpha=\beta=\gamma=\frac{1}{3}$

Areal co-ords of G are $\left(\frac{1}{3},\frac{1}{3},\frac{1}{3}\right)$





另一种方法是先证明ΔDNP～ΔPFM～ΔLPE(用平行线证明)

然后设 DP=xa,NP=xb,DN=xc,LE=ya,EP=yb,LP=yc,PM=za,FM=zb,FP=zc

$$\text{ratio} = \frac{xyz\bullet abc}{(x+y)(x+z)(y+z)\bullet abc} = \frac{xyz}{2xyz+xy(x+y)+xz(x+z)+yz(y+z)}$$

$$\leq \frac{xyz}{2xyz+2x^{\frac{3}{2}}y^{\frac{3}{2}}+2x^{\frac{3}{2}}z^{\frac{3}{2}}+2y^{\frac{3}{2}}z^{\frac{3}{2}}} \leq \frac{xyz}{2xyz+6\sqrt[3]{x^3y^3z^3}} = \frac{1}{8}$$

(1)BMO1-2010-6

Let a, b and c be the lengths of the sides of a triangle. Suppose that $ab+bc+ca=1$. Show that $(a+1)(b+1)(c+1)<4$.

思路

To prove:

$(a+1)(b+1)(c+1)<4$

Working

$abc+ab+ac+bc+a+b+c+1<4$

$ab+bc+ca=1$

$2-a-b-c-abc>0$

$1-a-b-c+ab+bc+ca-abc$

$=(1-a)(1-b)(1-c)$

To prove

$(1-a)(1-b)(1-c)>0$

Working:

$c<a+b,\quad ab+bc+ca=1$

$c^2<c(a+b)=1-ab<1$

$1-c>0$

2.函数

(1)BMO1-2006-1

Find four prime numbers less than 100 which are factors of $3^{32}-2^{32}$.

思路

factorise;答案是 5,13,17,97

(2)BMO1-2007-1

Find the value of

$$\frac{1^4+2007^4+2008^4}{1^2+2007^2+2008^2}.$$

$$\frac{1+x^4+(x+1)^4}{1+x^2+(x+1)^2}$$

$$=\frac{1+x^4+x^4+4x^3+6x^2+4x+1}{1+x^2+x^2+2x+1}$$

$$=\frac{2x^4+4x^3+6x^2+4x+2}{2x^2+2x+2}$$

$$=x^2+x+1$$

思路

factorise,答案是 4030057

(2)BMO1-2009-5

Find all functions f, defined on the real numbers and taking real values, which satisfy the equation $f(x)f(y)=f(x+y)+xy$ for all real numbers x and y.

思路

$y=-x$: $\quad f(x)f(-x)=f(0)-x^2=1-x^2$

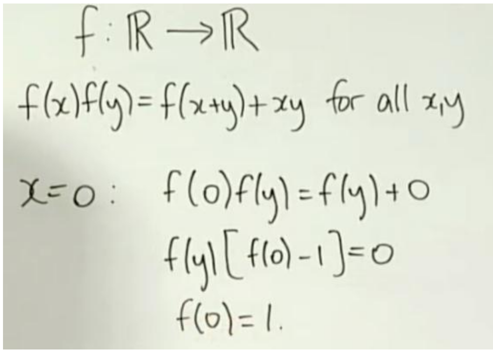
$x=1, y=-1$: $\quad f(1)f(-1)=0$

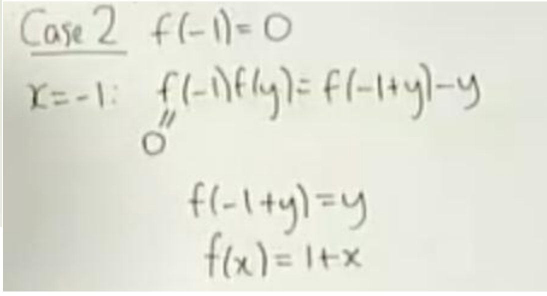
Case 1 $\quad f(1)=0$

$x=1$: $\quad 0=f(1)f(y)=f(1+y)+y$

$f(1+y)=-y$

$f(x)=-(x-1)=1-x$





(2)BMO1-2011-4

Initially there are m balls in one bag, and n in the other, where $m, n > 0$. Two different operations are allowed:

a) Remove an equal number of balls from each bag;
b) Double the number of balls in one bag.

Is it always possible to empty both bags after a finite sequence of operations?

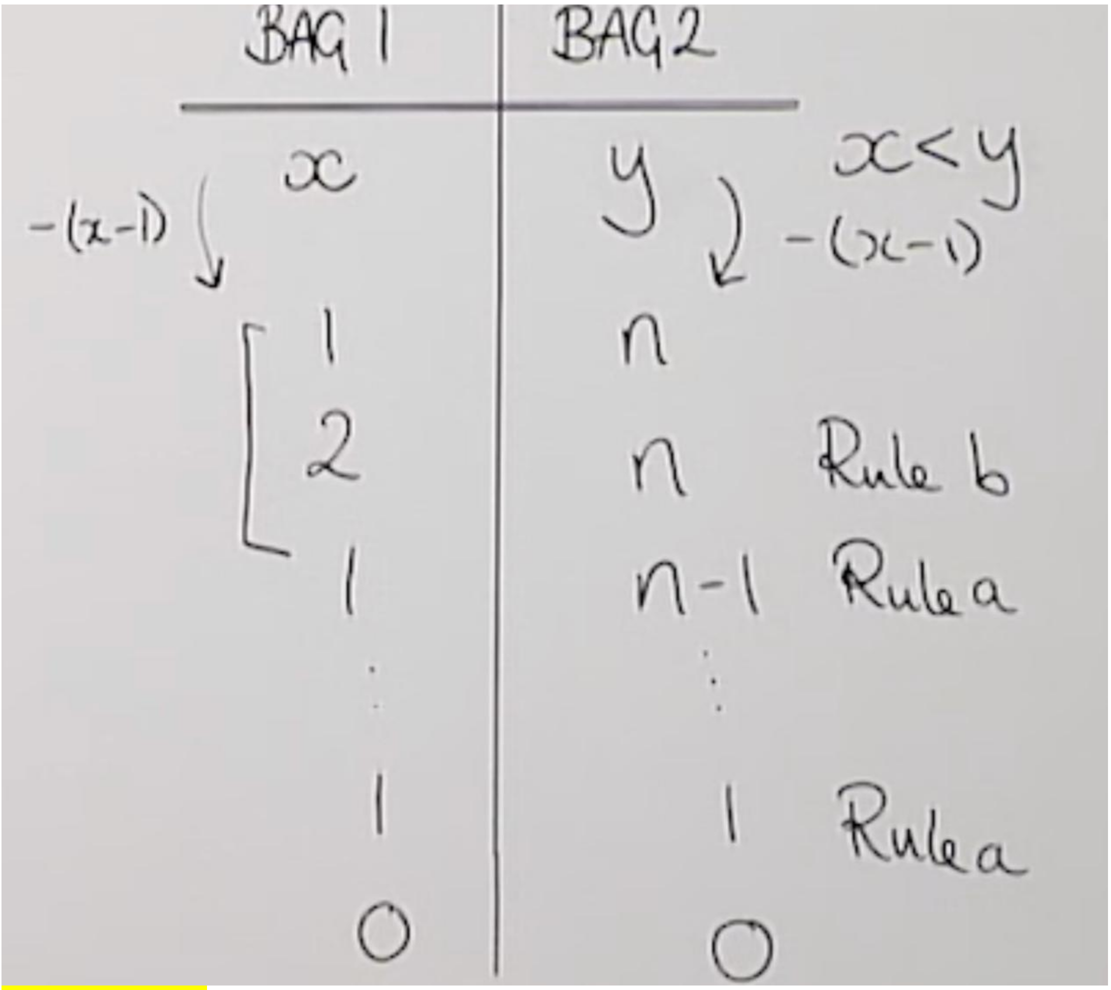
Operation b) is now replaced with

b′) Triple the number of balls in one bag.

Is it now always possible to empty both bags after a finite sequence of operations?

思路

第一种情况下是每次减 1,第二种是每次减 2



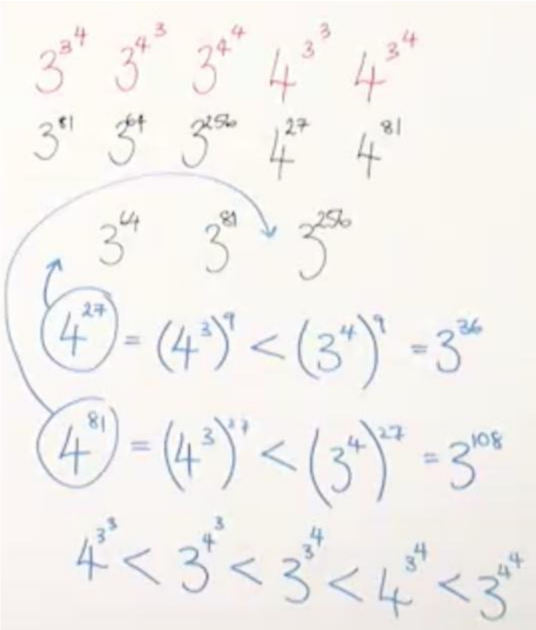
(2)BMO1-2014-1

Place the following numbers in increasing order of size, and justify your reasoning:

$$3^{3^4}, 3^{4^3}, 3^{4^4}, 4^{3^3} \text{ and } 4^{3^4}.$$

Note that a^{b^c} means $a^{(b^c)}$.

思路

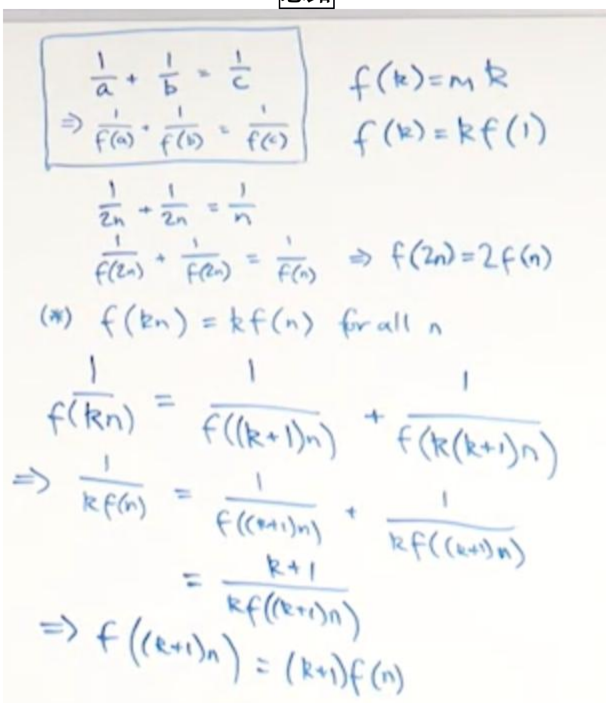


(2)BMO1-2014-6

Determine all functions $f(n)$ from the positive integers to the positive integers which satisfy the following condition: whenever a, b and c are positive integers such that $1/a + 1/b = 1/c$, then

$$1/f(a) + 1/f(b) = 1/f(c).$$

思路



(2)BMO1-2016-2

For each positive real number x, we define $\{x\}$ to be the greater of x and $1/x$, with $\{1\} = 1$. Find, with proof, all positive real numbers y such that

$$5y\{8y\}\{25y\} = 1.$$

思路

本质上就是分类讨论

$\{x\}$?

if $x \geq 1$, $\frac{1}{x} \leq 1$, so $\{x\} = x$

if $x < 1$, $\frac{1}{x} > 1$, so $\{x\} = \frac{1}{x}$

$5y\{8y\}\{25y\} = 1$.

$8y<1 \quad 25y<1 \quad y<\frac{1}{25}$.

$8y<1 \quad 25y\geq 1 \quad y<\frac{1}{8} \quad y \geq \frac{1}{25}$

~~$8y \geq 1 \quad 25y<1$~~

$8y \geq 1 \quad 25y \geq 1 \quad y \geq \frac{1}{8}$

$5y\{8y\}\{25y\} = 1$

$y<\frac{1}{25}$

$5y \quad \frac{1}{8y} \quad \frac{1}{25y} = 1$

$y = 40y^2 \qquad y = \frac{1}{40}$

$5y\{8y\}\{25y\} = 1$.

$\frac{1}{25} \leq y < \frac{1}{8}$

$5\cancel{y} \quad \frac{1}{8\cancel{y}} \quad 25y = 1$

$125y = 8 \qquad y = \frac{8}{125}$.

$5y\{8y\}\{25y\} = 1$.

$\frac{1}{8} \leq y$.

$5y \cdot 8y \cdot 25y = 1$

$1000y^3 = 1$ ✗

$10y = 1 \qquad y = \frac{1}{10}$

(2)BMO1-2019-6

A function f is called *good* if it assigns an integer value $f(m,n)$ to every ordered pair of integers (m,n) in such a way that for every pair of integers (m,n) we have:

$$2f(m,n) = f(m-n, n-m) + m + n = f(m+1, n) + f(m, n+1) - 1.$$

Find all good functions.

3.数列

(1)BMO1-2011-2

Consider the numbers $1, 2, \ldots, n$. Find, in terms of n, the largest integer t such that these numbers can be arranged in a row so that all consecutive terms differ by at least t.

思路

$1, 2, 3, 4, \ldots, m+1$

and insert

$m+2, m+3, \ldots, 2m+1$

to give

$1, m+2, 2, m+3, 3, \ldots, m, 2m+1, m+1.$

Interleave

$1, 2, 3, \ldots, m$

with

$m+1, m+2, \ldots, 2m$

to give

$m+1, 1, m+2, 2, m+3, \ldots, m-1, 2m, m.$

so for each integer $n \geq 2$, we have

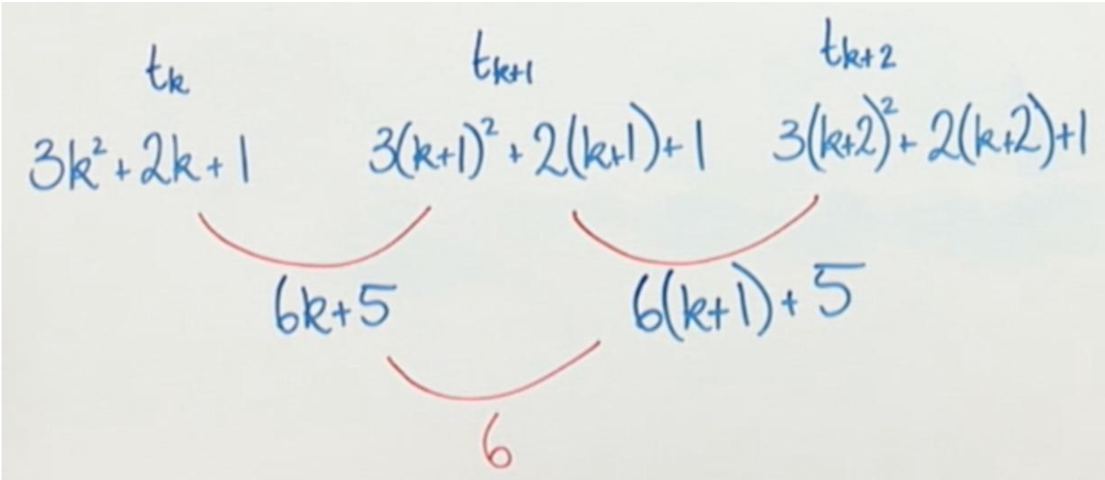
$$t = \left\lfloor \frac{n}{2} \right\rfloor.$$

(1)BMO1-2015-3

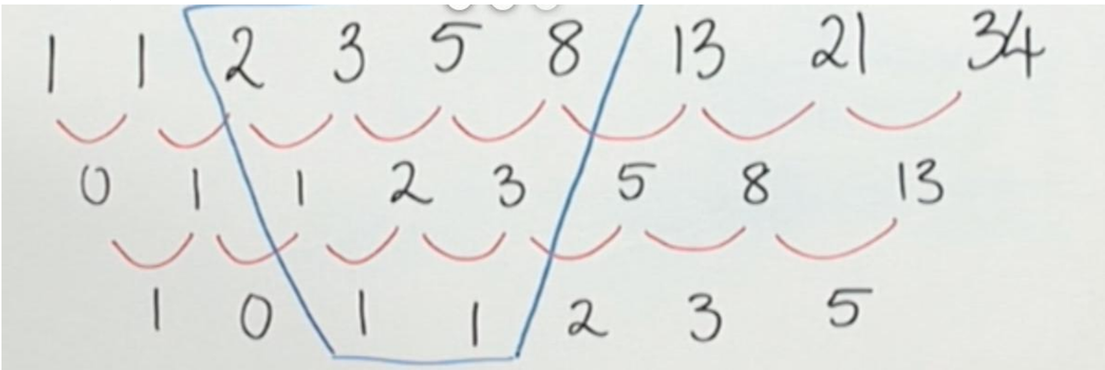
Suppose that a sequence $t_0, t_1, t_2, \ldots$ is defined by a formula $t_n = An^2 + Bn + C$ for all integers $n \geq 0$. Here A, B and C are real constants with $A \neq 0$. Determine values of A, B and C which give the greatest possible number of successive terms of the sequence which are also successive terms of the Fibonacci sequence. *The Fibonacci sequence is defined by* $F_0 = 0$, $F_1 = 1$ *and* $F_m = F_{m-1} + F_{m-2}$ *for* $m \geq 2$.

思路

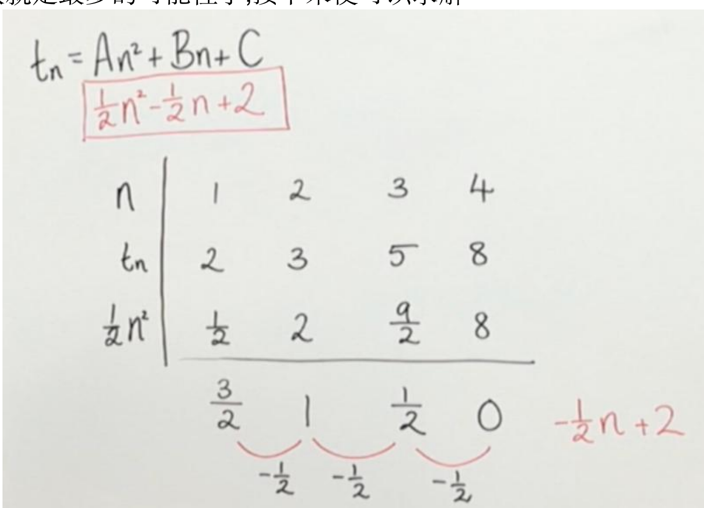
首先我们先来看t_k的规律,即差的差是常数(我们假设$t_k=3k^2+2k+1$)



也就是说,我们要在 Fibonacci 数列中找到差的差是常数项的



因此,中间 4 个数就是最多的可能性了,接下来便可以求解



(1)BMO1-2015-6

A positive integer is called *charming* if it is equal to 2 or is of the form 3^i5^j where i and j are non-negative integers. Prove that every positive integer can be written as a sum of different charming integers.

思路

1, 2, 3, 5, 9, 15, 25, 27, 45

$1 = 1$
$2 = 2$
$3 = 3$
$4 = 3 + 1$
$5 = 5$
$6 = 5 + 1$
$7 = 5 + 2$
$8 = 5 + 3$
$9 = 9$
$10 = 9 + 1$
$11 = 9 + 2$
$12 = 9 + 3$
$13 = 9 + 3 + 1$
$14 = 9 + 5$
$15 = 15$
$16 = 15 + 1$
⋮
$25 = 25$

Step 1: Show that if N is charming and every number up to N can be written as a sum of different charming numbers, then so can every number up to 2N-1.

Step 2: Show that if N is charming, then the next charming number is less than or equal to 2N.

Step 2
Let $N > 2$ be charming.
Then $N = 3^i 5^j$ for some $i, j \geqslant 0$
(where i, j not both 0).
If $i \geqslant 1$, then $3^{i-1} 5^{j+1} = \frac{5N}{3} \leq 2N$.
If $i = 0$, then $j \geqslant 1$ and $3^{i+2} 5^{j-1} = \frac{9N}{5} \leq 2N$.

(1)BMO1-2017-4

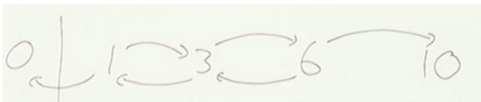
Consider sequences $a_1, a_2, a_3, \ldots$ of positive real numbers with $a_1 = 1$ and such that

$$a_{n+1} + a_n = (a_{n+1} - a_n)^2$$

for each positive integer n. How many possible values can a_{2017} take?

思路

我们首先先试几项



随后我们发现,每一次 n+1 实际上可以向前或后变化一位数字
我们也可以证明这一点
由此,便可推出结论

t_n : n^{th} triangular number
$a_1 = t_1$
$a_2 = t_2$
$a_3 = t_1$ or t_3
$a_4 = t_2$ or t_4
$a_5 = t_1$ or t_3 or t_5
⋮
$a_{2017} = t_1$ or t_3 or t_5 or ... or t_{2017}

(1)BMO1-2019-2

A sequence of integers $a_1, a_2, a_3, \ldots$ satisfies the relation:

$$4a_{n+1}^2 - 4a_n a_{n+1} + a_n^2 - 1 = 0$$

for all positive integers n. What are the possible values of a_1?

思路

factorise 之后可得

$$a_{n+1} = \frac{a_n \pm 1}{2}$$

a_n odd for all n.

$$a_{n+1} = \frac{a_n + 1}{2} \qquad a_{n+1} = \frac{a_n - 1}{2}$$

因为a_{n+1}剩下的两个结果中肯定有一个是 odd(因为两个数之差为 1)
所以说a_1是 odd 即可

(1)BMO1-2019-4

There are 2019 penguins waddling towards their favourite restaurant. As the penguins arrive, they are handed tickets numbered in ascending order from 1 to 2019, and told to join the queue. The first penguin starts the queue. For each $n > 1$ the penguin holding ticket number n finds the greatest $m < n$ which divides n and enters the queue directly behind the penguin holding ticket number m. This continues until all 2019 penguins are in the queue.

(a) How many penguins are in front of the penguin with ticket number 2?

(b) What numbers are on the tickets held by the penguins just in front of and just behind the penguin holding ticket 33?

思路

第一步是先试一试
第二步是,如果一个新企鹅在后面,设其号码牌为 33p
那么,会不会存在 11p 让它不待在后面?
诸如此类

5.不定方程的整数解(丢番图方程)

(1)BMO1-2005-1

Let n be an integer greater than 6. Prove that if $n-1$ and $n+1$ are both prime, then $n^2(n^2+16)$ is divisible by 720. Is the converse true?

思路

720 = 16×9×5 , 连续的三个数必定有 2 与 3 的倍数,分别找出在满足 prime 条件下 mod 2/3/5 的大小,然后带入式子看看这些情况下满不满足被 720 整除
remember, a|c and a|c, so ab|c
证伪只需要举出例子即可

(2)BMO1-2006-6

Let n be an integer. Show that, if $2+2\sqrt{1+12n^2}$ is an integer, then it is a perfect square.

$\boxed{\text{思路}}$

首先第一步是先判断$\sqrt{1+12n^2}$ 是整数(不可能是$\frac{n}{2}$)
然后化简,基本操作,最后一步得到 $3n^2 = c(c-1)$
因为 c（c-1）co-prime,n 不可能同时被这两项整除
所以说设 $n = x^2y^2$,则 c,(c-1)这俩一个是 x^2,一个是 $3y^2$
如果 c 是 $3y^2$,那么 c-1 就是 $3y^2-1 = x^2$
但是因为平方 mod 3 后=0 或 1,不可能为 2
所以说 $3y^2-1$ 永远不等于 x^2
c 是 x^2 那么 $4c=a=4x^2$,证毕
这告诉我们不要想当然,$n = x^2y^2$
解有无限多,只要右图两边一直平方下去就能得到新的解

$2+2\sqrt{z}\in\mathbb{Z}$
$\sqrt{z}=\frac{y}{2}$
$\Rightarrow z=\frac{y^2}{4}$
$\therefore \sqrt{1+12n^2}\in\mathbb{Z}$

$2+2\sqrt{1+12n^2}=a\in\mathbb{Z}$
$1+\sqrt{1+12n^2}=b$
$1+12n^2=(b-1)^2$
$12n^2=(2c-1)^2-1=4c^2-4c$
$3n^2=c(c-1)$
$4c=a$

$c=x^2$
$c-1=3y^2$
$x^2-3y^2=1$
$(x-y\sqrt{3})(x+y\sqrt{3})=1$
$x=2,\ y=1$
$(2-\sqrt{3})(2+\sqrt{3})=1$
$(7-4\sqrt{3})(7+4\sqrt{3})=1$
$x=7,\ y=4$

(1)BMO1-2007-2

Find all solutions in positive integers x, y, z to the simultaneous equations

$$x+y-z=12$$
$$x^2+y^2-z^2=12.$$

$\boxed{\text{思路}}$

symmetric,只要考虑半边就行
注意 66 因式分解时可以分成负数,但是如果分成负数的话 x 或 y 就不是正数了(必须点明)

$x+y-z=12$ ①
$x^2+y^2-z^2=12$ ②
$x^2+2xy+y^2-z^2=12+2xy$
$(x+y-z)(x+y+z)=12+2xy$
$\therefore 12x+12y+12z=12+2xy$
$\therefore xy-6x-6y-6z+6=0$

$x+y-z=12$ ①
$x^2+y^2-z^2=12$ ②
$xy-6x-6y-6z+6=0$
$xy-12x-12y+78=0$
$xy-12x-12y+144=66$
$(x-12)(y-12)=66$ ③

x	$x-12$	$y-12$	y	z
13	1	66	78	79
14	2	33	45	47
15	3	22	34	37
18	6	11	23	29

$(x,y,z) = (13,78,79)$
$= (14,45,47)$
$= (15,34,37)$
$= (18,23,29)$

(1)BMO1-2009-1

Find all integers x, y and z such that

$$x^2+y^2+z^2=2(yz+1) \text{ and } x+y+z=4018.$$

$\boxed{\text{思路}}$

$$x^2+y^2+z^2=2(yz+1)$$
$$x+y+z=4018$$
$$y^2+z^2-2yz=(y-z)^2$$
$$x^2+(y-z)^2=2$$

(1)BMO1-2009-6

Long John Silverman has captured a treasure map from Adam McBones. Adam has buried the treasure at the point (x,y) with integer co-ordinates (not necessarily positive). He has indicated on the map the values of x^2+y and $x+y^2$, and these numbers are distinct. Prove that Long John has to dig only in one place to find the treasure.

思路

注意,我们不需要证明出解,我们只需要证明出对于一组解仅有一组对应坐标即可

(x,y) (x',y') two different points

$$x^2+y=x'^2+y'$$
$$x+y^2=x'+y'^2$$

Case 1 $x+x'=y+y'=1$

$x'=1-x$ $y'=1-y$

$$x^2+y=(1-x)^2+(1-y)$$

$x+y=1$ $y+y'=1$

$x=y'$ $x'=y$

$$x^2+y=x'^2+y'$$
$$x+y^2=x'+y'^2 \quad x\neq x'$$
$$y-y'=x'^2-x^2=(x'-x)(x'+x)$$
$$x-x'=y'^2-y^2=(y'-y)(y'+y)$$
$$x-x'=-(x'-x)(x'+x)(y'+y)$$
$$(x'+x)(y'+y)=1$$

Case 2 $x+x'=y+y'=-1$

$x'=-1-x$ $y'=-1-y$

$$x^2+y=(-1-x)^2+(-1-y)$$
$$x=y$$

(1)BMO1-2010-1

One number is removed from the set of integers from 1 to n. The average of the remaining numbers is $40\frac{3}{4}$. Which integer was removed?

思路

$$x=\frac{n(n+1)}{2}-\frac{163(n-1)}{4}$$
$$1\leq x\leq n$$

$$2n(n+1)-163(n-1)\geq 4$$
$$2n^2-161n+159\geq 0$$
$$(2n-159)(n-1)\geq 0$$

$$2n(n+1)-163(n-1)\leq 4n$$
$$2n^2-165n+163\leq 0$$
$$(2n-163)(n-1)\leq 0$$

写出表达式后带入范围!

(1)BMO1-2010-2

Let s be an integer greater than 6. A solid cube of side s has a square hole of side $x < 6$ drilled directly through from one face to the opposite face (so the drill removes a cuboid). The volume of the remaining solid is numerically equal to the total surface area of the remaining solid. Determine all possible integer values of x.

思路

$$s^3 - x^2 s = 6s^2 - 2x^2 + 4xs$$

$$\Rightarrow (s-2)x^2 + 4sx + 6s^2 - s^3 = 0$$

$$\Rightarrow (x+s)((s-2)x + 6s - s^2) = 0$$

$$\Rightarrow (s-2)x + 6s - s^2 = 0$$

$$(s-2)x + 6s - s^2 = 0$$

$$x = \frac{s^2 - 6s}{s-2}$$

$$= \frac{(s-2)(s-4) - 8}{s-2}$$

$$= s - 4 - \frac{8}{s-2}$$

答案:5

(1)BMO1-2011-1

Find all (positive or negative) integers n for which $n^2 + 20n + 11$ is a perfect square. *Remember that you must justify that you have found them all.*

思路

$$m^2 = n^2 + 20n + 11$$

$$= n^2 + 20n + 100 - 89$$

$$= (n+10)^2 - 89$$

$$(n+10)^2 - m^2 = 89$$

$$(n+10+m)(n+10-m) = 89$$

$n+10+m$	$n+10-m$	$2n+20$	n
89	1	90	35
-89	-1	-90	-55
1	89	90	35
-1	-89	-90	-55

(1)BMO1-2011-5

Prove that the product of four consecutive positive integers cannot be equal to the product of two consecutive positive integers.

思路

两个连续的 square 中间不可能有第三个 square

$$n(n+1)(n+2)(n+3) = m(m+1)$$

$$(n^2+3n)(n^2+3n+2) = m(m+1)$$

$$(n^2+3n+1)^2 - 1 = m(m+1)$$

$$(n^2+3n+1)^2 = m^2 + m + 1$$

$$m^2 < (n^2+3n+1)^2 < (m+1)^2$$

(1)BMO1-2012-3

Find all real numbers x, y and z which satisfy the simultaneous equations $x^2 - 4y + 7 = 0$, $y^2 - 6z + 14 = 0$ and $z^2 - 2x - 7 = 0$.

思路

再次强调!求解丢番图方程后一定要验算!

$$x^2-4y+7=0$$
$$y^2-6z+14=0$$
$$z^2-2x-7=0$$

$$x^2-2x+y^2-4y+z^2-6z+14=0$$
$$(x-1)^2+(y-2)^2+(z-3)^2=0$$
$$x=1 \quad y=2 \quad z=3 \quad ?$$

(1)BMO1-2012-4

Find all positive integers n such that $12n-119$ and $75n-539$ are both perfect squares.

$\boxed{\text{思路}}$

n 太烦人了!所以说我们想要去掉 n

$$4a^2-25b^2=(300n-2156)-(300n-2975)$$
$$=819$$

注意,两个 factors 之间差的应该是 10b 的倍数

$$(2a+5b)(2a-5b)=3\times3\times7\times13$$
$$\cancel{=819\times1}$$
$$=273\times3$$
$$=117\times7$$
$$\cancel{=91\times9}$$
$$=63\times13$$
$$\cancel{=39\times21}$$

$$\left.\begin{aligned}2a+5b&=273\\2a-5b&=3\end{aligned}\right\}\ \left.\begin{aligned}a&=69\\b&=27\end{aligned}\right\}\ n=\frac{212}{3}$$

$$\left.\begin{aligned}2a+5b&=117\\2a-5b&=7\end{aligned}\right\}\ \left.\begin{aligned}a&=31\\b&=11\end{aligned}\right\}\ n=20$$

$$\left.\begin{aligned} 2a+5b&=63 \\ 2a-5b&=13 \end{aligned}\right\} \left.\begin{aligned} a&=19 \\ b&=5 \end{aligned}\right\} n=12$$

(1)BMO1-2013-1

Calculate the value of

$$\frac{2014^4+4\times 2013^4}{2013^2+4027^2}-\frac{2012^4+4\times 2013^4}{2013^2+4025^2}.$$

思路

因为每一项都跟 2013 或多或少有点关系,所以说让 2013=n

$$\frac{5n^4+4n^3+6n^2+4n+1}{5n^2+4n+1} \qquad \frac{5n^4-4n^3+6n^2-4n+1}{5n^2-4n+1}$$

结果是$(n^2+1)-(n^2+1)=0$

(1)BMO1-2014-2

Positive integers p, a and b satisfy the equation $p^2+a^2=b^2$. Prove that if p is a prime greater than 3, then a is a multiple of 12 and $2(p+a+1)$ is a perfect square.

思路

$$p^2+a^2=b^2$$
$$p^2=b^2-a^2$$
$$=(b+a)(b-a)$$
$$\cancel{b+a=b-a=p}$$
$$\begin{cases} b+a=p^2 \\ b-a=1 \end{cases} \Rightarrow \quad a=\frac{p^2-1}{2} \quad b=\frac{p^2+1}{2}$$
$$2a=(p-1)(p+1)$$
$$p-1,\ p,\ p+1$$
$$\Rightarrow 8 \mid 2a$$
$$\Rightarrow 3 \mid 2a$$

$$2(p+a+1)=2p+(p^2-1)+2$$
$$=p^2+2p+1$$
$$=(p+1)^2$$

(1)BMO1-2014-4

Let x be a real number such that $t=x+x^{-1}$ is an integer greater than 2. Prove that $t_n=x^n+x^{-n}$ is an integer for all positive integers n. Determine the values of n for which t divides t_n.

思路

首先写出一个 general 的形式

$$t = x + x^{-1},\quad t_n = x^n + x^{-n}$$

$$t t_k = (x+x^{-1})(x^k + x^{-k})$$

$$= x^{k+1} + x^{-(k+1)} + x^{k-1} + x^{-(k-1)}$$

$$t t_k = t_{k+1} + t_{k-1}$$

$$t_0 = x^0 + x^{-0} = 2$$

然后,如果 m|a,m|b,则 m|a-b

$$t t_{k+2} = t_{k+3} + t_{k+1}$$

$$t(t_{k+2} - t_k) = t_{k+3} - t_{k-1}$$

$$t_{12} - t_4 = (t_{12} - t_8) + (t_8 - t_4)$$

$$4 \mid i-j \Rightarrow t \mid t_i - t_j$$

$t \mid t_k \iff k$ is odd

$t_0 = 2$ ✗

$t_1 = t$ ✓

$t_2 = t t_1 - t_0$ ✗

$t_3 = t t_2 - t_1$ ✓

(1)BMO1-2016-3

Determine all pairs (m, n) of positive integers which satisfy the equation $n^2 - 6n = m^2 + m - 10$.

思路

$$n^2 - 6n = m^2 + m - 10$$

$$\Rightarrow (n-3)^2 = m^2 + m - 1$$

$$\Rightarrow (n-3)^2 - m^2 = m - 1$$

$(n-3)^2 - m^2 = m - 1$

Case 1 : $m = 1$

Eq becomes $(n-3)^2 = 1$

$\Rightarrow n - 3 = \pm 1$

$\Rightarrow n = 2$ or $n = 4$

$(n-3)^2 - m^2 = m - 1$

Case 2 : $m \geqslant 2$

Have $(m+1)^2 - m^2$

$= 2m + 1 > m - 1$

– no solutions

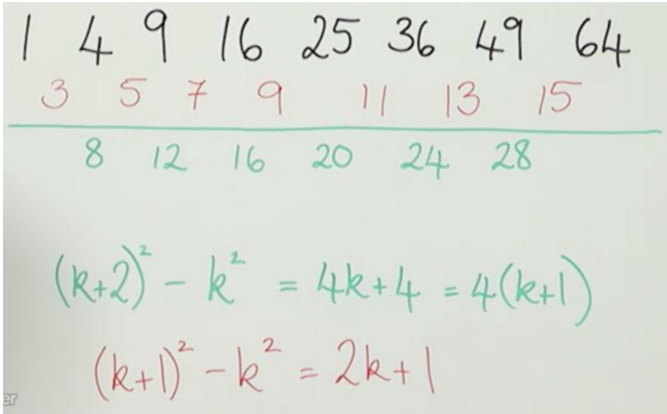
(1)BMO1-2016-4

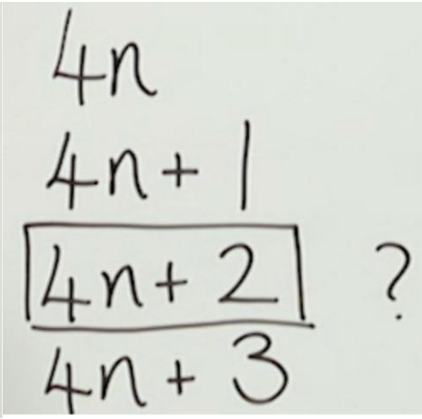
Naomi and Tom play a game, with Naomi going first. They take it in turns to pick an integer from 1 to 100, each time selecting an integer which no-one has chosen before. A player loses the game if, after their turn, the sum of all the integers chosen since the start of the game (by both of them) cannot be written as the difference of two square numbers. Determine if one of the players has a winning strategy, and if so, which.

思路

我们要先探寻平方差具有什么性质

我们发现,平方差不可能是 4n+2 的倍数
因为(a+b)(a-b)只可能是 2 和 4 的倍数,而不能是 2 但不是 4 的倍数





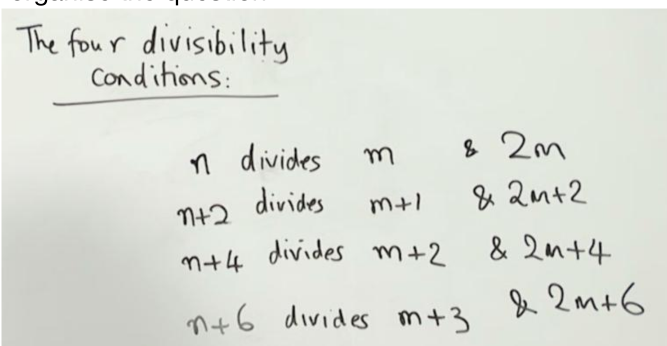
所以说,A 先走,抽 4n,如果 B 抽了 4n+1,A 抽 4n+3,如果 B 抽了 4n,A 就抽 4n

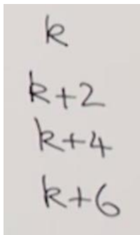
(1)BMO1-2016-6

Consecutive positive integers $m, m+1, m+2$ and $m+3$ are divisible by consecutive odd positive integers $n, n+2, n+4$ and $n+6$ respectively. Determine the smallest possible m in terms of n.

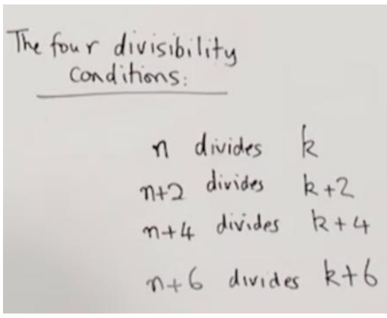
思路

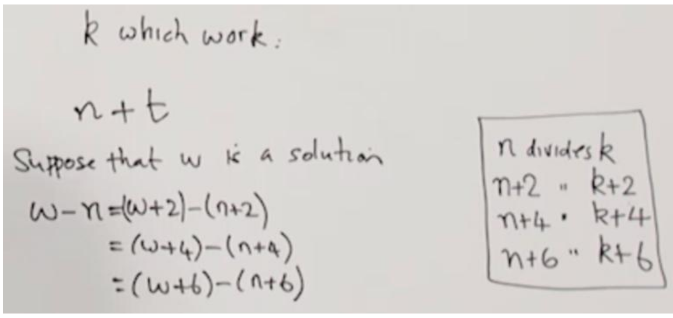
首先,这道题可以用中国剩余定理,但是这里的话我们尝试丢番图
首先我们 re-organise the question



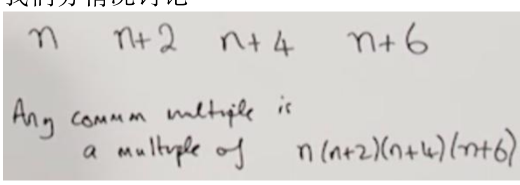


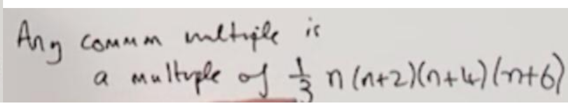
所以说问题转换成了





因此,我们需要找到 k 的所有可能性,然后从里面找到 minimum even 的答案
我们可以看到,k 的第一种结果便是 n(虽然说不符合题意)
我们注意到,如果 3|n, lcm(n,n+2,n+4,n+6)=$\frac{1}{3}$n(n+2)(n+4)(n+6)
我们分情况讨论





Case that 3 does not divide n

$$\frac{1}{2}\left(n + n(n+2)(n+4)(n+6)\right) \quad (\text{is an integer})$$

Case that 3 does divide n.

$$\frac{1}{2}\left(n + \frac{1}{3}\cdot n(n+2)(n+4)(n+6)\right) \quad (\text{is an integer})$$

(1)BMO1-2018-3

Ares multiplies two integers which differ by 9. Grace multiplies two integers which differ by 6. They obtain the same product T. Determine all possible values of T.

思路

$$a(a-9) = g(g-6)$$

$$\left(a-\frac{9}{2}\right)^2 - \frac{81}{4} = (g-3)^2 - 9$$

$$(2a-9)^2 - (2g-6)^2 = 45$$

$$A^2 - G^2 = 45$$

$$(A-G)(A+G) = 45$$

± 1, ± 45

± 3, ± 15

± 9, ± 5

A	23	-23	9	-9	7	-7
a	16	-7	9	0	8	-1
a-9	7	-16	0	9	-1	8
product	112	112	0	0	-8	-8

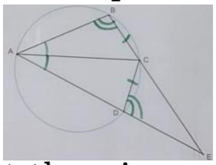
G	22	-22	6	-6	2	-2
g	14	-8	6	0	4	2
g-6	8	-14	0	-6	-2	-4
Product	112	112	0	0	-8	-8

几何部分

1.三角形多边形

(1)BMO1-2005-3

In the cyclic quadrilateral $ABCD$, the diagonal AC bisects the angle DAB. The side AD is extended beyond D to a point E. Show that $CE = CA$ if and only if $DE = AB$.

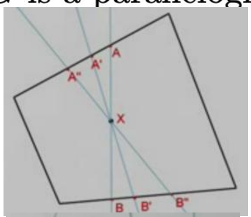


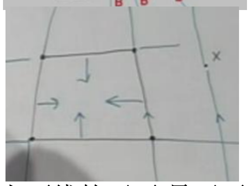
思路

圆内接四边形等角对等边

(2)BMO1-2005-5

Let G be a convex quadrilateral. Show that there is a point X in the plane of G with the property that every straight line through X divides G into two regions of equal area if and only if G is a parallelogram.





思路

难点是用 X 证明 G 是平行四边形,我们只来关注这一条

分成两部分:证明 X 在四边形之内,证明四边形是平行四边形

如果要证明四边形是平行四边形,我们要用到不同的定义

在这里我们选择对边平行,因为边的长度受分割线的影响太大

假设一个四边形和一个 X,如右图

用面积法易证$A_{\Delta A'A''X} = A_{\Delta B'B''X}$

则$\frac{1}{2}\cdot A'X\cdot A''X\cdot\angle A'XA''=\frac{1}{2}\cdot B'X\cdot B''X\cdot\angle B'XB''$

同理,易得$A'X\cdot A''X=B'X\cdot B''X$,$AX\cdot A''X=BX\cdot B''X$, $A'X\cdot AX=B'X\cdot BX$

所以$A'X= A''X$,$B'X = B''X$,$AX=BX$,所以说对边平行(另一对同理)

顺便要提一嘴说 X 必须要在最右边线的左边,最左边线的下面,最上面线的下面,最下面线的上面就是在四边形中间

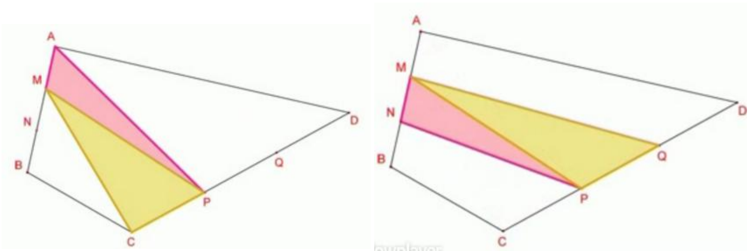
(4)BMO1-2006-2

In the convex quadrilateral $ABCD$, points M, N lie on the side AB such that $AM = MN = NB$, and points P, Q lie on the side CD such that $CP = PQ = QD$. Prove that

$$\text{Area of } AMCP = \text{Area of } MNPQ = \frac{1}{3} \text{ Area of } ABCD.$$

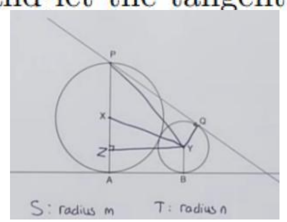
思路

有关于第二部分的证明



(5)BMO1-2006-2

Two touching circles S and T share a common tangent which meets S at A and T at B. Let AP be a diameter of S and let the tangent from P to T touch it at Q. Show that $AP = PQ$.



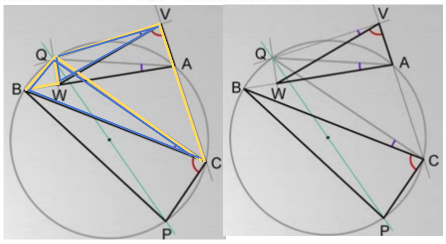
思路

证明两边相等时,除了用全等/相似/四点共圆等外

因为本题垂直较多,可以考虑设数列勾股定理

(6)BMO1-2007-3

Let ABC be a triangle, with an obtuse angle at A. Let Q be a point (other than A, B or C) on the circumcircle of the triangle, on the same side of chord BC as A, and let P be the other end of the diameter through Q. Let V and W be the feet of the perpendiculars from Q onto CA and AB respectively. Prove that the triangles PBC and AWV are similar. [*Note: the circumcircle of the triangle ABC is the circle which passes through the vertices A, B and C.*]



思路

有两个四点共圆,通过 BQ 导角

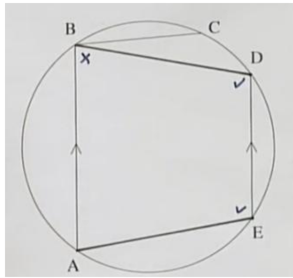
或者连接 BQ,根据四点共圆,∠BQA=∠MQV,所以说∠BQM=∠AQV

因为 RtΔBQW 和 RtΔAQV,$\frac{BQ}{QW}=\frac{CQ}{QV}$,因此$\Delta BCQ \sim \Delta WVQ$

剩下的导两个角就行了

(6)BMO1-2009-2

Points A, B, C, D and E lie, in that order, on a circle and the lines AB and ED are parallel. Prove that $\angle ABC = 90°$ if, and only if, $AC^2 = BD^2 + CE^2$.



思路

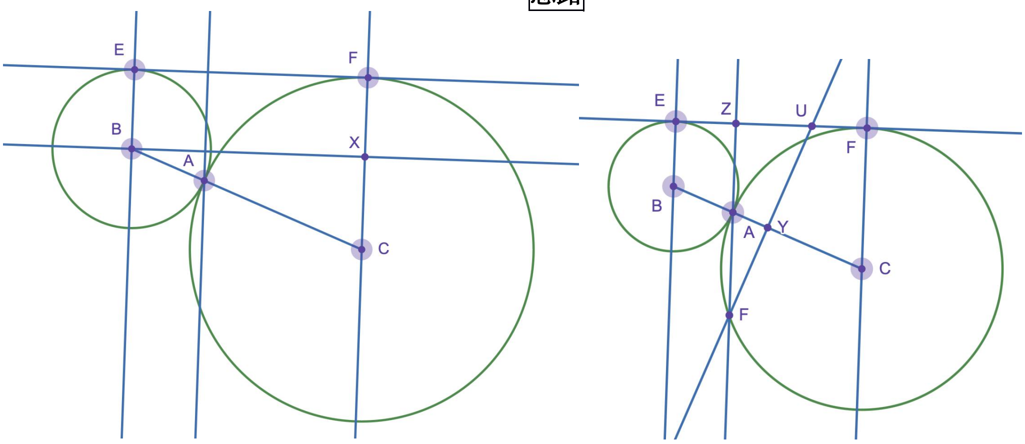
把有关线 AC/BD/CE 转化成直角三角形

因为∠ABD+∠D=∠ABD+∠E=180,∠D=∠E,AE=BD

(6)BMO1-2009-4

Two circles, of different radius, with centres at B and C, touch externally at A. A common tangent, not through A, touches the first circle at D and the second at E. The line through A which is perpendicular to DE and the perpendicular bisector of BC meet at F. Prove that $BC = 2AF$.

思路

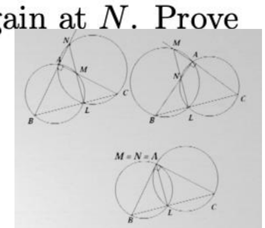


Let BE=a, CF=b, 可以算出 EF

$\Delta ZUF \sim \Delta YAF$, 用 AF 表达出然后算出 AF 就行

(6)BMO1-2010-3

Let ABC be a triangle with $\angle CAB$ a right-angle. The point L lies on the side BC between B and C. The circle ABL meets the line AC again at M and the circle CAL meets the line AB again at N. Prove that L, M and N lie on a straight line.



思路

圆内接四边形,以直径为弦所构成的角为 90 度

主要一定要考虑完全所有情况

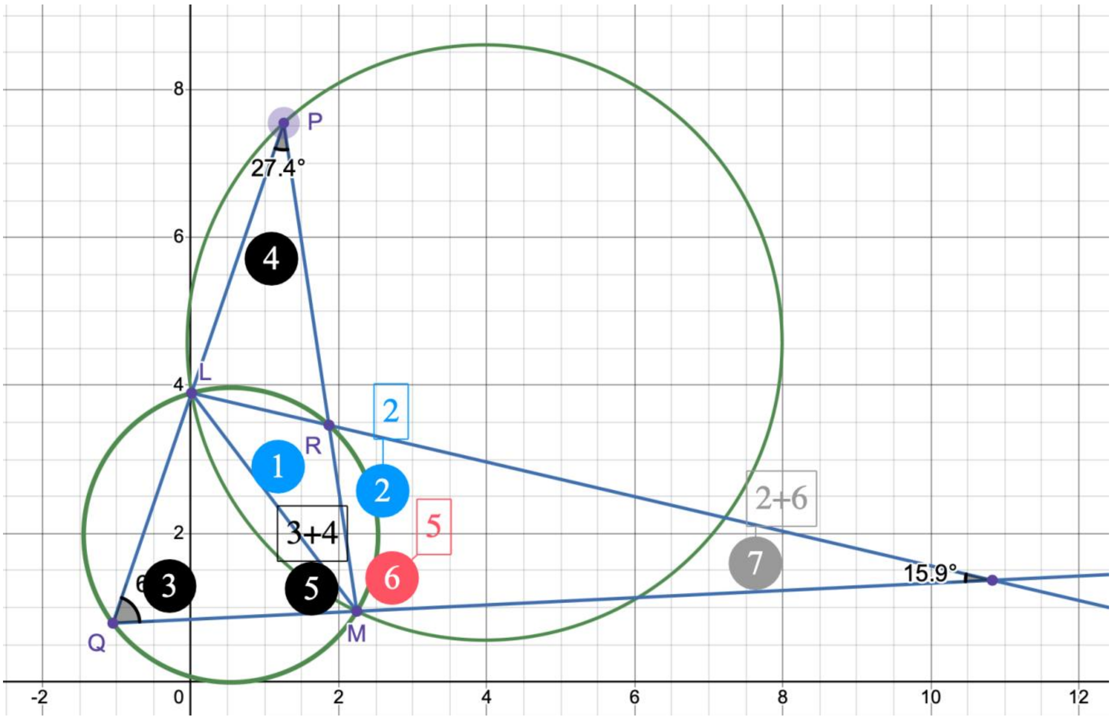
(6)BMO1-2010-5

Circles S_1 and S_2 meet at L and M. Let P be a point on S_2. Let PL and PM meet S_1 again at Q and R respectively. The lines QM and RL meet at K. Show that, as P varies on S_2, K lies on a fixed circle.

思路

我们要证明其在圆上,只需要证明其角度恒不变

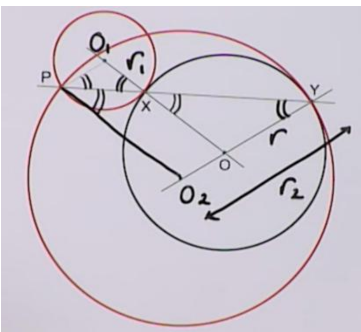
这里面,要证明角不变,那么最后角的组成角肯定也不变,如果要想组成角不变,弦肯定是 ML



(6)BMO1-2011-3

Consider a circle S. The point P lies outside S and a line is drawn through P, cutting S at distinct points X and Y. Circles S_1 and S_2 are drawn through P which are tangent to S at X and Y respectively. Prove that the difference of the radii of S_1 and S_2 is independent of the positions of P, X and Y.

思路



$$\frac{r}{XY} = \frac{r_1}{PX} = \frac{r_2}{PY}$$

$$r_1 = \frac{r \cdot PX}{XY}, \quad r_2 = \frac{r \cdot PY}{XY}$$

$$r_2 - r_1 = \frac{r \cdot PY - r \cdot PX}{XY} = \frac{r}{XY}(PY - PX)$$

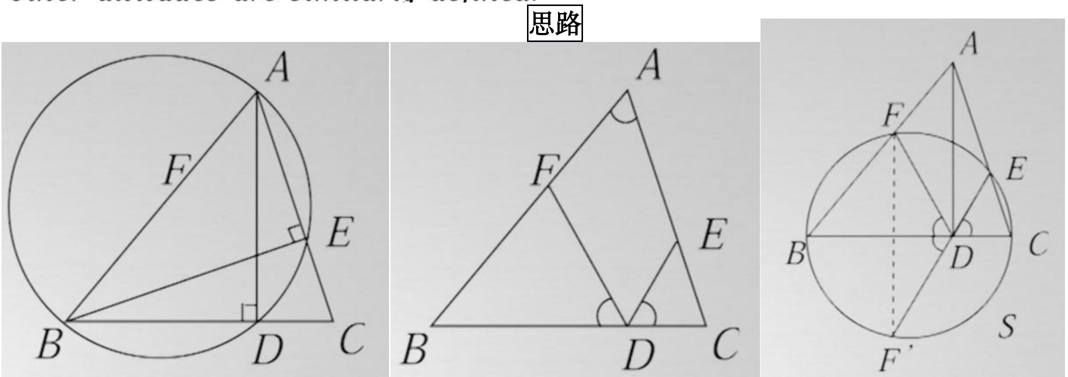
三角函数或相似均可

(6)BMO1-2011-3

Let ABC be an acute-angled triangle. The feet of the altitudes from A, B and C are D, E and F respectively. Prove that $DE + DF \leq BC$ and determine the triangles for which equality holds.

The altitude from A is the line through A which is perpendicular to BC. The foot of this altitude is the point D where it meets BC. The other altitudes are similarly defined.

思路



通过圆的性质倒角,然后导边

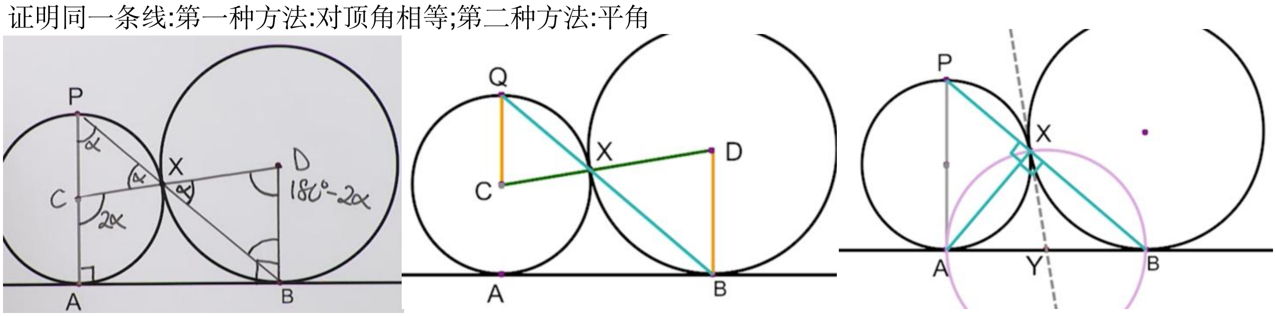
证明三角形边长的不等式,可以尝试用圆

(6)BMO1-2012-2

Two circles S and T touch at X. They have a common tangent which meets S at A and T at B. The points A and B are different. Let AP be a diameter of S. Prove that B, X and P lie on a straight line.

思路

证明同一条线:第一种方法:对顶角相等;第二种方法:平角



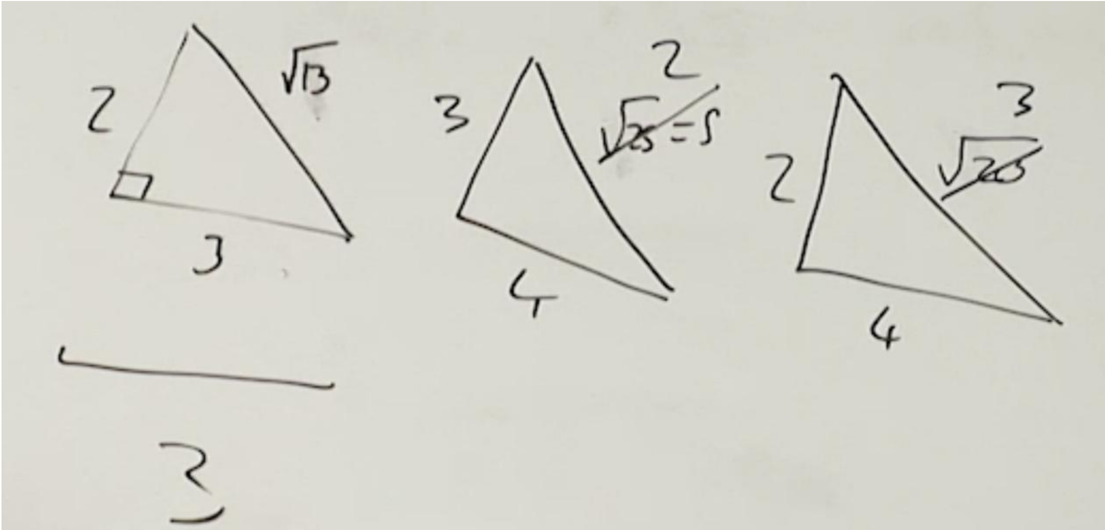
(6)BMO1-2012-5

A triangle has sides of length at most 2, 3 and 4 respectively. Determine, with proof, the maximum possible area of the triangle.

思路

注意,面积最大时不一定周长最大.因此,我们要在固定一边为底时,让高尽可能大

因此,让一条已知边为底,另一条尽可能为直角边

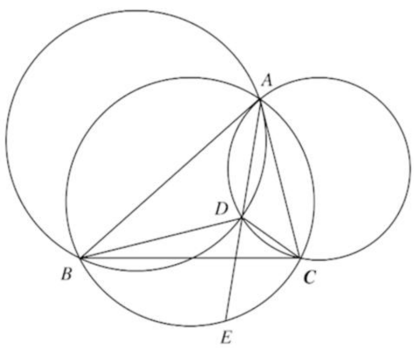


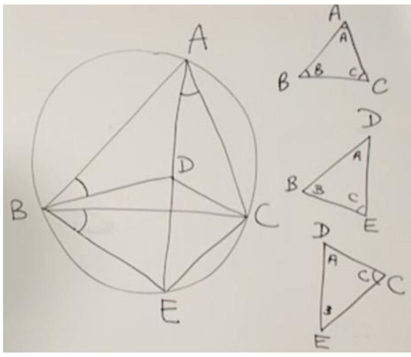
(6)BMO1-2012-6

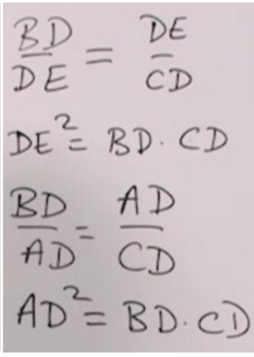
Let ABC be a triangle. Let S be the circle through B tangent to CA at A and let T be the circle through C tangent to AB at A. The circles S and T intersect at A and D. Let E be the point where the line AD meets the circle ABC. Prove that D is the midpoint of AE.

思路

第一种方法,倒角证明相似



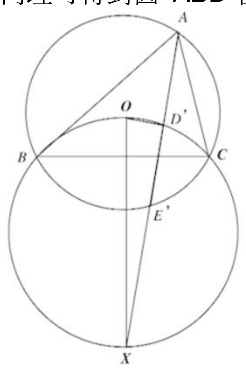


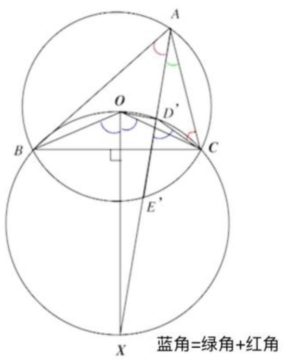


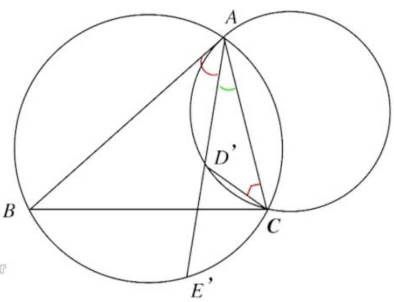
第二种方法,O 是圆心,我们接下来只需要证明 D'是 D 即可

在圆 ACD'在 A 点与 AB tangent,我们用红角=红角即可证明

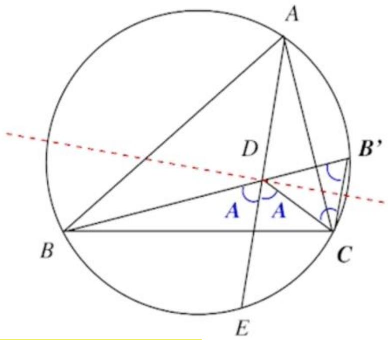
同理可得到圆 ABD'在 A 点与 AC tangent,因此 D'是 D







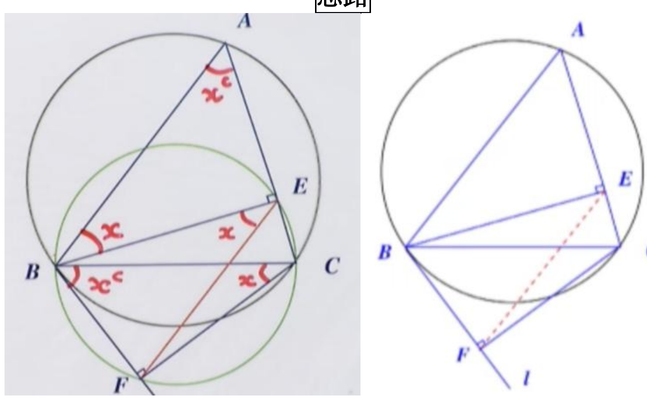
第三种方法,用方法二相似的方法证明两倍角,然后构建等腰三角形后被垂直平分



(6)BMO1-2013-2

In the acute-angled triangle ABC, the foot of the perpendicular from B to CA is E. Let l be the tangent to the circle ABC at B. The foot of the perpendicular from C to l is F. Prove that EF is parallel to AB.

思路



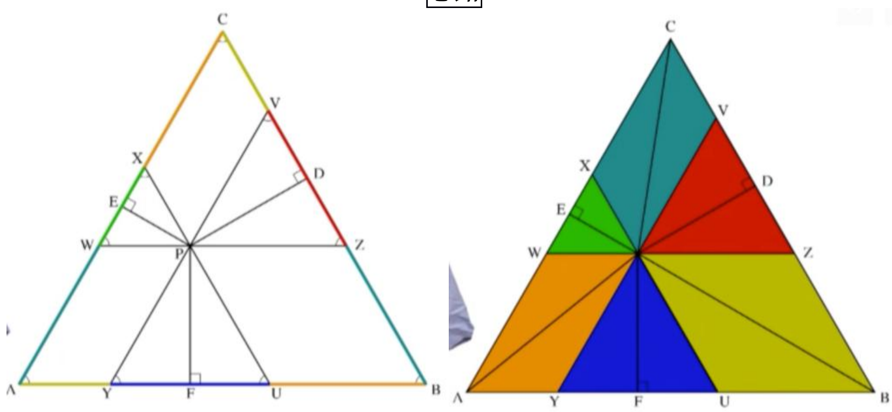
(6)BMO1-2013-5

Let ABC be an equilateral triangle, and P be a point inside this triangle. Let D, E and F be the feet of the perpendiculars from P to the sides BC, CA and AB respectively. Prove that

a) $AF + BD + CE = AE + BF + CD$ and
b) $[APF] + [BPD] + [CPE] = [APE] + [BPF] + [CPD]$.

The area of triangle XYZ is denoted $[XYZ]$.

思路



(6)BMO1-2013-6

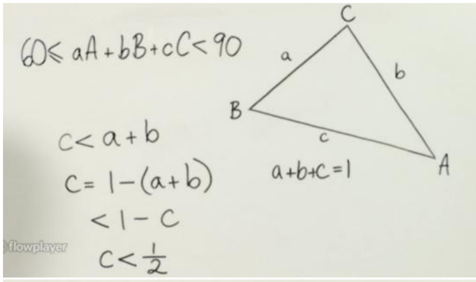
The angles A, B and C of a triangle are measured in degrees, and the lengths of the opposite sides are a, b and c respectively. Prove that

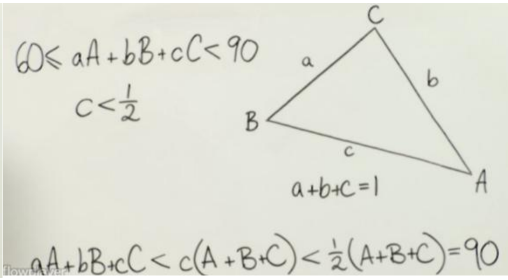
$$60 \leq \frac{aA + bB + cC}{a + b + c} < 90.$$

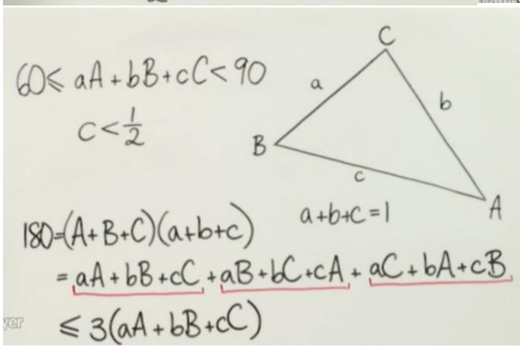
思路

首先假设 a≤b≤c,那么 A≤B≤C

因为 a/b/c 的大小不影响结论,所以说假设 a+b+c=1







因为

$a_1, a_2, \ldots, a_n$

$a_1 \leq a_2 \leq \cdots \leq a_n$

$s = a_1b_1 + a_2b_2 + \cdots + a_nb_n$

Another collection of unsorted real numbers

$b_1, b_2, \ldots, b_n$

s maximized when $b_1 \leq b_2 \leq \cdots \leq b_n$

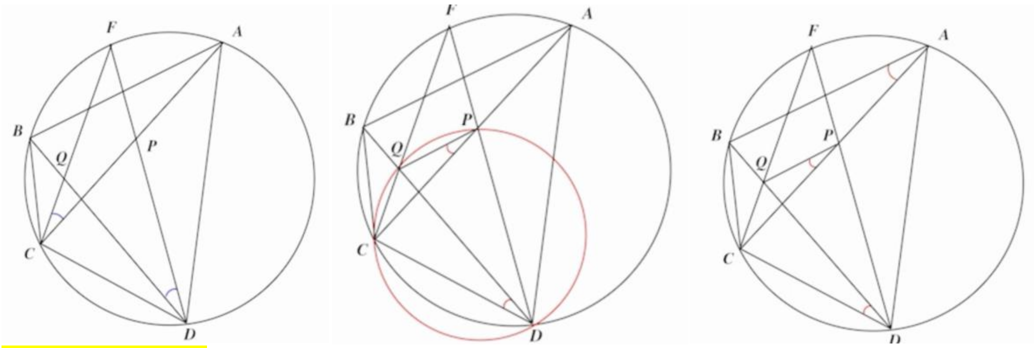
s minimized when $b_1 \geq b_2 \geq \cdots \geq b_n$

(6)BMO1-2014-5

Let $ABCD$ be a cyclic quadrilateral. Let F be the midpoint of the arc AB of its circumcircle which does not contain C or D. Let the lines DF and AC meet at P and the lines CF and BD meet at Q. Prove that the lines PQ and AB are parallel.

思路

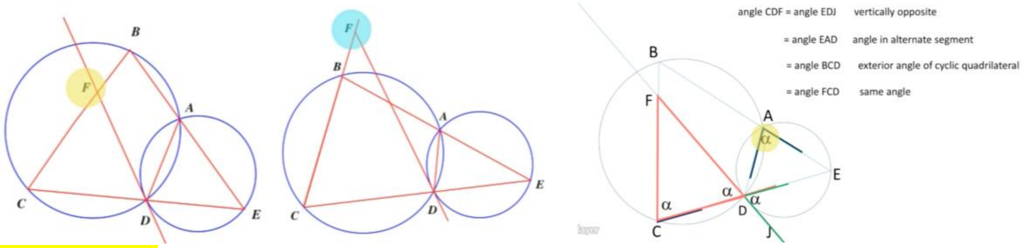
因为 F 为中点,所以说两角相等



(6)BMO1-2015-2

Let $ABCD$ be a cyclic quadrilateral and let the lines CD and BA meet at E. The line through D which is tangent to the circle ADE meets the line CB at F. Prove that the triangle CDF is isosceles.

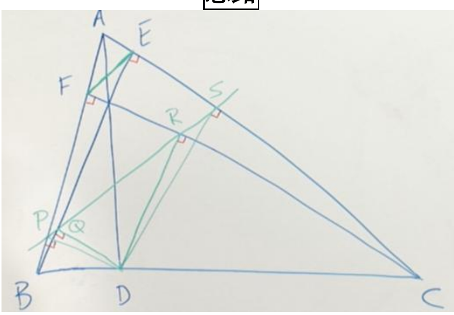
思路



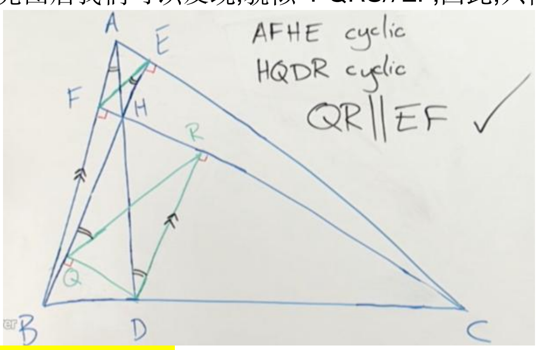
(6)BMO1-2015-5

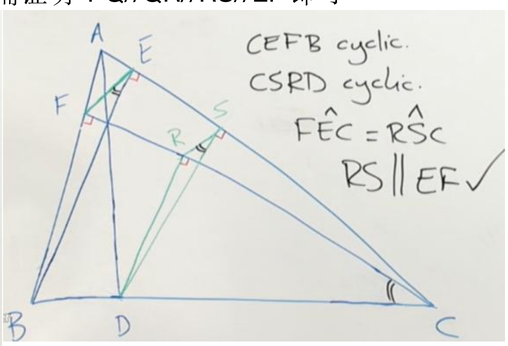
Let ABC be a triangle, and let D, E and F be the feet of the perpendiculars from A, B and C to BC, CA and AB respectively. Let P, Q, R and S be the feet of the perpendiculars from D to BA, BE, CF and CA respectively. Prove that P, Q, R and S are collinear.

思路



画完图后我们可以发现,貌似 PQRS//EF,因此,只需证明 PQ//QR//RS//EF 即可





(6)BMO1-2016-4

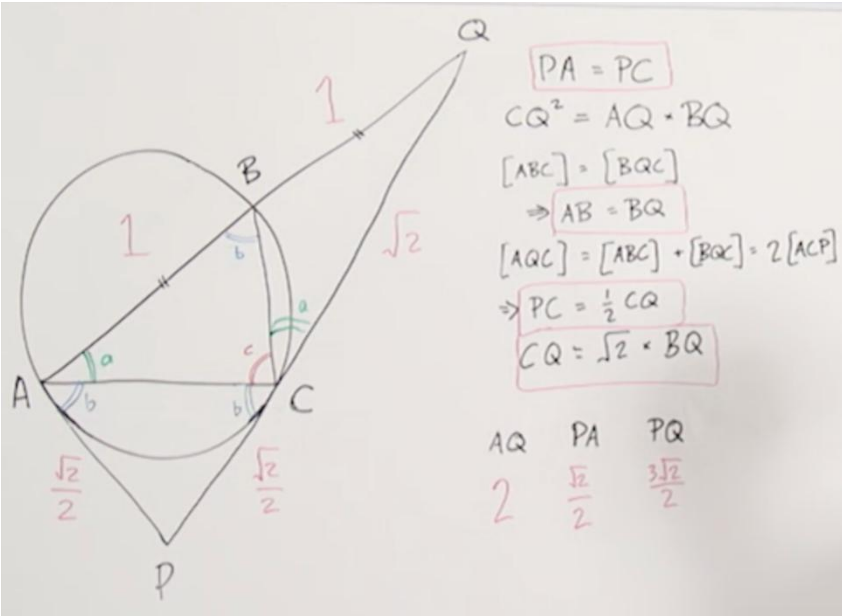
Let ABC be a triangle with $\angle A < \angle B < 90^\circ$ and let Γ be the circle through A, B and C. The tangents to Γ at A and C meet at P. The line segments AB and PC produced meet at Q. It is given that

$$[ACP] = [ABC] = [BQC].$$

Prove that $\angle BCA = 90^\circ$. *Here $[XYZ]$ denotes the area of triangle XYZ.*

思路

经过导角,我们可知 PA=PC,且 a+b 应该为 90
因此,我们只要证明ΔAQP 为 RTΔ即可

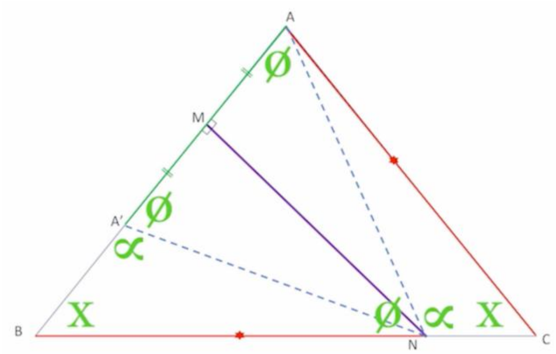


(6)BMO1-2017-3

The triangle ABC has $AB = CA$ and BC is its longest side. The point N is on the side BC and $BN = AB$. The line perpendicular to AB which passes through N meets AB at M. Prove that the line MN divides both the area and the perimeter of triangle ABC into equal parts.

思路

证明全等即可,多种方法



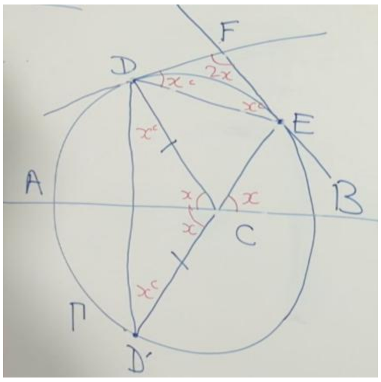
(6)BMO1-2018-4

Let Γ be a semicircle with diameter AB. The point C lies on the diameter AB and points E and D lie on the arc BA, with E between B and D. Let the tangents to Γ at D and E meet at F. Suppose that $\angle ACD = \angle ECB$.

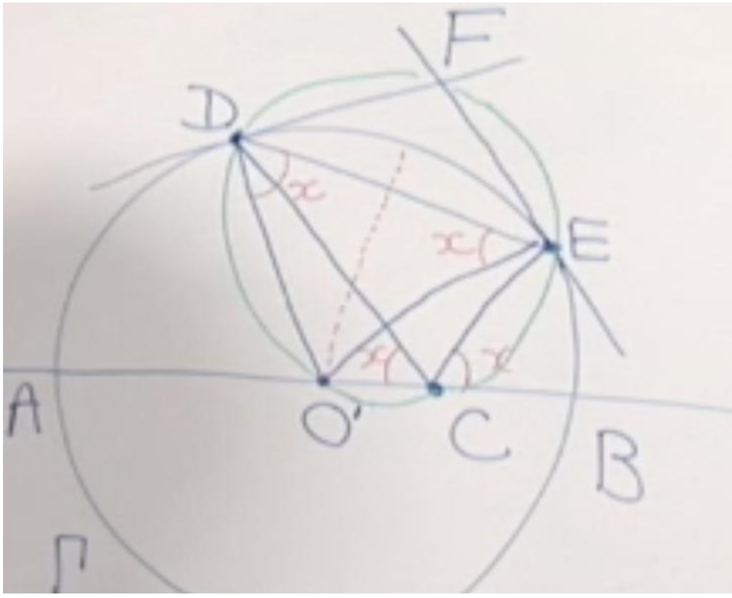
Prove that $\angle EFD = \angle ACD + \angle ECB$.

$\boxed{思路}$

第一种思路是把两个角导到一块



第二种思路是证明 CDFE 共圆即可



我们只需要证明 O'就是圆心即可,得到 O'为圆心后通过两个与切线的直角夹角就可以得到
所以说,我们首先假设 CDE 共圆,通过导角便可知 O'为圆心,即可证明五点共圆

(6)BMO1-2018-5

Two solid cylinders are mathematically similar. The sum of their heights is 1. The sum of their surface areas is 8π. The sum of their volumes is 2π. Find all possibilities for the dimensions of each cylinder.

$\boxed{思路}$

$$h(1+x)=1$$
$$r(r+h)(1+x^2)=4$$
$$r^2(1-x+x^2)=2$$
$$h=\frac{1}{1+x}$$
$$r=\sqrt{\frac{2}{1-x+x^2}}$$

$$\sqrt{\frac{2}{1-x+x^2}}\left(\sqrt{\frac{2}{1-x+x^2}}+\frac{1}{1+x}\right)(1+x^2)=4$$
$$\sqrt{2}\left(\sqrt{2}(1+x)+\sqrt{1-x+x^2}\right)(1+x^2)=4(1+x^3)$$
$$\left(2(1+x)+\sqrt{2(1-x+x^2)}\right)(1+x^2)=4(1+x^3)$$
$$\begin{aligned}\sqrt{2(1-x+x^2)}(1+x^2)&=4(1+x^3)-2(1+x)(1+x^2)\\&=4+4x^3-2-2x-2x^2-2x^3\\&=2-2x-2x^2+2x^3\\&=2(1-x-x^2+x^3)\end{aligned}$$

$$x^6-3x^5-5x^4+10x^3-5x^2-3x+1=0$$
$$x^3-3x^2-5x+10-5x^{-1}-3x^{-2}+x^{-3}=0$$
$$t=x+x^{-1}$$
$$t^3-3t^2-8t+16=0$$
$$(t-4)(t^2+t-4)=0$$
$$\underline{t=4} \qquad \cancel{t=\frac{-1\pm\sqrt{17}}{2}}$$

$$\left(\sqrt{x}-\frac{1}{\sqrt{x}}\right)^2\geqslant 0$$
$$x+\frac{1}{x}\geqslant 2$$

$$t=x+x^{-1}$$
$$x=2\pm\sqrt{3}$$
$$h=\frac{3\pm\sqrt{3}}{6}$$
$$r=\frac{3\pm\sqrt{3}}{3}$$

(6)BMO1-2019-3

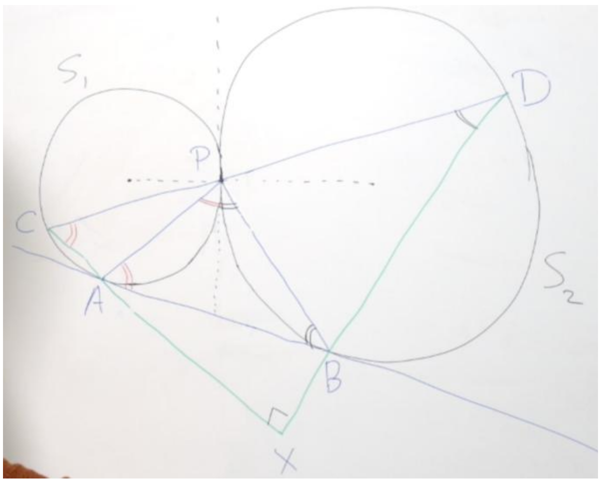
Two circles S_1 and S_2 are tangent at P. A common tangent, not through P, touches S_1 at A and S_2 at B. Points C and D, on S_1 and S_2 respectively, are outside the triangle APB and are such that P is on the line CD.

Prove that AC is perpendicular to BD.

$\boxed{\text{思路}}$

要想证明∠X=90,证明∠C+∠D=90 即可

一定要用上全部条件！！！



数论部分

1.整除

(1)BMO1-2013-3

A number written in base 10 is a string of 3^{2013} digit 3s. No other digit appears. Find the highest power of 3 which divides this number.

思路

答案:3^{2014}

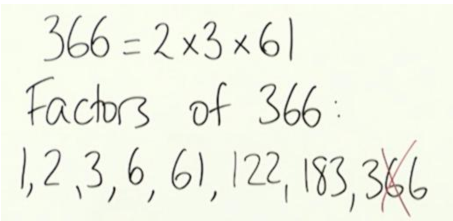
(1)BMO1-2017-1

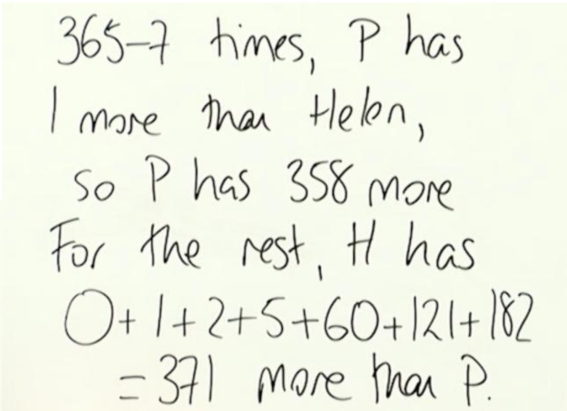
Helen divides 365 by each of $1, 2, 3, \ldots, 365$ in turn, writing down a list of the 365 remainders. Then Phil divides 366 by each of $1, 2, 3, \ldots, 366$ in turn, writing down a list of the 366 remainders. Whose list of remainders has the greater sum and by how much?

思路

我们发现,如果一个数能够整除 336,那么这个数的 remainder 骤降到 0

否则 336remainder 大 1

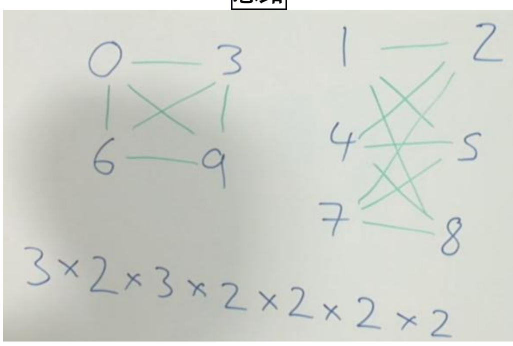




(1)BMO1-2018-1

A list of five two-digit positive integers is written in increasing order on a blackboard. Each of the five integers is a multiple of 3, and each digit 0,1,2,3,4,5,6,7,8,9 appears exactly once on the blackboard. In how many ways can this be done? *Note that a two-digit number cannot begin with the digit 0.*

思路

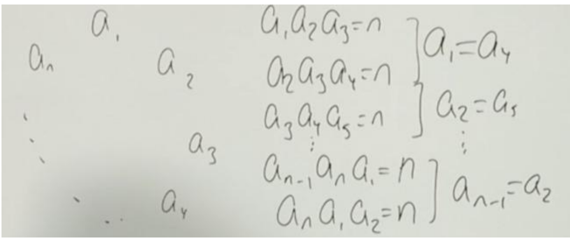


(1)BMO1-2018-2

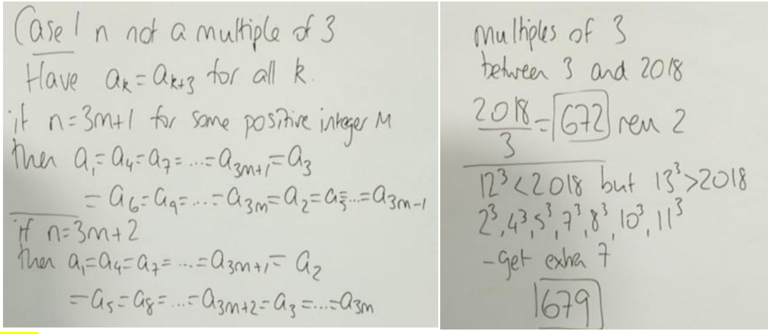
For each positive integer $n \geq 3$, we define an n-ring to be a circular arrangement of n (not necessarily different) positive integers such that the product of every three neighbouring integers is n. Determine the number of integers n in the range $3 \leq n \leq 2018$ for which it is possible to form an n-ring.

思路

loop around



随后我们便可以分类讨论:n 是否被 3 整除

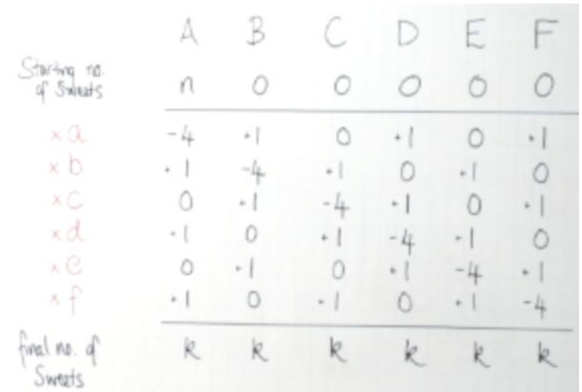


(1)BMO1-2019-5

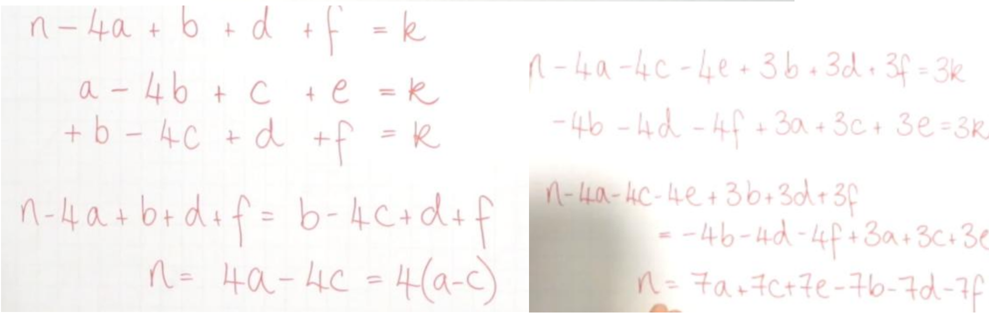
Six children are evenly spaced around a circular table. Initially, one has a pile of $n > 0$ sweets in front of them, and the others have nothing. If a child has at least four sweets in front of them, they may perform the following move: eat one sweet and give one sweet to each of their immediate neighbours and to the child directly opposite them. An arrangement is called *perfect* if there is a sequence of moves which results in each child having the same number of sweets in front of them. For which values of n is the initial arrangement perfect?

思路

我们首先画一个表格整理一下变化



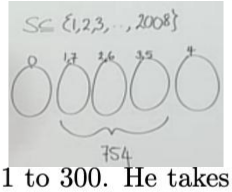
	A	B	C	D	E	F
Starting no. of Sweets	n	0	0	0	0	0
×a	-4	+1	0	+1	0	+1
×b	+1	-4	+1	0	+1	0
×c	0	+1	-4	+1	0	+1
×d	+1	0	+1	-4	+1	0
×e	0	+1	0	+1	-4	+1
×f	+1	0	+1	0	+1	-4
final no. of Sweets	k	k	k	k	k	k



2.同余

(1)BMO1-2007-4

Let S be a subset of the set of numbers $\{1, 2, 3, ..., 2008\}$ which consists of 756 distinct numbers. Show that there are two distinct elements a, b of S such that $a + b$ is divisible by 8.



思路

先 mod 分组,分完组后就是鸽笼定理

(3)BMO1-2017-6

Matthew has a deck of 300 cards numbered 1 to 300. He takes cards out of the deck one at a time, and places the selected cards in a row, with each new card added at the right end of the row. Matthew must arrange that, at all times, the mean of the numbers on the cards in the row is an integer. If, at some point, there is no card remaining in the deck which allows Matthew to continue, then he stops.

When Matthew has stopped, what is the smallest possible number of cards that he could have placed in the row? Give an example of such a row.

思路

相当于取完 n-1 个 cards 之后,我们对 sum 进行 mod n,然后补余数

因此,如果我们可以用完用来补的余数项的话就停止了

因为18^2>300,因此其每组用来补的余数项只有 17 个(maximum)

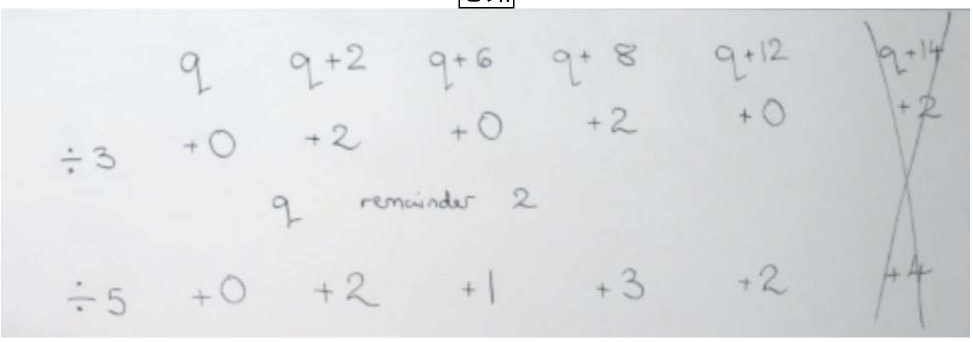
当我们取完 17 个数然后想要第 18 个余数项的时候就不可能了,所以说 18 不可能

答案:17

(1)BMO1-2019-1

Show that there are at least three prime numbers p less than 200 for which $p + 2$, $p + 6$, $p + 8$ and $p + 12$ are all prime. Show also that there is only one prime number q for which $q + 2$, $q + 6$, $q + 8$, $q + 12$ and $q + 14$ are all prime.

思路



3.算数基本定理及约数定理

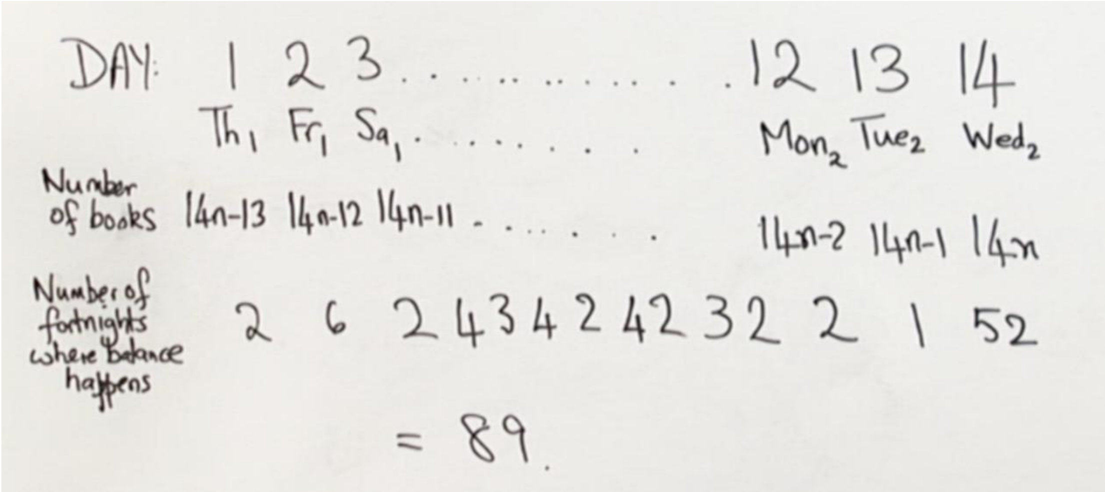
(1)BMO1-2015-1

On Thursday 1st January 2015, Anna buys one book and one shelf. For the next two years, she buys one book every day and one shelf on alternate Thursdays, so she next buys a shelf on 15th January 2015. On how many days in the period Thursday 1st January 2015 until (and including) Saturday 31st December 2016 is it possible for Anna to put all her books on all her shelves, so that there is an equal number of books on each shelf?

思路

分成 52 个 14 天和剩下的 3 天,对于每一组 14 天,有 n 个 shelf

要想 n 整除书的数量,n 要能够整除后面常数项,即求后面常数项的因数个数



4.裴蜀定理

5.最大公因数与最小公倍数

6.n 进制

7.中国剩余定理

概率部分

1.排列组合概率

(1)BMO1-2005-2

Adrian teaches a class of six pairs of twins. He wishes to set up teams for a quiz, but wants to avoid putting any pair of twins into the same team. Subject to this condition:

ii) In how many ways can he split them into three teams of four?

思路

第一种方法是因为是 3 组 4 人,所以说每两组之间必定有两组 twins 都同时存在,那么一共有 6C2×4C2 种分法,但是因为 order of group doesn't matter,所以说要除以 6 来取消重复;然后就是每一组都有 2 种选择进行分配自己组的名额了

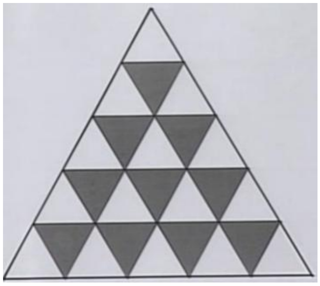
第二种方法是假设 3 组 6 人,就是说现在有 6 组三胞胎,最后我们把老三拿走就行;现在我们先确定老三,那么每组有 2 个老三,一共有 6C2×4C2 种分法,但是因为组的顺序不重要,再除以 6 来消除重复;随后再确定老大老二在另外两组的位置;最后我们把老三拿走就行(不影响结果)

答案:980

(2)BMO1-2005-4

The equilateral triangle ABC has sides of integer length N. The triangle is completely divided (by drawing lines parallel to the sides of the triangle) into equilateral triangular cells of side length 1.

A continuous route is chosen, starting inside the cell with vertex A and always crossing from one cell to another through an edge shared by the two cells. No cell is visited more than once. Find, with proof, the greatest number of cells which can be visited.



思路

因为要证明,所以说我们要通过理论去证明结论

与图形有关的情况下,涂色法是个不错的选择

走的顺序是:..-黑-白-黑-白-...

为了 maximise,黑色要走完,开头结尾都是白

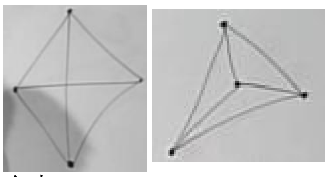
然后问题就转换成了有多少个黑色三角形

答案:n^2-n+1

(3)BMO1-2005-6

Let T be a set of 2005 coplanar points with no three collinear. Show that, for any of the 2005 points, the number of triangles it lies strictly within, whose vertices are points in T, is even.

思路



这种复杂题,先考虑简单情况

假设有四个点,那么一共有两种组合方式(右图)

但是对于这两种组合方式,不管多余的点放在哪里

都会在 0 个或 2 个三角形中间

所以说,我们把 2005 个点看成 1 个点(设为 S)+2004 个点

2004 个点中可以找出 n 组 4 个点来,然后在每一组内,S 都在偶数个三角形内(0 或 2)

将这些 n 组的结果相加后除以 2001(每个三角形被重复计算了 2001 次),结果一定是偶

(不可能是小数因为不符合现实意义)

(3)BMO1-2006-3

The number 916238457 is an example of a nine-digit number which contains each of the digits 1 to 9 exactly once. It also has the property that the digits 1 to 5 occur in their natural order, while the digits 1 to 6 do not. How many such numbers are there?

思路

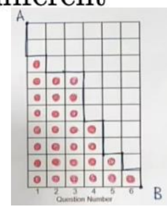
插空法,一个一个来

先排好 12345,然后 6 有 5 个空,7 有 7 个,8 有 8 个,9 有 9 个

答案：2520

(3)BMO1-2009-3

Isaac attempts all six questions on an Olympiad paper in order. Each question is marked on a scale from 0 to 10. He never scores more in a later question than in any earlier question. How many different possible sequences of six marks can he achieve?



思路

将分数分布转化成 path

或者用伪隔板法,每一个题目后面的那个分数代表该题分数

则一共有 10 个下和 6 个右,随后排列组合即可

答案:8008

(3)BMO1-2010-4

Isaac has a large supply of counters, and places one in each of the 1×1 squares of an 8×8 chessboard. Each counter is either red, white or blue. A particular pattern of coloured counters is called an *arrangement*. Determine whether there are more arrangements which contain an even number of red counters or more arrangements which contain an odd number of red counters. *Note that* 0 *is an even number.*

思路

$$O_{64} = \binom{64}{1}2^{63} + \binom{64}{3}2^{61} + \cdots + \binom{64}{63}2^1$$

$$E_{64} = \binom{64}{0}2^{64} + \binom{64}{2}2^{62} + \cdots + \binom{64}{64}$$

$$E_{64} - O_{64} = \binom{64}{0}2^{64} - \binom{64}{1}2^{63} + \cdots - \binom{64}{63}2 + \binom{64}{64}$$

$$= (2-1)^{64}$$

$$= 1^{64}$$

(3)BMO1-2012-1

Isaac places some counters onto the squares of an 8 by 8 chessboard so that there is at most one counter in each of the 64 squares. Determine, with justification, the maximum number that he can place without having five or more counters in the same row, or in the same column, or on either of the two long diagonals.

$\boxed{\text{思路}}$

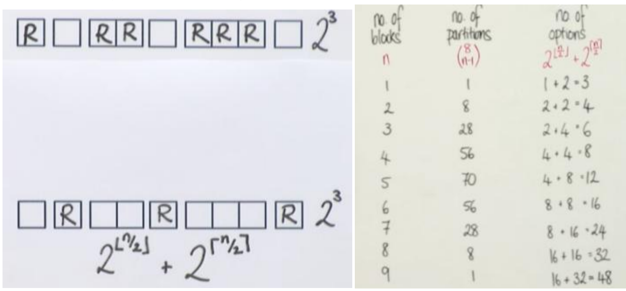
答案：32

(3)BMO1-2013-4

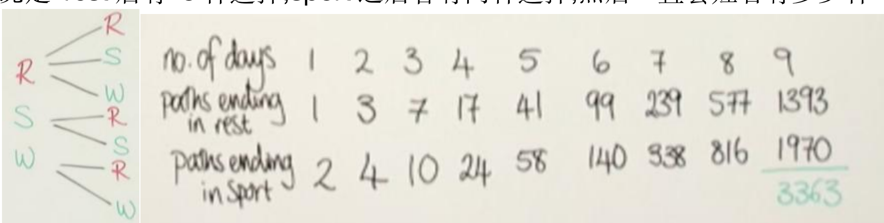
Isaac is planning a nine-day holiday. Every day he will go surfing, or water skiing, or he will rest. On any given day he does just one of these three things. He never does different water-sports on consecutive days. How many schedules are possible for the holiday?

$\boxed{\text{思路}}$

第一种思路是分成多少段时间,然后考虑每段时间的选择



第二种情况是 rest 后有 3 种选择,sport 之后各有两种选择,然后一直套娃看有多少种可能性



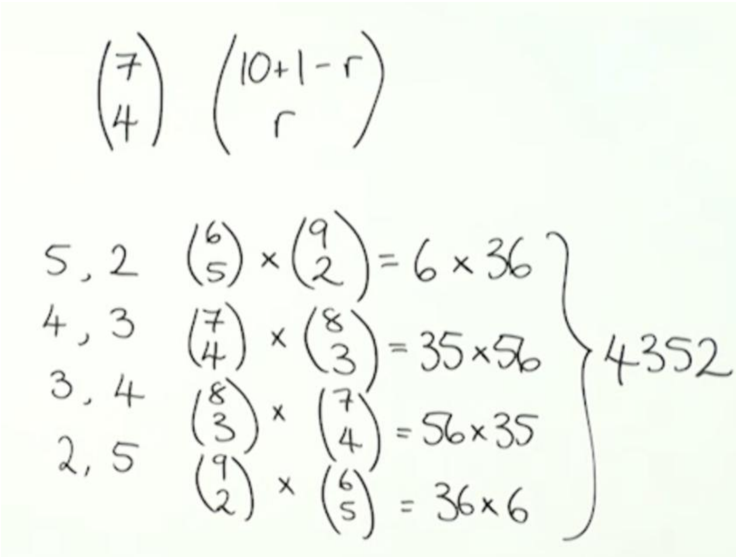
(3)BMO1-2014-3

A hotel has ten rooms along each side of a corridor. An olympiad team leader wishes to book seven rooms on the corridor so that no two reserved rooms on the same side of the corridor are adjacent. In how many ways can this be done?

$\boxed{\text{思路}}$

看成每个人预定两个房间且只睡在左边的房间,为了能住在第 10 间放我们虚构一个第 11 间

因此,本质上便是从 11-4 个元素中选择 3 个空房间



(3)BMO1-2015-4

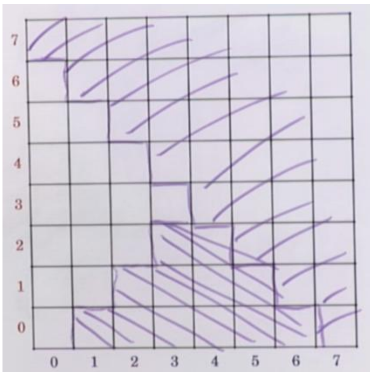
James has a red jar, a blue jar and a pile of 100 pebbles. Initially both jars are empty. A move consists of moving a pebble from the pile into one of the jars or returning a pebble from one of the jars to the pile. The numbers of pebbles in the red and blue jars determine the *state* of the game. The following conditions must be satisfied:

a) The red jar may never contain fewer pebbles than the blue jar;
b) The game may never be returned to a previous state.

What is the maximum number of moves that James can make?

思路

我们把我们的操作在一个坐标轴上表示出来
假设 x 轴代表 blue jar 的球,y 轴代表 red jar 的球
放入/去除 red jar 的球表示上下移动,放入/去除 blue jar 的球表示右左移动
然后,我们根据条件,把不符合的格子划去,随后该问题就变成了一个类似下图的走格子问题
答案:2550



(3)BMO1-2016-1

The integers $1, 2, 3, \ldots, 2016$ are written down in base 10, each appearing exactly once. Each of the digits from 0 to 9 appears many times in the list. How many of the digits in the list are odd? *For example, 8 odd digits appear in the list* $1, 2, 3, \ldots, 11$.

思路

注意,我们只要看 odd 即可,因此我们在前面补 0 对结果没有影响

00	10	20	30	40	50	60	70	80	90
01	11	21	31	41	51	61	71	81	91
02	12	22	32	42	52	62	72	82	92
03	13	23	33	43	53	63	73	83	93
04	14	24	34	44	54	64	74	84	94
05	15	25	35	45	55	65	75	85	95
06	16	26	36	46	56	66	76	86	96
07	17	27	37	47	57	67	77	87	97
08	18	28	38	48	58	68	78	88	98
09	19	29	39	49	59	69	79	89	99

half of each of the **hundreds, tens** and **units** digits are odd.

and so we have $\frac{3000}{2}$ that is 1500 odd digits in the integers from 0 to 999.

And so, when we consider the integers from 1 to 1999 as four digit integers, we see that

we have 1500 odd digits from 1 to 999 then (1000 + 1500) odd digits from 1000 to 1999.

So, for 1 to 2016 we have these 4000 odd digits plus the extra from 2000 onwards, 15 of these, giving a grand total of

4015 odd digits in the integers from 1 to 2016.

(3)BMO1-2017-2

In a 100-day period, each of six friends goes swimming on exactly 75 days. There are n days on which at least five of the friends swim. What are the largest and smallest possible values of n?

思路

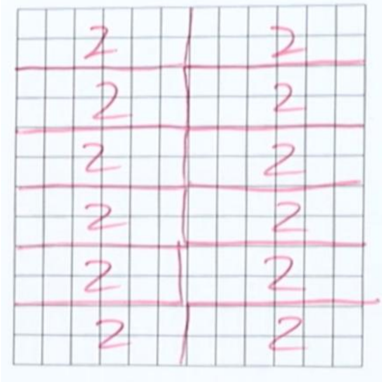
鸽笼原理,记得最后要验证极端情况是否可行
答案:90,25

(3)BMO1-2017-2

If we take a 2×100 (or 100×2) grid of unit squares, and remove alternate squares from a long side, the remaining 150 squares form a 100-*comb*. Henry takes a 200×200 grid of unit squares, and chooses k of these squares and colours them so that James is unable to choose 150 uncoloured squares which form a 100-comb. What is the smallest possible value of k?

思路

首先,200×200 太大了,我们先考虑一个小点的情况
我们发现,每个单位区域内至少要有 2 个格子
经过实验,我们发现当对角填涂时最为有效
答案:400

(3)BMO1-2018-6

Ada the ant starts at a point O on a plane. At the start of each minute she chooses North, South, East or West, and marches 1 metre in that direction. At the end of 2018 minutes she finds herself back at O. Let n be the number of possible journeys which she could have made. What is the highest power of 10 which divides n?

$\boxed{\text{思路}}$

相当于我走了 r 个 north 的话就得走 r 个 south,因此,总个数为

$$\sum_{r=0}^{1009} {}^{2018}C_{2r} \times {}^{2r}C_{r} \times {}^{2018-2r}C_{1009-r}$$

但是这一表达式是用求和写的,我们希望不用加法
我们发现,我们可以用 x+y 和 x-y 这两组数据表达一个坐标
而且因为结局肯定是 x+y=x-y=0,因此 x+y 中有 1009 个+1,x-y 中有 1009 个+1

You change each of x+y and x−y by ±1 at each step, and you can make these choices independently because each of the four possibilities is realized by Ada the ant moving in one of the four cardinal directions.

Change in x+y	Change in x−y	Do this by moving
+1	+1	East
+1	−1	North
−1	+1	South
−1	−1	West

$$\left({}^{2018}C_{1009}\right)^2$$

$$\left({}^{2018}C_{1009}\right)^2 = 10^4 \times *$$

$${}^{2018}C_{1009} = \frac{2018\times 2017\times \cdots \times 3\times 2\times 1}{(1009\times 1008\times \cdots \times 3\times 2\times 1)^2} = 5^2 \times * = 10^2 \times *$$

$$\frac{2015}{5} = 403 \qquad \frac{1005}{5} = 201$$

$$\frac{2000}{25} = 80 \qquad \frac{1000}{25} = 40$$

$$\frac{2000}{125} = 16 \qquad \frac{1000}{125} = 8$$

$$\frac{1875}{625} = 3 \qquad \frac{625}{625} = 1$$

$$502 \qquad 250$$