

Q1

Level 1.5 - 1

How many values of $x \in \mathbb{R}$ are there satisfying the equation

$$x^2 - \sqrt{(x-1)^3} - \sqrt{x-1} + 2(1-x) = 0?$$

$$x^2 - 2x + 2 - \sqrt{(x-1)^3} - \sqrt{x-1} = 0$$

$$u^4 - u^3 - u + 1 = 0, u = \sqrt{x-1}$$

Q2

Level 1.5 - 3

Prove that $\ln\left(\frac{x+1}{x}\right) > \frac{1}{x+1}, \forall x > 0$ **x=0 + 求导**可得 **x=0** 时, **LHS>RHS**; 求导, **LHS>RHS**所以说 **LHS>RHS****Think about Talyor Expansion yet fail****Q3**

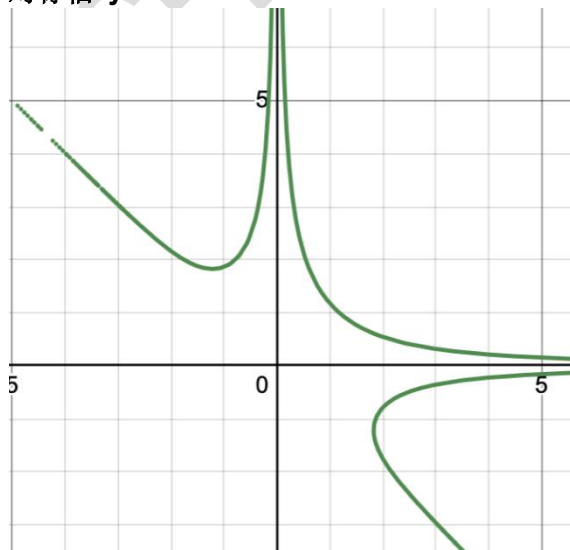
Level 1.5 - 7

Is $(8!)^{\frac{1}{8}} > (9!)^{\frac{1}{9}}$? Or is $(8!)^{\frac{1}{8}} < (9!)^{\frac{1}{9}}$? Or are they even the same设 **8! = x****Q4**

Level 1.5 - 10

Sketch the following curve and identify all of the symmetries, if any:

$$x^2 y^2 = \frac{3}{x+y}$$

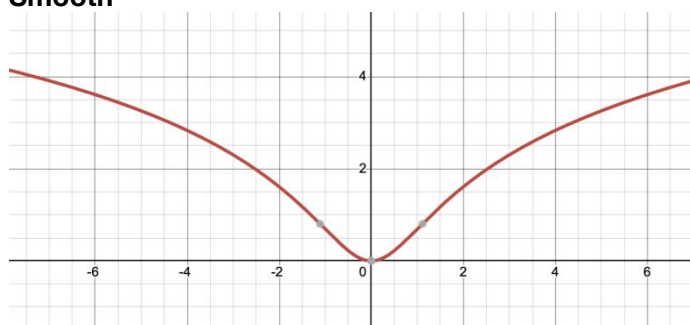
渐近线: **y=-x, x=0, y=0**对称轴: **y=x**

Q5

Level 1,5 - 30

Sketch $y = \ln(1 + x^2)$ and explain its shape.

Smooth



Q6

Level 2 - 1

A fair die is rolled four times. Find the probability that each of the final three rolls is at least as large as the roll preceding it.

考虑六个小球之间隔板

0 0 || 0 0 | 0 0 | means 2 2 4 6

然后算有多少种隔板可能

Q7

Level 2 - 19

Sketch the graph of

$$y = \frac{x^2 e^{-x}}{1+x}, \text{ for } -\infty < x < \infty.$$

Show that the value of

$$\int_0^{\infty} \frac{x^2 e^{-x}}{1+x} dx$$

lies between 0 and 1.

Upper boundary, 考虑 $\frac{(x^2+x)e^{-x}}{1+x}$ 的积分

Q8**Level 2 - 21**

Binary number

1. Convert certain binary number to decimal one.
2. Guess about decimal binary number.
3. Convert 0.101010...into decimal number by geometric series.
4. Convert $\frac{1}{5}$ into binary number

0.2*2=0.4 0.4*2=0.8 0.8*2=0.6+1 0.6*2=0.2+1, 0011 循环***2 相当于 往前移一位****Q9****Level 1 - 22**

$$\text{Prove } \frac{1}{1 \times 2} + \frac{1}{2 \times 3} + \frac{1}{3 \times 4} + \cdots + \frac{1}{n(n+1)} = 1 - \frac{1}{n+1}$$

$$\sum_{n=1}^{\infty} \frac{1}{n(n+1)} = 1$$

$$\text{Prove } 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \cdots - \frac{1}{2n} = \frac{1}{n+1} + \frac{1}{n+2} + \cdots + \frac{1}{2n}$$

$$\sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n} = \log_e 2$$

$$\left[1 + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{2n} \right] - 2 \left[\frac{1}{2} + \frac{1}{4} + \frac{1}{6} + \cdots + \frac{1}{2n} \right]$$

$$\left[1 + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{2n} \right] - \left[1 + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{n} \right]$$

Q10

Level 2 - 25

$a^2 - 2^b = 1$, a and b are integers, find the solutions.

$$a^2 - 1 = (a + 1)(a - 1) = 2^b$$

Q11

Level 2 - 26

What is the probability of 2 people born in the same month if there are 13 people? If there are 12 people?

1-P(every month has one person)

Q12

Level 2 - 27

Randomly choose 5 integers, there must be 3 of them where the sum of these 3 integers is divisible by 3.

Hence prove: Randomly choose 17 integers, the sum of 9 of them can be divisible by 9.

Write general conclusion.

先把 17 个分组: 5 个+5 个+5 个+2 个 得到 **3a, 3b, 3c + 剩下 8 个**

再把 8 个分组: 5 个+3 个 得到 **3d + 剩下 5 个**

最后用 5 个得到 **3e**

然后用 **3a 3b 3c 3d 3e** 得到 **3(3f)**

Q13

Level 1.5 - 45

How many solutions does the equation $x^{-2} \cdot e^x = 1$ have?

化成 $x^2 = e^x$ 画图! 一个解!

Q14

Level 1.5 - 48

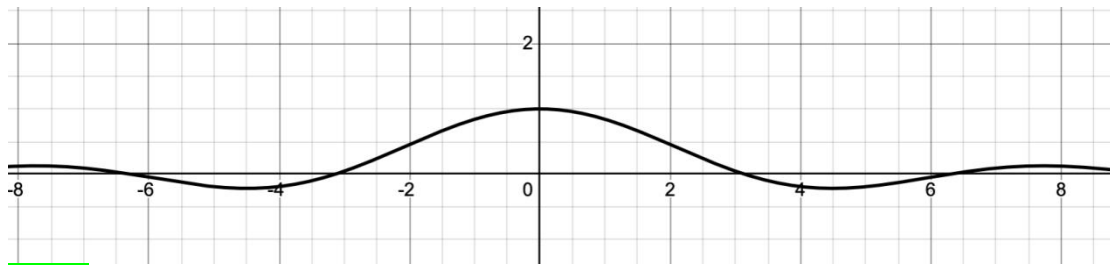
$a < 100$ and a is integer. find a where $100 \mid a(a - 1)$.

Only possibility: one mod 4, one mod 25

Q15

Level 1.5 - 31

Sketch $y = \frac{\sin x}{x}$.



Q16

Level 1.5 - 73

Draw $y = \frac{\ln x}{1-x}$



Q17

Level 2 - 55

a triangle a bee is at A initially same chance/probability to go either other 2 points What's the propability after n^{th} jump for the bee to ger back to A

迭代, $A_n = \frac{1}{2}B_{(n-1)} + \frac{1}{2}C_{(n-1)} = \frac{1}{2}(1 - A_{(n-1)})$

Q18

Level 1.5 - 83

A square cake has a square hole on it. If the hole can be any size, direction at any part of the square cake. Use one straight cut to make it into two pieces with same area (2D geometric question, find a general way).

过中心

Q19

Level 2 - 64

There are 100 levels, with on/off two states. There are 100 frogs passing, the first jump to 1, 2, 3... 100, the second frog 2, 4, 6... 98, ..., n th frog $n, 2n, \dots$. After 100 frogs completing all 100 levels, how many on, and how many off?

考虑因数个数, 非奇数则偶数

也就是说考虑只用 $p_i^{2a_i}$ 能组成多少数, 因为一旦指数是奇数那么该数字就有偶数个因数

Q20

Level 2 - 65

A cuboid with sides a, b, c .

$$\text{Volume} = 2$$

$$\text{Surface area} = 10$$

$$\text{Total length} = 16$$

What is the value of a, b and c ?

化成 polynomial 做题

Q21

Level 2 - 67

$\cos x = Ax + B$ has two solutions

Prove: $\sin x = A$ has at least one solution.

method 1: 求导, $-\sin x = A$ 有至少一个解, so $\sin x = A$ 有至少一个解(odd function)

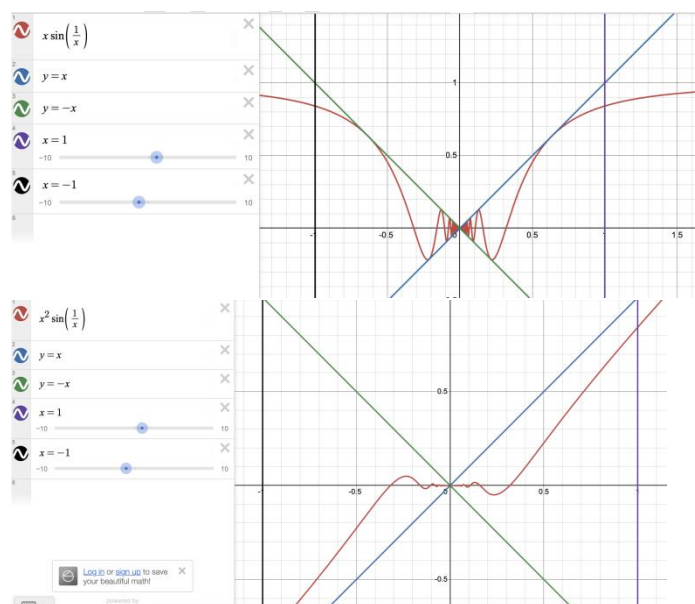
method 2: prove $-1 \leq A \leq 1$, 那么画图 $\cos x$ 然后用 line 证明说只有可能这样

Q22

Level 2 - 68

Draw $x^2 \sin\left(\frac{1}{x}\right)$

What about its limit?



对于 $x \sin \frac{1}{x}$: consider as oscillation, even function, 考虑趋近 $y=1$ 和怎么趋近(有 stationary 吗?)

对于 $x^2 \sin \frac{1}{x}$: consider as oscillation, odd function, 考虑趋近 $y=\infty$ (based on aforementioned) 和怎么趋近(有 stationary 吗?)

Q23

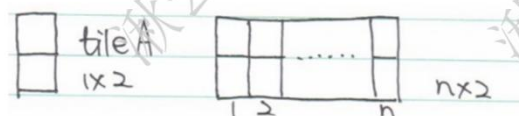
Level 2 - 73

On a plane with n points, can one always start from any point and connect all points in sequence without crossing?

如果是曲线连接的话就比较直接, 就是以任意一点做圆心然后画半径为 r_i 的圆然后把点的距离从小到大排列就行

Q24

Level 2 - 77



Let α_n be the number of ways to use tile A to cover on area of size $1 \times n$.

Find α_n .

迭代, $A_n = A_{(n-1)} + A_{(n-2)}$

Q25

Level 2 - 81

$f(1) = 2$, $f(-2) = -19$, then, what is the remainder of $\frac{f(x)}{(x-1)(x+2)}$?

$$\frac{f(x)}{(x-1)(x+2)} = \frac{1}{3} \left(\frac{f(x)}{x-1} - \frac{f(x)}{x+2} \right)$$

Q26

Level 2.5 - 1

Determine the last two digits of the number

$$2^5 + 2^{5^2} + 2^{5^3} + \dots + 2^{5^{1991}}$$

written in decimal notation.

每一项都是 32: 这告诉我们不要把题目想的太难

$$a^{b^c} = (a^b)^c$$

Q27

Level 2.5 - 4

If

$$a_1 a_2 + a_2 a_3 + \dots + a_{n-1} a_n + a_n a_1 = 0, a_i \in \{1, -1\},$$

prove that $4 \mid n$.

当所有 $a_i = 1$, LHS = n

随便 change one sign, old LHS - new LHS = 4

Q28

Level 2.5 - 11

Given that $(\sqrt[3]{4} - \sqrt[3]{2})(\sqrt[3]{a} + \sqrt[3]{b} + \sqrt[3]{c}) = 2$, what does abc equal?

$$(\sqrt[3]{4} - \sqrt[3]{2})(\sqrt[3]{a} + \sqrt[3]{b} + \sqrt[3]{c}) = 2$$

$$\sqrt[3]{a} + \sqrt[3]{b} + \sqrt[3]{c} = \frac{2}{\sqrt[3]{4} - \sqrt[3]{2}} = \frac{2(\sqrt[3]{4}^2 + \sqrt[3]{4}\sqrt[3]{2} + \sqrt[3]{2}^2)}{(\sqrt[3]{4} - \sqrt[3]{2})(\sqrt[3]{4}^2 + \sqrt[3]{4}\sqrt[3]{2} + \sqrt[3]{2}^2)}$$

Q29

Level 2.5 - 15

If each face of a cube is colored with one of 6 different colors, how many ways can it be done?

The correct option is **C 30**

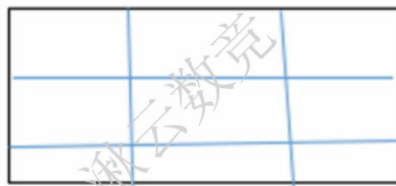
First paint any colour on any face. Now the opposite face can be painted in 5 ways (with anyone of the remaining 5 colours). Now, the remaining 4 faces can be painted with the remaining 4 colours in $(4-1)!$ ways. (circular permutations)
∴ Ans = $5 \times (4 - 1)! = 30$ ways.

Q30

Level 2.5 - 34

There is a white board on a wall which has 9 small squares in it. If I have nine colors to paint it, how many different combinations can I get? Does the number change if I can turn it around?

What if the white board becomes a glass?



Handwritten solution for Q30:

左右对称 q^6

上下对称 q^6

上下左右对称 q^4

不对称: $q^9 - (q^6 + q^6 - q^4)$

共: $\frac{q^9 - (q^6 + q^6 - q^4)}{4} + \frac{q^6}{2} + \frac{q^6}{2} + q^4$

第一种情况: \square 重复有 q^2 个

第二种情况: \square 重复有 $q^3 - q^1$ 个

第三种情况: \square 重复有 $q^5 - q^3$ 个

第四种情况: \square 重复有 $q^9 - q^1$ 个

共: $q^2 + (q^3 - q^1) + \frac{q^5 - q^3}{2} + \frac{q^9 - q^1}{4}$

$\therefore 96870249$

Q31

Level 2.5 - 38

Assume a is a fixed positive integer.

$$\left\lceil \frac{x}{a} \right\rceil = \left\lceil \frac{x}{a-1} \right\rceil$$

where x is a positive integer.

How many values of x can be taken?

$$\text{if } \lceil \frac{x}{a} \rceil = \lceil \frac{x}{a-1} \rceil = k$$

$$(k-1)(a-1) < x \leq k(a-1)$$

$$(k-1)a < x \leq ka$$

$$\underline{(k-1)a} < x \leq ka$$

$$-a < x - ka \leq -k$$

$$a > ka - x \geq k$$

Q32

Level 2.5 - 57

1. What is the largest semicircle that can fit inside a square?
2. Write out the effect of changes in the length and width of a rectangle on the radius of a semicircle. Draw the corresponding functional graph of r .

就是说最优的情况是圆能同时切住两边, 这样能占据更大的空间, 你必须满足圆心在另一条对角线上, 且直径和对角线平行

如果圆心不在另一条对角线上, 那么你肯定到两条边的距离是不相等的

如果直径和对角线不平行, 那你的直径肯定要比平行的情况要短

