Q1

(c) $y' = (e^y - x)^{-1}$

HINT: 尝试求倒数试试?

$$y' = \frac{1}{e^y - x}$$

$$\frac{dy}{dx} = \frac{1}{e^y - x}$$

$$\frac{dx}{dy} = e^y - x$$

$$\frac{dx}{dy} + x = e^y$$

$$\frac{d}{dy}\left[e^y x\right] = e^{2y}$$

$$e^y x = \frac{1}{2}e^{2y} + C$$

$$x = \frac{1}{2}e^{y} + Ce^{-y} = A\sinh y + B\cosh y$$

Q2

(c) $y' = (x - y)^2$

HINT: 尝试 u=(x-y)试试?

$$\frac{dy}{dx} = (x-y)^2 \qquad u = x-y, \quad \frac{du}{dx} = 1 - \frac{dy}{dx}$$

$$1 - \frac{du}{dx} = u^2 \qquad \frac{dy}{dx} = 1 - \frac{du}{dx}$$

$$\frac{du}{dx} = 1 + u^2$$

$$\int \frac{1}{1-u^2}\,du = \int 1\,dx$$

$$\tanh^{-1} u = x + C$$

Q3

19) Let f be a differentiable function. Evaluate

$$\lim_{x \to a} \frac{af(x) - xf(a)}{x - a}.$$

HINT: 怎么表示 x when x is approaching to a?

$$x = a + \Delta x$$

$$\frac{af(a+\Delta x) - (a+\Delta x)f(a)}{\Delta x}$$

$$= \frac{af(a+\Delta x) - af(a)}{\Delta x} - \frac{\Delta x f(a)}{\Delta x}$$

$$= af'(a) - f(a)$$

Q4

20) Find the derivative of $f(x) = (1-x)(2-x)(3-x)\ldots(n-x)$ at $x = 1$.

HINT:
derivative of one term * remaining original terms.
what will happen if (1-x) is including in the remaining original terms?

$[(1-x)]'\,(2-x)(3-x)\cdots(n-x)$

because 如果求导别的term，要乘(1-x), leading to 0.

so answer: $-1\cdot(2-1)(3-1)\cdots(n-1)$

Q5

(q) $\left(\frac{1}{n}\left(\frac{1}{n^2}\right)^2 + \frac{1}{n}\left(\frac{2}{n^2}\right)^2 + \ldots + \frac{1}{n}\left(\frac{n}{n^2}\right)^2\right)$ as $n \longrightarrow \infty$

HINT: 黎曼和

when $n \to \infty$

$\frac{1}{n^2}\left(\frac{1}{n}\left(\frac{1}{n}\right)^2 + \frac{1}{n}\left(\frac{2}{n}\right)^2 + \cdots + \frac{1}{n}\left(\frac{n}{n}\right)^2\right)$

$= \frac{1}{n^2}\left[\Delta x f(x_1) + \Delta x f(x_2) + \cdots + \Delta x f(x_n)\right]$

$= \frac{1}{n^2}\int_0^1 x^2\,dx$

Q6

29) Let $f(x) = x + 2x^2\sin(\frac{1}{x})$. Show that $f'(0) = 1$. You may assume $f(0) = 0$.

HINT: DEFINITION

$f'(0)$

$= \frac{f(h)-f(0)}{h}$

$= \frac{h + 2h^2\sin\frac{1}{h}}{h}$

$= 1 + 2\underset{0}{\underline{h}}\,\underset{-1\sim 1}{\underline{\sin\frac{1}{h}}} \longrightarrow 1$

Q7

32) Is $\int_1^\infty \frac{1}{x}\sin\left(\frac{1}{x}\right)dx$ bounded or unbounded?

HINT: substitution $\frac{1}{x}$

$u = \frac{1}{x}$ $\frac{du}{dx} = -\frac{1}{x^2} = -u^2$. $-\frac{1}{u^2} du = dx$

LHS $= \int_1^0 u \sin u \left(-\frac{1}{u^2}\right) du$

$= \int_0^1 \frac{\sin u}{u} du \longrightarrow$ bounded

Q8

32) (a) Verify that

$$y(x) = \int_x^{\infty} e^{-t^2}\,dt$$

satisfies the differential equation $y'' + 2xy' = 0$.

(b) By using an integrating factor on the DE, derive the general solution in integral form.

HINT: f(x,$\frac{dy}{dx}$) = 0

$$\frac{d^2y}{dx^2} + 2x\frac{dy}{dx} = 0$$

$$\frac{dy}{dx} = p. \quad \frac{dp}{dx} + 2xp = 0$$

$$[e^{x^2}p]' = 0$$

$$e^{x^2}p = C$$

$$p = Ce^{-x^2}$$

$$\frac{dy}{dx} = Ce^{-x^2}$$

Q9

33) Integrate $x^5e^{-x^2}$ from negative to positive infinity.

HINT: odd function

Q10

Show also that $2^{91} - 1$ is not prime.

HINT: modular 如果不行的话尝试 factorise

$2^{91} - 1 = u^{13} - 1$ where $u = 2^7$.

$= (u-1)(u^{12} + u^{11} + \cdots + 1)$

Q11

3) Suppose that we have some positive integers (not necessarily distinct) whose sum is 100. How large can their product be?

HINT: if n>3, we can break it down to make the product larger

for all $a \in \mathbb{Z}^*$, $a \geq 4$, we can break it down into either $(\frac{a+1}{2} - 1, \frac{a+1}{2})$ or $(\frac{a}{2}, \frac{a}{2})$. to maximise product

we want: $2a + 3b = 100$, max $2^a \cdot 3^b$

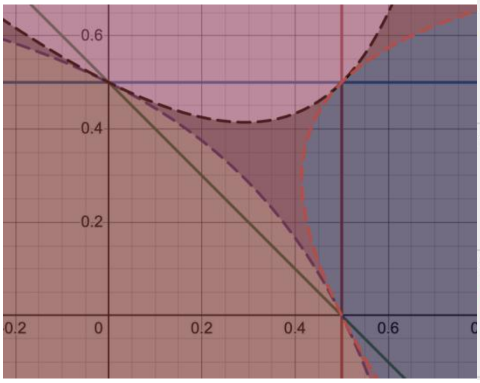
notice: $2^3 < 3^2$, while $2 \cdot 3 = 3 \cdot 2$

as many 3 as possible: $b = 32$ $a = 2$.

Q12

4) A thin rod is broken into three pieces. What is the probability that a triangle can be formed from the three pieces? What about if we want the triangle to be acute (e.g. all the internal angles are less than 90 degrees)?

HINT: $x^2+y^2>z^2$ for acute angle



4 $x^2+y^2>(1-x-y)^2$

5 $x^2+(1-x-y)^2>y^2$

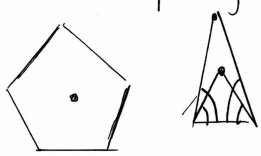
6 $y^2+(1-x-y)^2>x^2$

Q13

9) Consider a regular polygon with an odd number of sides. If I pick 3 vertices at random and form a triangle, what is the probability that the centre of the polygon is inside the triangle?

HINT: start by smaller cases

Start with pentagon:



actually, for each initial side
if its two vertices connect with center
we get the 'base angle'
if its two vertices connect with third
vertex, we get the 'true angle'.
then 'true angle' > 'base angle'

if first and second vertices are adjacent, third one has 1 possibility

if first and second vertices are seperated by 2 sides, third one has 2 possibilities

....

if first and second vertices are seperated by n sides, (total 2n+1 vertices), third one has n possibilities

totally: $\underbrace{(1+2+\cdots+n)}_{\text{aforementioned}} \times \underbrace{2}_{\text{seperation clockwise/anticlockwise}} \times \underbrace{n}_{\text{different first vertex}} \,/\, \underbrace{6}_{\text{each triangle be counted 6 times}}$

Q14

19) A biased coin has probability of landing heads equal to p. If the coin is tossed n times and let X be the number of times it lands heads in total, find $\mathbb{P}(X = k)$. Suppose now that the value of p is unknown. However, it's observed that k heads are obtained after tossing the coin n times. What is the value of p which makes this event most likely? That is, what value of p maximises $\mathbb{P}(X = k)$?

HINT: consider Normal Distribution: peak at np

Q15

20) If we flip a coin and generate a sequence of length n, what is the probability that the number of heads is even?

HINT: symmetric, binomial expansion

- if n is odd:

either even H + odd T ①

or odd H + even T. ②

$\because$ for each situation in ①, turn H into T. turn T in H, we get a corresponding situation in ②, with same possibility.

$\therefore P(\text{even } H) = \frac{1}{2}$.

- if n is even

$$P(\text{even } H) = \frac{{}^nC_n + {}^nC_{n-2} + \cdots + {}^nC_2 + {}^nC_0}{2^n}$$

$$P(\text{odd } H) = \frac{{}^nC_{n-1} + {}^nC_{n-3} + \cdots + {}^nC_3 + {}^nC_1}{2^n}$$

~~P(~~

$\therefore P(\text{even } H) - P(\text{odd } H) =$

$$= \frac{{}^nC_0 - {}^nC_1 + {}^nC_2 - {}^nC_3 + \cdots + {}^nC_{n-2} - {}^nC_{n-1} + {}^nC_n}{2^n}$$

$= 0$

$\therefore P(\text{even } H) = \frac{1}{2}$.

Q16

21) If we flip a coin and generate a sequence of length n, what is the probability we do not see two heads in a row? Set up a recursion equation and then solve it to find u_n.

HINT: start with smaller cases, Fibonacci sequence

assume initial n toss satisfies our condition

$\because$ for $n+1$ toss:

if n^{th} toss is H, we want T for $(n+1)^{th}$

if n^{th} toss is T, we want H/T for $(n+1)^{th}$

$\because$ after trials, notice that H can only be followed by T, whereas T can be followed by both — like a Fibonacci sequence.

		number of H	number of
1st	H · T		
2nd	T · H T	1 (F_1)	2 · (F
3rd	H T · T · H T	2 (F_2)	3 $(F_3$
4th	T H T · H T · T · H T	3 (F_3)	5 (F_4)

$$\therefore U_{n+1} = U_n \times \Big(\underbrace{\frac{F_n}{F_n + F_{n-1}} \times 1}_{n^{th} \text{ is } T} + \underbrace{\frac{F_{n-1}}{F_n + F_{n-1}} \times \frac{1}{2}}_{n^{th} \text{ is } H} \Big)$$

Q17

32) A fair die is thrown m times. Show that the probability of getting an even number of sixes is $\frac{1}{2}\left(1 + \left(\frac{2}{3}\right)^n\right)$.

HINT: binomial expansion

$P(\text{even } 6) =$

$$\sum_{i \text{ is even}} \left(\frac{1}{6}\right)^i \left(\frac{5}{6}\right)^{n-i} \binom{n}{i}$$

$P(\text{odd } 6) =$

$$\sum_{i = \text{odd}} \left(\frac{1}{6}\right)^i \left(\frac{5}{6}\right)^{n-i} \binom{n}{i}$$

$P(\text{even } 6) - P(\text{odd } 6) =$

$$\left(\frac{1}{6}\right)^0 \left(\frac{5}{6}\right)^n \binom{n}{0} - \left(\frac{1}{6}\right)^1 \left(\frac{5}{6}\right)^{n-1} \binom{n}{1} + \cdots$$

$$= \left(\frac{5}{6} - \frac{1}{6}\right)^n = \left(\frac{2}{3}\right)^n$$

$P(\text{even } 6) + P(\text{odd } 6) = 1$

Q18

35) Let u_n be the probability that n tosses of a fair coin contain no run of 4 heads. Find a recurrence relation for u_n and use it to show $u_8 = \frac{208}{256}$.

HINT: P = 1 - P inverse

if consumer P_n as the probability of n tosses containing runs of 4 heads:

$$P_n = \underbrace{P_{n-1}}_{\text{前}(n-1)\text{项不满足}} + \underbrace{(1 - P_{n-4}) \cdot \left(\frac{1}{2}\right)^4}_{\text{---- HHHH}}$$

Q19

39) A coin is tossed repeatedly, with probability of getting heads $p = 1 - q$. Find the expected length of the initial run (this is a run of heads if the first toss gives heads, and of tails otherwise).

HINT: find expected length = find when we end

to find out expected length, we are actually finding out when we will end.

$$\underbrace{p}_{\text{possibility that we firstly have a H}} \Big(1 + \underbrace{\tfrac{1}{q}}_{\text{expected toss until we have a T}}\Big) + \underbrace{q}_{\text{possibility that we firstly have a T}} \Big(1 + \underbrace{\tfrac{1}{p}}_{\text{expected toss until we have a H}}\Big)$$

Q20

38) The random variable X has binomial distributions with parameter n and p. Find the expected value (i.e. mean) and variance of X.

HINT: DEFINITION

X	0	1	2	$\cdots$	n
$P(X=x)$	$p^0q^n\binom{n}{0}$	$p^1q^{n-1}\binom{n}{1}$	$p^2q^{n-2}\binom{n}{2}$	$\cdots$	$p^nq^0\binom{n}{n}$

$$E(X)=\sum_{i=0}^{n} i p^i q^{n-i}\binom{n}{i}$$
$$=p\sum_{i=1}^{n} i p^{i-1} q^{n-i}\binom{n}{i}$$
$$=p\frac{d}{dp}\left[(p+q)^n\right]$$
$$=np(p+q)^{n-1}=np$$

$$Var(X)=\sum_{i=0}^{n} i^2 p^i q^{n-i}\binom{n}{i}-[E(X)]^2$$
$$=p^2\sum_{i=2}^{n} i(i-1)p^{i-2}q^{n-i}\binom{n}{i}+p\sum_{i=1}^{n} i p^{i-1}q^{n-i}\binom{n}{i}$$
$$-(np)^2$$
$$=p^2\frac{d^2}{dp^2}\left[(p+q)^n\right]+np-(np)^2$$
$$=n(n-1)p^2(p+q)^{n-2}+np-(np)^2$$
$$=n(n-1)p^2+np-n^2p^2$$
$$=np(\cancel{np}-p+1-\cancel{np})=npq$$

Q21

37) The random variable X is distributed geometrically with parameter p. Find the expected value (i.e. mean) and variance of X.

HINT: DEFINITION

X	1	2	3	$\cdots$
$P(X=x)$	p	pq	pq^2	$\cdots$

$$E(X)=\sum_{i=1}^{\infty} i p q^{i-1}=p\sum_{i=1}^{\infty} i q^{i-1}$$
$$=p\frac{d}{dq}\left(\sum_{i=0}^{\infty} q^i\right)=p\frac{d}{dq}\left(\frac{1}{1-q}\right)$$
$$=\frac{1}{p}$$

$$Var(X)=\sum_{i=1}^{\infty} i^2 p q^{i-1}-[E(X)]^2$$
$$=pq\sum_{i=2}^{\infty} i(i-1)q^{i-2}+p\sum_{i=1}^{\infty} i q^{i-1}-\left(\frac{1}{p}\right)^2$$
$$=pq\frac{d^2}{dq^2}\left(\sum_{i=0}^{\infty} q^i\right)+\frac{1}{p}-\left(\frac{1}{p}\right)^2$$
$$=pq\frac{d^2}{dq^2}\left(\frac{1}{1-q}\right)+\frac{1}{p}-\left(\frac{1}{p}\right)^2$$
$$=pq\cdot\frac{2}{(1-q)^3}+\frac{1}{p}-\frac{1}{p^2}$$
$$=\frac{2pq+p^2-p}{p^3}=\frac{2q+p-1}{p^2}=\frac{q}{p^2}$$

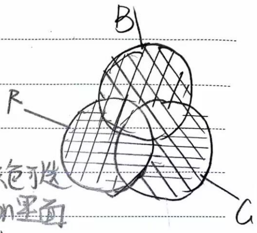
Q22

41) A fair die having two faces coloured blue, two red and two green, is thrown repeatedly. Find the probability that not all colours occur in the first k throws. Deduce that if N is the random variable which takes the value n if all three colours occur in the first n throws, but only two occur in the first $n-1$ throws (i.e. it requires the nth throw to finally hit all three colours), then the expected value of N is $\frac{11}{2}$.

HINT: 容斥原理

$$\underbrace{\left(\frac{2}{3}\right)^k\times 3}_{\text{规定2种}}-\underbrace{\left(\frac{1}{3}\right)^k\times 6}_{\text{规定1种}}$$

(有可能2种颜色被
包combination里面
只一种颜色)



/// 选RB
\\\ 选BG
≡ 选RG

Q23

42) The probability of obtaining a head when a biased coin is tossed is p. The coin is toss repeatedly until n heads occur in a row. Let X be the total number of tosses required for this to happen. Find the expected value of X.

HINT: start anew

$$E(X_n)=p(E(X_{n-1})+1)+q(E(X_{n-1})+1+E(X_n))$$
$$E(X_n)=E(X_{n-1})+1+qE(X_n)$$
$$pE(X_n)=E(X_{n-1})+1$$
$$E(X_n)=\frac{E(X_{n-1})+1}{p}$$

so $\begin{pmatrix}\frac{1}{p}&\frac{1}{p}\\0&1\end{pmatrix}\begin{pmatrix}E(X_{n-1})\\1\end{pmatrix}=\begin{pmatrix}E(X_n)\\1\end{pmatrix}$

$$\det\begin{pmatrix}\frac{1}{p}-\lambda&\frac{1}{p}\\0&1-\lambda\end{pmatrix}=\left(\frac{1}{p}-\lambda\right)(1-\lambda)=0$$

$\lambda_1=\frac{1}{p}$ $\lambda_2=1$

$$\begin{pmatrix}\frac{1}{p}&\frac{1}{p}\\0&1\end{pmatrix}\begin{pmatrix}1\\m\end{pmatrix}=\begin{pmatrix}\frac{1}{p}-\frac{m}{p}\\m\end{pmatrix}=\frac{1}{p}\begin{pmatrix}1\\m\end{pmatrix}$$

$m=0$

$$\begin{pmatrix}\frac{1}{p}&\frac{1}{p}\\0&1\end{pmatrix}\begin{pmatrix}1\\m\end{pmatrix}=\begin{pmatrix}\frac{1}{p}-\frac{m}{p}\\m\end{pmatrix}=\begin{pmatrix}1\\m\end{pmatrix}$$

$m=1-p$

$$\begin{pmatrix}1&-1\\1-p&0\end{pmatrix}^{-1}\begin{pmatrix}\frac{1}{p}&\frac{1}{p}\\0&1\end{pmatrix}^n\begin{pmatrix}1&1\\1-p&0\end{pmatrix}=\begin{pmatrix}1&0\\0&\frac{1}{p}\end{pmatrix}^n$$

$$\begin{pmatrix}E(X_n)\\1\end{pmatrix}=\begin{pmatrix}\frac{1}{p}&\frac{1}{p}\\0&1\end{pmatrix}\begin{pmatrix}E_0\\1\end{pmatrix}=\begin{pmatrix}1&1\\1-p&0\end{pmatrix}\begin{pmatrix}1&0\\0&\frac{1}{p}\end{pmatrix}^n\begin{pmatrix}1&1\\1-p&0\end{pmatrix}^{-1}\begin{pmatrix}E(X_0)\\1\end{pmatrix}$$

$$E(X_n) = q(1+E(X_n)) + pq(2+E(X_n)) + p^2q(3+E(X_n)) + \cdots + p^{n-1}q(n+E(X_n)) + p^n \cdot n$$

$$= \sum_{i=1}^{n}(ip^{i-1}q) + qE(X_n)(1+p+p^2+\cdots+p^{n-1}) + p^n \cdot n$$

$$= q\frac{d}{dp}\left(\sum_{i=0}^{n} p^i\right) + qE(X_n)\frac{1-p^n}{1-p} + p^n n$$

$$= q \cdot \frac{d}{dp}\left(\frac{1-p^{n+1}}{1-p}\right) + (1-p^n)E(X_n) + p^n n$$

Q24

45) Two identical decks of cards, each containing N cards, are shuffled randomly. We say that a k-matching occurs if the two decks agree in exactly k places. Show that the probability that there's a k-matching is

$$\frac{1}{k!}\left(1 - \frac{1}{1!} + \frac{1}{2!} - \frac{1}{3!} + \dots + \frac{(-1)^{N-k}}{(N-k)!}\right).$$

HINT: disarrange, 容斥

这种问题还是disarrange，这样考虑

我们要k个配对，那就disarrange $(N-k)$个

① 先将所有的都排好，然后选$(N-k)$个disarrange：

$${}^{N}C_{N-k} \cdot (N-k)! = \frac{N!}{(N-k)!\,k!}(N-k)! = \frac{N!}{k!}$$

② 容斥原理，有些element可能没有被disarrange：

先考虑1个element已知（该被disarrange）但未disarrange：

$${}^{N}C_{N-k} \cdot {}^{N-k}C_1 \cdot (N-k-1)!$$

$$= \frac{N!}{(N-k)!\,k!} \cdot \frac{(N-k)!}{(N-k-1)!\,1!} \cdot (N-k-1)! = \frac{N!}{k!}\,\frac{1}{1!}$$

③ repeat，共有

$$\frac{N!}{k!}\left(1 - \frac{1}{1!} + \frac{1}{2!} - \cdots + \frac{(-1)^{N-k}}{(N-k)!}\right)$$ 个情况（k个配对）

概率易得（共$N!$情况）

Q25

1. Consider a 12 hour digital clock (a clock which shows times from 00:00 to 11:59). What is the probability that there is at least one digit '1' displayed at a random time? What about exactly one '1'?

HINT: 考虑清楚(1) hour 的十位和个位是不是 independent (2) 容斥原理是用来求什么的

Solution: Consider the possible hours displayed: 00, **01**, 02, 03, 04, 05, 06, 07, 08, 09, **10**, **11**. So we need one of the 9 hours without a '1' in. Then if we consider the minutes, we have 01, 10-19, 21, 31, 41 and 51 with a '1' in. So P(No '1's) $= \frac{9}{12} \cdot \frac{(60-15)}{60} = \frac{3}{4} \cdot \frac{3}{4} = \frac{9}{16}$.

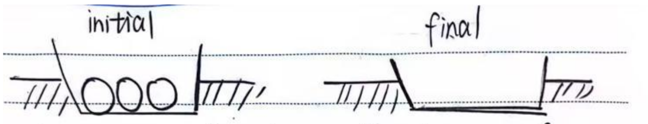
So we have P(At least one '1') $= 1 - \frac{9}{16} = \frac{7}{16}$.

For exactly one '1' we need either no 1s in the hours and one 1 in the minutes or one 1 in the hours and none in the minutes. There are $\times 14$ ways to have no 1s in the hours and one 1 in the minutes and 2×45 ways to have one 1 in the hours and none in the minutes. So P(Exactly one '1') $= \frac{9 \cdot 14 + 2 * 45}{60 \cdot 12} = \frac{3}{10}$.

Q26

1) A person is on a lake, in a rowing boat full of rocks. The rocks are thrown into the lake. What happens to the water level?

HINT: compare V



$F_{upthrust1} = G_B + G_s$ $\quad$ $F_{upthrust2} = G_B$

$= \rho g V_1$ $\quad$ $= \rho g V_2$

we want to compare: V_1 & $V_2 + V_s$

$\because \rho < \rho_s$

$\therefore \rho V_1 g = G_B + G_s$

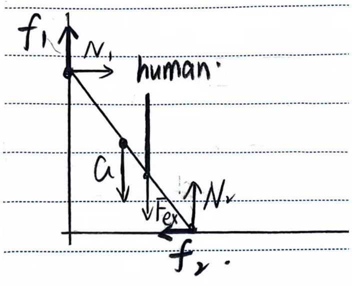
$\rho (V_2 + V_s) g < G_B + G_s$

$\therefore V_1 > V_2 + V_s$ $\quad$ $\therefore$ water level decreases

Q27

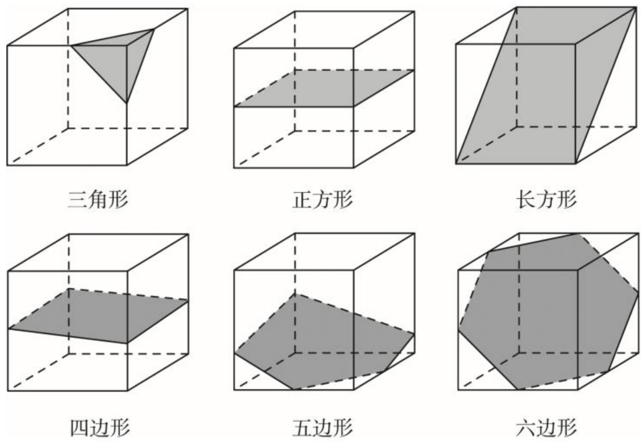
4) A ladder rests against a wall and a man stands on it. Label all the forces acting on the ladder and deduce four equations that relate them.

HINT: gravity, external force, normal force, friction



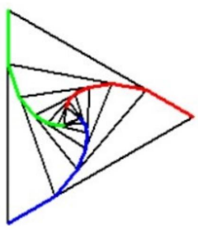
Q28

7) If I place a cube in water, what shape does it make on the surface?



Q29

15) Suppose that A, B and C are three points in a plane, such that $AB = AC = BC = 1$. At each point in time, A is moving toward B, B is moving toward C, and C is moving toward A, all with speed $v = 50$. Shown below is what the movement would look like.



The red curve represents the path from point A, the green curve the path from point B, and the blue curve the path from point C. At what time t will all the points reach the centre of the triangle?

Q30

12) If you're standing on a pair of scales in a lift, what happens to the reading when you start moving up a floor? Why?

it detects the force from humans acting on device.

human's resultant force is upwards, meaning supporting force from device greater than gravity

since same but opposite, larger force from human acting on device

Q31

18) A beam of light falls vertically at the point $x = x_0$ on an arbitrary curve $y = f(x)$. Find the gradient of the reflected beam of light.

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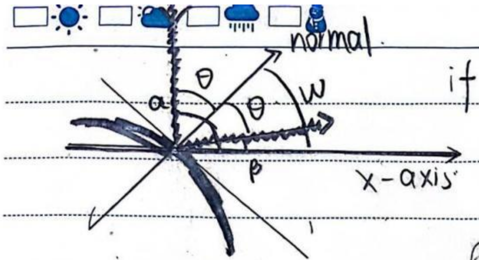


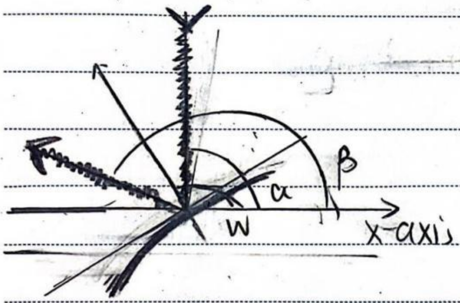
if $-\frac{1}{f'(x_0)} = k > 0$

$2w = \alpha + \beta$

$w = \arctan k$

$\beta = 2\arctan k - \frac{\pi}{2}$

gradient $= \tan\beta$



if $k < 0$

$2w = \alpha + \beta$

$w = \arctan k + \pi$

$\beta = 2\arctan k + 2\pi - \frac{\pi}{2}$

gradient $= \tan\beta$