Q1

10. Cam Peterhouse (2016)

$$f(x) = \begin{cases} x^2 \sin\frac{1}{x}, & x \neq 0 \\ 0, & x = 0 \end{cases}$$

what is the value of $f'(0)$?

HINT: definition

$$f'(0) = \lim_{h \to 0} \frac{h^2 \sin\frac{1}{h} - 0}{h}$$

$$= 0$$

Q2

15. Ox Oriel (2019)

Find:

a. $\frac{d}{dx} e^{-x} \sin x$

b. $\frac{d}{dx} e^{-x} \cos x$

c. $\frac{d}{dx} |e^{-x} \sin x|$

HINT: deal with absolute value by using |f(x)| = $\sqrt{\mathbf{f}(\mathbf{x})^2}$

c.

$$\frac{d}{dx} |e^{-x} \sin x| =$$

$$= \frac{d}{dx}\left(\sqrt{(e^{-x} \sin x)^2}\right)$$

$$= \frac{1}{2} \cdot \frac{1}{\sqrt{(e^{-x} \sin x)^2}} \cdot 2(e^{-x} \sin x) \cdot (-e^{-x} \sin x + e^{-x} \cos x)$$

$$= \frac{(e^{-2x} \sin x) \cdot (-\sin x + \cos x)}{|e^{-x} \sin x|}$$

Q3

6. Ox Trinity (2015)

$2a^2 - b^2 = 1$ (a, b are integers)

Find the solutions.

HINT: 根据已知 roots 推导出新的 roots

if $2m^2-n^2=1$, $2p^2-q^2=1$

$(2m^2-n^2)(2p^2-q^2)=1$

$4m^2p^2-2n^2p^2-2m^2q^2+n^2q^2=1$

$4m^2p^2 \pm 4mnpq+n^2q^2-2n^2p^2 \mp 4mnpq-2m^2q^2=1$

$(2mp \pm nq)^2-2(np \pm mq)^2=1$

if we have (m,n), (p,q) for $2a^2-b^2=1$, we may find two roots for $a^2-2b^2=1$

If $A^2-2B^2=1$, $2C^2-D^2=1$

$(A^2-2B^2)(2C^2-D^2)=1$

$2A^2C^2-4B^2C^2-A^2D^2+2B^2D^2=1$

$2A^2C^2 \pm 4ABCD+2B^2D^2-4B^2C^2 \mp 4ABCD-A^2D^2=1$

$2(AC \pm BD)^2-(2BC \pm AD)^2=1$

if we have (A,B) for $a^2-2b^2=1$ and (C,D) for $2a^2-b^2=1$, we may find two roots for $2a^2-b^2=1$

$2a^2-b^2=1$

$a^2-2b^2=1$

Q4

4. Ox St. Catherine (2014)

$\lceil a \rceil$ gives the smallest integer that is greater or equal to a;

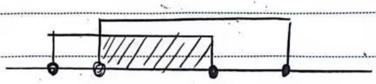
for instance, $\lceil 2 \rceil = 2$, $\lceil 2.5 \rceil = 3$. For $\left\lceil \frac{x}{a} \right\rceil = \left\lceil \frac{x}{a+1} \right\rceil$, state whether it has a solution or not. If yes, how many solutions?

HINT: boundary

$$\left\lceil \frac{x}{a} \right\rceil = \left\lceil \frac{x}{a+1} \right\rceil = k$$

$a(k-1) < x \le ak$ for ①

$(a+1)(k-1) < x \le (a+1)k$ for ②



$(a+1)(k-1) < x \le ak$

$ak + k - a - 1 < x \le ak$.

as long as $k \le a$

number of root for each k:

$\cancel{ak} - \cancel{ak} + k + a + 1 = a + 1 - k$

$$\sum_{k=0}^{a} [a+1-k]$$

Q5

12. Cam Christ's (2016)

Polynomial 所有系数为 1，0，-1，问 degree 为 2016 的这样 polynomial 所有的解为整数的个数为多少?

HINT: 韦达定理

focusing on last non-zero coefficient term

roots are -1, 1, 0

notice that $(x^2-1)^2 = x^4-2x^2+1$ ✗

$(x^2-1)\cdot(x+1) = x^3-x+x^2-1$ ✓

$(x^2-1)(x-1) = x^3-x-x^2+1$ ✓

possibility

number of	$x=1$	$x=-1$	$x=0$
	0	0	2016
	1	0	2015
	0	1	2015
	1	1	2014
	2	1	2013
	1	2	2013

totally 6 possibility

Q6

19. Ox Brasenose (2017)

Find the range of $\frac{a}{b+c}+\frac{b}{a+c}+\frac{c}{a+b}$

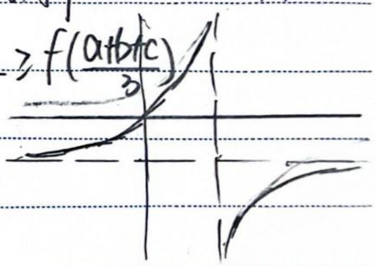
HINT: 二阶导

if $a+b+c=1$

$\frac{a}{1-a}+\frac{b}{1-b}+\frac{c}{1-c}$

assume $f(x)=\frac{x}{1-x}$, 因为 $f(x)$ convex

因此 $\frac{f(a)+f(b)+f(c)}{3} \geq f\left(\frac{a+b+c}{3}\right)$



这样 min value is

$3\cdot\frac{\frac{1}{3}}{1-\frac{1}{3}} = \frac{3}{2}$

Q8

20. Ox Oriel (2017)

a. WTP: $1+x+x^2+\cdots+x^n=\frac{1-x^{n+1}}{1-x}$

b. WTP: $1+x+x^2+\cdots+x^n+\cdots=\frac{1}{1-x}$

c. WTP: $1-x^2+x^4-x^6+\cdots=\frac{1}{1+x^2}$

d. WTP: $\arctan x=x-\frac{x^3}{3}+\frac{x^5}{5}-\frac{x^7}{7}+\cdots$

e. Find expression for $x+2x^2+3x^3+\cdots$

f. Find expression for $x+4x^2+9x^3+16x^4+\cdots$

g. Hence, get $\frac{1}{2}+\frac{2}{4}+\frac{3}{8}+\cdots+\frac{n}{2^n}+\cdots$ and $\frac{1}{2}+\frac{4}{4}+\frac{9}{8}+\cdots\frac{n^2}{2^n}+\cdots$

h. According to $(e^x)^1=e^x$, show that $e^x=1+x+\frac{x^2}{2!}+\cdots+\frac{x^n}{n!}+\cdots$

i. According to $e^{ix}=\cos x+i\sin x$, show that $\sin x=x-\frac{x^3}{3!}+\frac{x^5}{5!}-\cdots$ and $\cos x=1-\frac{x^2}{2!}+\frac{x^4}{4!}-\cdots$

j. Hence show that $\frac{\sin x}{x}=(1-\frac{x^2}{\pi^2})(1-\frac{x^2}{4\pi^2})(1-\frac{x^2}{9\pi^2})\cdots$

HINT: 积分和求导

d. $\tan \text{RHS} = \tan\left(x-\frac{x^3}{3}+\frac{x^5}{5}+\cdots\right)$
$= \tan\left[\int y'\,dx\right]$
$= \tan\left[\int 1+x^2+x^4+x^6+\cdots dx\right]$
$= \tan\left[\int \frac{1}{1+x^2}dx\right]$
$= \tan[\arctan x]$
$= x = \tan \text{LHS}$

e. $(1-x)\sum ix^i = \frac{x}{1-x}$
$\sum ix^i = \frac{x}{(1-x)^2}$

f. $(1-x)\sum i^2x^i = \sum(2i+1)x^i$
$= 2\sum ix^i - \sum x^i$
$= \frac{2x}{(1-x)^2} - \frac{1}{1-x}$
$\sum i^2x^i = \frac{2x}{(1-x)^3} - \frac{1}{(1-x)^2}$

e. $x[1+2x+3x^2+\cdots]$
$= x\frac{d}{dx}[1+x+x^2+x^3+\cdots] = x\frac{d}{dx}\left[\frac{1}{1-x}\right]$

f. $\sum_{i=1}^{\infty} i^2x^i = x^2\sum_{i=2}^{\infty} i(i-1)x^{i-2} + x\sum_{i=1}^{\infty} ix^{i-1}$
$= x^2\frac{d^2}{dx^2}\left[\sum_{i=0}^{\infty}x^i\right] + x\frac{d}{dx}\left[\sum_{i=0}^{\infty}x^i\right]$
$= x^2\frac{d^2}{dx^2}\left[\frac{1}{1-x}\right] + x\frac{d}{dx}\left[\frac{1}{1-x}\right]$

h. $y^{(n)} = 1+x+\frac{x^2}{2!}+\cdots$
$y' = 1+x+\frac{x^2}{2!}+\cdots = y$
$\Rightarrow y = e^x$

* a very interesting conclusion:

$\frac{\sin x}{x} = (1-\frac{x}{\pi})(1+\frac{x}{\pi})(1-\frac{x}{2\pi})(1+\frac{x}{2\pi})\cdots$
$= (1-\frac{x^2}{\pi^2})(1-\frac{x^2}{4\pi^2})(1-\frac{x^2}{9\pi^2})\cdots$
$= 1-\frac{x^2}{3!}+\frac{x^4}{5!}-\cdots$ Taylor

$-(\frac{1}{\pi^2}+\frac{1}{4\pi^2}+\frac{1}{9\pi^2}+\cdots) = -\frac{1}{\pi^2}\sum_{n=1}^{\infty}\frac{1}{n^2} = -\frac{1}{3!}$ 根与系数

Q9

21. Ox Wadham (2017)

a. Show that $\frac{x}{y}+\frac{y}{x}\geq 2$

b. Find all x, y where $x, y \in \mathbb{R}$ in which $\frac{x}{y}+\frac{y}{x}= integer$

HINT: integral polynomial only has integer rational roots

$u=\frac{x}{y}$ $u+\frac{1}{u}=\frac{u^2+1}{u}=k$, k integer

$u^2-ku+1=0$

notice that the only rational root for integral polynomial is integer

so u may be integer

u can only be 1 due to $\frac{u^2+1}{u}=k$

Q10

29. Ox St. Hugh's (2019)

Solve $\sqrt{3-x}-\sqrt{1+x}>\frac{1}{2}$

HINT: 3-x + 1+x = 4

Solve the inequality $\sqrt{3-x}-\sqrt{x+1}>\frac{1}{2}$.

三角换元是解决数学问题中的一个很重要方法，在计算积分、解不等式、求数列通项、找函数极值等等问题中都可以找到三角换元的身影. 通常情况下，如果题目中隐含了某个三角恒等式，我们就可以考虑进行相应的三角换元.

我们来看昨天的题目：

Solve the inequality $\sqrt{3-x}-\sqrt{x+1}>\frac{1}{2}$.

在这道题目中，最明显的是有一个平方和为常数的关系，即

$$\left(\sqrt{3-x}\right)^2+\left(\sqrt{x+1}\right)^2=4$$

这容易让我们联想到 $\sin^2\theta+\cos^2\theta=1$ 这个三角恒等式，所以我们也许可以尝试做如下三角换元

$$\sqrt{3-x}=2\sin\theta,\quad \sqrt{x+1}=2\cos\theta,\quad \theta\in\left[0,\frac{\pi}{2}\right]$$

因为两个根号都大于等于零，所以我们把 θ 限制在了第一象限. 接下来我们便得到

$$\begin{aligned}
&2\sin\theta-2\cos\theta>\frac{1}{2}\\
\Longrightarrow\ &2\sin\theta>2\cos\theta+\frac{1}{2}>0\\
\Longrightarrow\ &(2\sin\theta)^2>\left(2\cos\theta+\frac{1}{2}\right)^2\\
\Longrightarrow\ &32\cos^2\theta+8\cos\theta-15<0\\
\Longrightarrow\ &0\leq\cos\theta<\frac{\sqrt{31}-1}{8}
\end{aligned}$$

因为 $x=4\cos^2\theta-1$, 我们便得 $-1\leq x<1-\frac{\sqrt{31}}{8}$.

Q11

13. Cam (2015)

$x(t) = t - \sin t$, $y(t) = 1 - \cos t$, Sketch $y(x)$

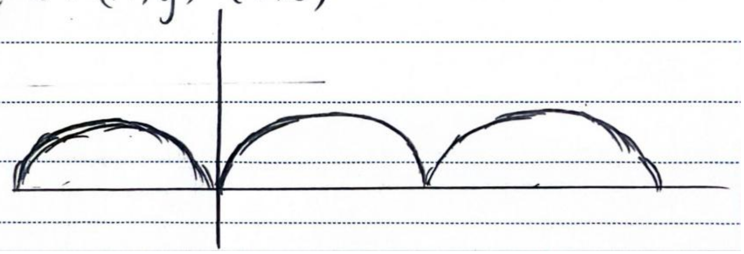
HINT: initial condition+derivatives

$x = t - \sin t \qquad y = 1 - \cos t$

$\frac{dx}{dt} = 1 - \cos t \qquad \frac{dy}{dt} = \sin t$

$\frac{dy}{dx} = \frac{\sin t}{1 - \cos t} = \frac{2\cos\frac{t}{2}\sin\frac{t}{2}}{2\cos^2\frac{t}{2}} = \cot\frac{t}{2}$

$t = 0,\ (x, y) = (0, 0)$



Q12

23. Cam Pembroke (2017)

a. Sketch $y = (4x^4 - 13x^2 + 3)e^{-x^2}$

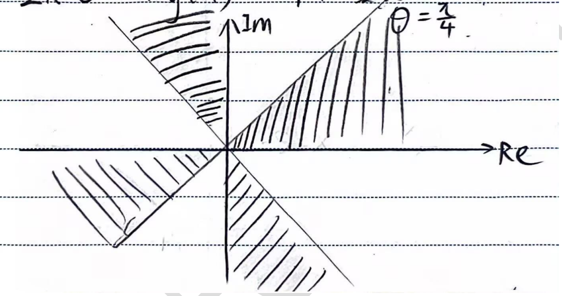
b. Sketch $\mathrm{Im}(Z^4) > 0$

$\mathrm{Im}(x^4) > 0$

$2k\pi < \arg(x^4) < \pi + 2k\pi$

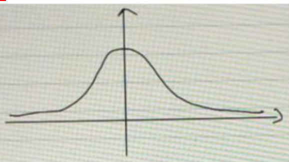
$\arg(x^4) = 4\arg(x)$

$\frac{1}{2}k\pi < \arg(x) < \frac{\pi}{4} + \frac{\pi}{2}k\pi$



Q13

③



1) find the function looks like this.

2) what we need to change to make the graph's gradient sharper but the point on y-axis stays the same

3) let $\int_{-\infty}^{\infty} e^{-x^2} = A$

what change needs to make to make the area equals to 1 but the maximum point stays the same.

HINT: all possible way of operations

several possible ways of operation

- $\frac{f(x)}{k}$ $\quad \frac{A}{k}$
- $f(x)+k$ $\quad A + k \cdot$ [range of domain]
- $f(x+k)$ $\quad A$
- ✓ $f(kx)$ $\quad \frac{A}{k}$

Q14

17. Cam Peterhouse(2016)

3*6 的座位，10 个女生，7 个男生做，要求同一排同一列不全是男生或者女生，问有几种做法

HINT: find a bijection

firstly label 7 boy: A B C D E F G

distribute them into 3 row / 6 column seperately

we find a bijection!

[number of boy combination] = [number of boy combination in row] × [number of boy combination in column]

find each term by using inclusion and exclusion principle.

Q15

5. Ox St. Cats (2015)

State the following statements are true or false:

a. For all positive real number y, there is a positive real number x that 1/x<y.

b. There exists a positive real number x, where 1/x>y for all positive real number

c. For all positive real number y, there is a positive real number x such that x>z, where $-y < \frac{1}{x} < y$

d. For all positive real number y, there is a positive real number x such that x>z, where $x^2 \sin x > y$.

e. There exists a positive real number x such that x>z, where x2sinx>y for all positive real number y.

HINT: interpret first

c.

$\because -y < \frac{1}{x} < y$, x is positive

$\therefore 0 < \frac{1}{x} < y$, $x > \frac{1}{y}$

$\therefore \begin{cases} x > \frac{1}{y} \\ x > z \end{cases}$ for any y, find a corresponding x s.t. satisfy this statement simultaneously ✓ true

Q16

9. Cam Christ's (2016)

一个 2016*2016 表格里所有项为 1 和-1，问所有 2*2 squares 和都为 0 的排列数

HINT: start with smaller cases

assume a_n represents the number of combinations for n×n squares

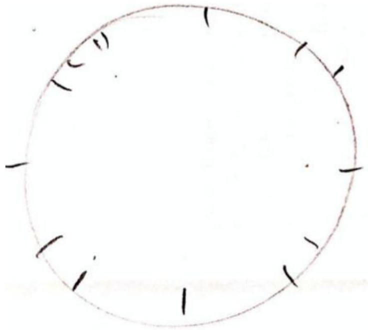
after trials:

$$\begin{array}{ccc} 1 & -1 & 1 \\ -1 & 1 & -1 \\ 1 & -1 & 1 \end{array}$$

we focus on boundary:

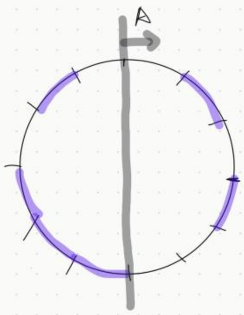
- when expand, if there exists consecutive 1 or -1 on boundary, we can get 2 distinct new square
- when expand, if there exists alternative numbers on boundary, we can get 3 distinct new square
- totally: $a_k = \underbrace{2 \cdot 3}_{\text{2 possible ways s.t. boundary consists of alternative numbers}} + \underbrace{(a_{k-1} - 2) \cdot 2}_{\text{remaining ways s.t. boundary consists of consecutive 1 or -1}}$

Q17



a clock with each position either red or blue, prove must have one symmetrical axis that divides them into 3 red 3 blue on each side

HINT: model & trials



① sketch & discuss each possibilities ✕

② algebaric

- we first draw a line, called A, and focus on the number of blue segments on its right, say a
- then we rotate the line clockwisely 30° each time, and count the number of blue segment on the right side of the line.
- eventually we rotate 180°, and now on the right side of the line, we have (6 − a) number of blue segment

☆ continuous so must exist a particular line s.t. there are 3 blue segments on both sides.

Q18

1. Cam (2014)

   $T \in [0,1]$,x 为$x^2 + 2x + T = 0$ 最大值

   a. $\mathbb{P}\left(x > -\frac{1}{3}\right)$

   b. $\mathbb{E}(x)$

   c. $Var(x)$

1. $X = \frac{-2 \pm \sqrt{4-4T}}{2}$

$X_{MAX} = -1 + \sqrt{1-T}$

$P(x > -\frac{1}{3})$

$-\frac{1}{3} < -1 + \sqrt{1-T}$

$\frac{2}{3} < \sqrt{1-T}$

$\frac{4}{9} < 1 - T$

$T < \frac{5}{9}$

2. $E(x) = \sum np = \frac{\int_0^1 (-1+\sqrt{1-T})\,dT}{\int_0^1 1\,dT}$

$= \left[-T - \frac{2}{3}(1-T)^{\frac{3}{2}}\right]_0^1$

$= -1 + \frac{2}{3} = -\frac{1}{3}$

3. $Var(x) = \sum n^2 p - [E(x)]^2$

$= \frac{\int_0^1 (-1+\sqrt{1-T})^2\,dT}{\int_0^1 1\,dT} - \frac{1}{9}$

$= \int_0^1 1 - 2\sqrt{1-T} + 1 - T\,dT - \frac{1}{9}$

$= \int_0^1 2 - T - 2\sqrt{1-T}\,dT - \frac{1}{9}$

$= \left[2T - \frac{1}{2}T^2 + \frac{4}{3}(1-T)^{\frac{3}{2}}\right]_0^1 - \frac{1}{9}$

$= 2 - \frac{1}{2} - \frac{4}{3} - \frac{1}{9} = \frac{36-9-24-2}{18} = \frac{1}{18}$

Q19

43. Ox Keble (2019)

x, y, z are prime

solve $x + y^2 = 4z^2$

$x = 4z^2 - y^2 = (2z-y)(2z+y)$

$2z - y = 1 \quad z = \frac{y+1}{2}$

$\begin{cases} y=3 & z=2 & x=7 \\ y=5 & z=3 & x=11 \end{cases}$

WTP: these are only roots

given that:

$0\times 2 \equiv 0 \mod 3$

$2\times 2 \equiv 4 \equiv 1 \mod 3$

$1 \times 2 \equiv 2 \mod 3$

① $\therefore$ since $2z \equiv y+1 \equiv 1 \mod 3$

$z \equiv 2 \mod 3$

$x \equiv 2z+y \equiv 1 \mod 3$

② $\therefore$ since $2z \equiv y+1 \equiv 2 \mod 3$

$z \equiv 1 \mod 3$

$x \equiv 2z+y \equiv 0 \mod 3$

③ since $2z \equiv y+1 \equiv 0 \mod 3$

$z \equiv 0 \mod 3$

$x \equiv 2z+y \equiv 2 \mod 3$

x	y	z
1	0	2
0	1	1
2	2	0

mod 3

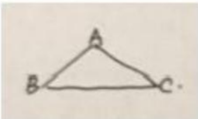
either $x=3$, $y=3$, or $z=3$

✗ ✓ ✓

Q20

2. Here is a triangle. A bee is at A initially. There is same chance to go either 2 points.

What's the probability after nth jump for the bee to get back to A?



① induction / re currency

② smaller cases

if P_n represent the possibility s.t. at A at nth jump

$$P_{n+1} = \underbrace{P_n \cdot 0}_{A \to B,C} + \underbrace{(1-P_n)\cdot \tfrac{1}{2}}_{B,C \to A}$$