INTEGRATION INTERVIEW QUESTIONS

Advice

This is an interview-style practice sheet. As you attempt each question, you will benefit greatly from saying your thought process out loud to yourself, as if speaking to a real-life interviewer. This allows you to practice communicating your ideas, which is a skill that interviewers assess for. When you get stuck, you may consult the "Hints" section at the end of this problem sheet, which contains hints for some of the questions. Once you have seen the hint, you can pretend that the interviewer gave you the hint and continue working through the question, saying aloud your thought process. You may wish to read only part of the hint and before going back to the question, rather than reading the entire hint all at once.

Useful Taylor Expansions

- $e^x = 1 + x + \frac{x^2}{2!} + ... = \sum_{k=0}^{\infty} \frac{x^k}{k!}$, converges for $|x| < \infty$
- $\ln(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} - ... = -\sum_{k=1}^{\infty} \frac{(-1)^k x^k}{k}$, converges for $|x| < 1$.
- $\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} + ... = \sum_{k=0}^{\infty} \frac{(-1)^k x^{2k}}{(2k)!}$, converges for $|x| < \infty$.
- $\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - ... = \sum_{k=0}^{\infty} \frac{(-1)^k x^{2k+1}}{(2k+1)!}$, converges for $|x| < \infty$.

Questions

1) Let $I_n = \int \tanh^n x dx$.

(a) Show that, for $n \geq 2$,

$$I_n = I_{n-2} - \frac{1}{n-1} \tanh^{n-1} x.$$

(b) Evaluate $\int_0^{\ln 2} \tanh^4 x dx$.

2) $I_n = \int_0^4 x^n \sqrt{4-x} dx$.

(a) Find a reduction formula for I_n, and state for which values of n it is valid.

(b) Evaluate $\int_0^4 x^3 \sqrt{4-x} dx$.

3) Let $I_n = \int \cos^n x dx$.

(a) Find a reduction formula for I_n. For which values of n is it valid?

(b) Evaluate

i. $\int_0^{2\pi} \cos^4 x dx$

ii. $\int_0^{2\pi} \cos^8 x dx$

4) Consider $I_n = \int_0^{\ln\sqrt{3}} \tanh^n x dx$, where $n \geq 1$.

(a) Show that $I_n = I_{n-2} - \frac{1}{n-1}\left(\frac{1}{2}\right)^{n-1}$, $n \geq 2$.

(b) Prove that $\lim_{n\to\infty} I_n = 0$.

(c) Hence show that

$$\sum_{r=1}^{\infty} \frac{1}{2r}\left(\frac{1}{2}\right)^{2r} = \ln \frac{2}{\sqrt{3}}.$$

5) (a) By considering the inequality

$$\int_0^t (f(x) + \lambda g(x))^2 dx \geq 0,$$

where λ is a constant, prove that for all functions $f(x)$ and $g(x)$,

$$\left(\int_0^t f(x)g(x)dx\right)^2 \leq \left(\int_0^t f(x)^2 dx\right)\left(\int_0^t g(x)^2 dx\right).$$

This is known as the Cauchy-Schwarz inequality.

(b) Hence show that

$$\int_0^1 (1+x^5)^{1/2} dx \leq \sqrt{\frac{7}{6}}.$$

6) Let $A = \int \frac{\sin x}{\sin x + \cos x} dx$ and $B = \int \frac{\cos x}{\sin x + \cos x} dx$. Find A and B.

7) Integrate $\sin^4 x \cos x$ and $\sin^6 x \cos^3 x$. In general, when will your method work?

8) Prove that for a continuous function,

$$\int_a^b f(x)dx = \int_a^b f(a+b-x)dx.$$

Hence, evaluate

(a) $\int_4^8 \frac{\ln(9-x)}{\ln(9-x)+\ln(x-3)} dx$

(b) $\int_0^{\frac{\pi}{2}} \frac{\sin^{2000}\theta}{\sin^{2000}\theta + \cos^{2000}\theta} d\theta$.

9) Evaluate $I = \int_0^1 \frac{1}{\sqrt{x}+\sqrt[3]{x}} dx$.

10) By considering the graph of the function $f(x) = x^{-s}$, show that

$$\frac{1}{s-1} < 1 + 2^{-s} + 3^{-s} + \ldots < \frac{s}{s-1}$$

whenever $s > 1$.

11) Evaluate

(a) $I = \int \frac{1}{1-\sin x} dx$ (b) $J = \int e^x \sin x dx$ (c) $K = \int \sqrt{e^{2x}+1} dx$

12) Which of the two is larger, $I = \int_0^1 \sqrt[4]{1-x^7} dx$ or $J = \int_0^1 \sqrt[7]{1-x^4}$?

13) Show that

$$\int_{\pi/2}^{\pi} \frac{x \sin x}{1+\cos^2 x} dx = \int_0^{\pi/2} \frac{(\pi - x) \sin x}{1+\cos^2 x}.$$

Hence find $I = \int_0^{\pi} \frac{x \sin x}{1+\cos^2 x} dx$.

14) Show that

$$\int_0^1 \frac{x^2}{\sqrt{1-x^2}} dx = \int_0^1 \sqrt{1-x^2} dx.$$

Hence find $I = \int \frac{x^2}{\sqrt{1-x^2}} dx$.

15) Evaluate the following expression:

$$\frac{d}{dt} \int_0^{2t^2} (xt)^4 dx.$$

16) Find $\int \frac{x(x-1)}{1-\sqrt{x}}$.

17) What are the intercepts on the x axis made by tangents to the curve $y = \int_0^x |t| dt$, $x \in \mathbb{R}$, which are parallel to the line $y = 2x$?

18) Compute the value of $\int_{-5}^{3} |x^2 - 3x + 2|(1-x) dx$.

19) Show that

$$\int_0^1 \left\{ \frac{1}{\sqrt{x}} \right\} dx = 2 - \frac{\pi^2}{6}.$$

Note: $\sum_{n=1}^{\infty} \frac{1}{n^2} = \frac{\pi^2}{6}$.

20) (a) Find the values of

i. $\int_{-1}^{1}(x^2 - x)dx$

ii. $\int_{-1}^{1}(x^3 + x^2 - 2x)dx$

(b) Sketch the graph of $y = x^2 - x$ and indicate which difference in areas is represented by your answer to part (a)(i).

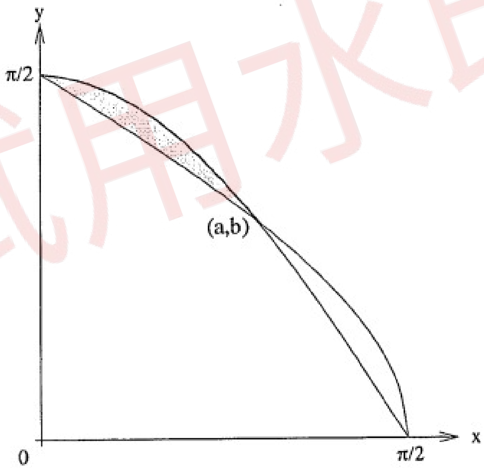
(c) Find the total area (measured positively) that lies between the graphs of $y = x^2 - x$ and $y = x^3 + x^2 - 2x$ between $x = -1$ and $x = 1$.

(d) The answers to (a)(i) and (a)(ii) are related in a particular way. Explain how the relationship can be seen *without* working out any integrals.

21) What is the value of the definite integral

$$\int_1^2 \frac{dx}{x + x^3} dx?$$

22) The curves $y = \frac{1}{2}\pi \cos x$ and $x = \frac{1}{2}\pi \cos y$ intersect at three distinct points $(0, \frac{1}{2}\pi)$, (a, b), $(\frac{1}{2}\pi, 0)$, as shown below.



(a) Explain why $a = b = \frac{1}{2}\pi \cos b$.

(b) Show that $\pi \sin b = \sqrt{\pi^2 - 4b^2}$.

(c) Show that the area of the shaded region is

$$\sqrt{\pi^2 - 4b^2} - \frac{\pi}{2} - b^2.$$

23) Let

$$f(x) = x^3 - 3x^2 + 2x.$$

(a) Sketch the curve $y = f(x)$ for the range $-1 < x < 3$, carefully labelling any turning points.

(b) The equation $f(x) = k$ has exactly one positive solution and exactly one negative solution. Find k.

For x in the range $0 \le x \le 2$, the functions $g(x)$ and $f(x)$ are defined by

$$g(x) = \int_0^x f(t)dt$$
$$h(x) = \int_0^x |f(t)|dt.$$

(c) Find the value X_1 of x in the range $0 \le x \le 2$ for which $g(x)$ is greatest. Calculate $g(X_1)$.

(d) Find the value X_2 of x in the range $0 \le x \le 2$ for which $h(x)$ is greatest.

24) (a) Find $\frac{dy}{dx}$ for each of the functions

$$y = \sin(\ln x)$$
$$y = x\sin(\ln x)$$
$$y = x\cos(\ln x)$$

(b) Sketch the following curves:

i. $y = \ln x$, for $1 \le x \le e^{\pi}$

ii. $y = \sin(\ln x)$, for $1 \le x \le e^{\pi}$

(c) Evaluae

$$\int_1^{e^{\pi}} \sin(\ln x)dx.$$

25) Which of the following integrals has the greatest value?

(a) $\int_0^{\pi/2} \sin^2 x \cos x dx$

(b) $\int_0^{\pi} \sin^2 x \cos x dx$

(c) $\int_0^{\pi/2} \sin x \cos^2 x dx$

(d) $\int_0^{\pi/2} \sin 2x \cos x dx$

26) (a) Calculate the derivative of the following functions:

i. $f(x) = 4x^{-1/4} + \frac{x}{(x^2-1)^{1/3}}$

ii. $f(x) = \exp[\sin^2(1/x)]$

(b) Solve the following integrals:

i. $\int x \sin x dx$

ii. $\int \frac{1}{\sqrt{1-x^2}} dx$

27) Calculate $\int_0^{\pi} (x \sin x)^2 dx$.

28) Let $I_n = \int_0^1 \frac{x^n}{\sqrt{x^3+1}} dx$. Show that $(2n-1)I_n + 2(n-2)I_{n-3} = 2\sqrt{2}$ for all $n \geq 3$. Compute I_8.

29) Evaluate

$$\int_0^1 \frac{dx}{x + \sqrt{1-x^2}}.$$

30) Integrate $\frac{1}{x^2}$ from -1 to 1. What integer power of x will make the area finite in an interval near $x = 0$?

31) Evaluate

$$\frac{d}{dx}\left(\int_0^x \frac{e^y}{1+y^2} dy\right) \quad \text{and} \quad \frac{d}{dx}\left(\int_0^{x^2} \frac{e^y}{1+y^2} dy\right).$$

32) Is $\int_1^{\infty} \frac{1}{x} \sin\left(\frac{1}{x}\right) dx$ bounded or unbounded?

33) Integrate $x^5 e^{-x^2}$ from negative to positive infinity.

34) Integrate $\frac{1}{x^2}$ from -1 to 1. What integer power of x will make the area finite in an interval near $x = 0$?

35) The function f is defined by

$$f(x) = \frac{e^x - 1}{e - 1}, \qquad x \geq 0,$$

and the function g is the inverse function to f, so that $g(f(x)) = x$. Sketch $f(x)$ and $g(x)$ on the same axes.

Verify, by evaluating each integral, that

$$\int_0^{\frac{1}{2}} f(x)dx + \int_0^k g(x)dx = \frac{1}{2(\sqrt{e}+1)},$$

where $k = \frac{1}{\sqrt{e}+1}$, and explain this result by means of a diagram.

36) (a) Show that, for $m > 0$,

$$\int_{1/m}^{m} \frac{x^2}{x+1} dx = \frac{(m-1)^3(m+1)}{2m^2} + \ln m.$$

(b) Show by means of a substitution that

$$\int_{1/m}^{m} \frac{1}{x^n(x+1)} dx = \int_{1/m}^{m} \frac{u^{n-1}}{u+1} du.$$

(c) Evaluate

i. $\int_{1/2}^{2} \frac{x^5+3}{x^3(x+1)} dx$

ii. $\int_{1}^{2} \frac{x^5+x^3+1}{x^3(x+1)} dx$

37) Differentiate $\sec t$ with respect to t.

(a) Use the substitution $x = \sec t$ to show that $\int_{\sqrt{2}}^{2} \frac{1}{x^3\sqrt{x^2-1}} dx = \frac{\sqrt{3}-2}{8} + \frac{\pi}{24}$

(b) Determine $\int \frac{1}{(x+2)\sqrt{(x+1)(x+3)}} dx$,

(c) Determine $\int \frac{1}{(x+2)\sqrt{x^2+4x-5}} dx$.

38) Evaluate the following integrals, in the different cases that arise according to the value of the positive constant a:

(a) $\int_0^1 \frac{1}{x^2+(a+2)x+2a} dx$.

(b) $\int_1^2 \frac{1}{u^2+au+a-1} du$.

39) Let

$$I = \int_0^a \frac{\cos x}{\sin x + \cos x} dx \qquad \text{and} \qquad J = \int_0^a \frac{\sin x}{\sin x + \cos x},$$

where $0 \le a < \frac{3\pi}{4}$. By considering $I+J$ and $I-J$, show that $2I = a + \ln(\sin a + \cos a)$.

Find also:

(a) $\int_0^{\frac{\pi}{2}} \frac{\cos x}{p \sin x + q \cos x} dx$ where p and q are positive numbers.

(b) $\int_0^{\frac{\pi}{2}} \frac{\cos x + 4}{3 \sin x + 4 \cos x + 25} dx$.

40) Show that, for any integer m,

$$\int_0^{2\pi} e^x \cos(mx)dx = \frac{1}{m^2+1}(e^{2\pi}-1).$$

(a) Expand $\cos(A+B)+\cos(A-B)$. Hence show that

$$\int_0^{2\pi} e^x \cos x \cos 6x dx = \frac{19}{650}(e^{2\pi}-1).$$

(b) Evaluate $\int_0^{2\pi} e^x \sin 2x \sin 4x \cos x dx$.

41) Show that

$$\int_{-1}^{1} |xe^x| dx = -\int_{-1}^{0} xe^x dx + \int_0^1 xe^x dx$$

and hence evaluate the integral.

Evaluate the following integrals:

(a) $\int_0^4 |x^3 - 2x^2 - x + 2| dx$;

(b) $\int_{-\pi}^{\pi} |\sin x + \cos x| dx$.

42) Show that

$$\int_0^{\ln n} \lfloor e^x \rfloor dx = n \ln n - \ln(n!).$$

43) The function f satisfies the condition $f'(x) > 0$ for $a \le x \le b$, and g is the inverse of f. By making a suitable change of variable, prove that

$$\int_a^b f(x)dx = b\beta - a\alpha - \int_\alpha^\beta g(y)dy,$$

where $\alpha = f(a)$ and $\beta = f(b)$. Interpret this formula geometrically, in the case where α and a are both positive.

Prove similarly and interpret (for $\alpha > 0$ and $a > 0$) the formula

$$2\pi \int_a^b xf(x)dx = \pi(b^2\beta - a^2\alpha) - \pi \int_\alpha^\beta [g(y)]^2 dy.$$

44) Given that $f''(x) > 0$ when $a \le x \le b$, explain with the aid of a sketch why

$$(b-a)f\left(\frac{a+b}{2}\right) < \int_a^b f(x)dx < (b-a)\frac{f(a)+f(b)}{2}.$$

By choosing suitable a, b and $f(x)$, show that

$$\frac{4}{(2n-1)^2} < \frac{1}{n-1} - \frac{1}{n} < \frac{1}{2}\left(\frac{1}{n^2} + \frac{1}{(n-1)^2}\right),$$

where n is an integer greater than 1.

Deduce that

$$4\left(\frac{1}{3^2} + \frac{1}{5^2} + \frac{1}{7^2} + \ldots\right) < 1 < \frac{1}{2} + \left(\frac{1}{2^2} + \frac{1}{3^2} + \frac{1}{4^2} + \ldots\right).$$

Show that

$$\frac{1}{2}\left(\frac{1}{3^2} + \frac{1}{4^2} + \frac{1}{5^2} + \frac{1}{6^2} + \ldots\right) < \frac{1}{3^2} + \frac{1}{5^2} + \frac{1}{7^2} + \ldots,$$

and hence show that

$$\frac{3}{2} < \sum_{n=1}^{\infty} \frac{1}{n^2} < \frac{7}{4}.$$

45) Give a rough sketch of the function $\tan^k \theta$ for $0 \le \theta \le \frac{\pi}{4}$ in the two cases $k = 1$ and $k \gg 1$. Show that for any positive integer n

$$\int_0^{\frac{\pi}{4}} \tan^{2n+1} \theta d\theta = (-1)^n \left(\frac{1}{2}\ln 2 + \sum_{m=1}^{n} \frac{(-1)^m}{2m}\right),$$

and deduce that

$$\sum_{m=1}^{\infty} \frac{(-1)^{m-1}}{2m} = \frac{1}{2}\ln 2.$$

Show similarly that

$$\sum_{m=1}^{\infty} \frac{(-1)^{m-1}}{2m-1} = \frac{\pi}{4}.$$

Hints

Question 29

As explained in the lessons, what's a good sub to try if we see $\sqrt{1-x^2}$ terms?

Question 30

There's a discontinuity at $x = 0$. How can we split up our integral to take this into account?

Question 32

Think about what the integrand looks like for large x and remember, a good way to show an integral/series converges is to show it's smaller than another convergent integral/series. And similarily, to show an integral/series diverges, it would be sufficient to show it's larger than a divergent integral/series.

Question 35

If you wanted to find the area bound between $y = f(x)$, the x axis and the lines $x = a$ and $x = b$, then we find $\int_a^b y\,dx$. How do we found the area bound between $y = f(x)$, the y-axis and the lines $y = a, y = b$?

Question 39

For (a) and (b), create new pairs of I and J integrals and deploy a similar strategy to the stem.

Question 40

IBP works for the stem, but try to do it another way. Rewrite $\cos(mx)$ as $\text{Re}(e^{imx})$ and try to figure out another way of computing this integral.

Question 41

Absolute value functions are piecewise functions - how do we deal with them?

Question 43

Another question on integrals and inverses, think about the hint for the previous question (understand how to find the integral of $y = f(x)$ between 2 vertical lines or 2 horizontal lines).

Question 44

For the stem, note the function is convex. How can we find an upper and lower bound for the integral (i.e. the area)?

Question 45

It's hard to integrate a power of tan(theta) directly - what could be good sub to use to transform the integral? Also note the RHS seems to have a series, so perhaps using a Taylor series for a function somewhere could generate this.

Differentiation Interview Practice Problems

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Useful Taylor Expansions

- $e^x = 1 + x + \frac{x^2}{2!} + \ldots = \sum_{k=0}^{\infty} \frac{x^k}{k!}$, converges for $|x| < \infty$
- $\ln(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \ldots = -\sum_{k=1}^{\infty} \frac{(-1)^k x^k}{k}$, converges for $|x| < 1$.
- $\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} + \ldots = \sum_{k=0}^{\infty} \frac{(-1)^k x^{2k}}{(2k)!}$, converges for $|x| < \infty$.
- $\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \ldots = \sum_{k=0}^{\infty} \frac{(-1)^k x^{2k+1}}{(2k+1)!}$, converges for $|x| < \infty$.

Questions

1) Sketch the following curves and comment on their derivatives.

(a) $|x^2 - 1|$

(b) $x^{1/3}$

(c) $x^{2/3}$

2) Given that $y = \int_0^x t^8 e^t \, dt$, find $\frac{dy}{dx}$.

3) Find $f(x)$ if $\int_0^x f(t)dt = 3f(x) + k$, where k is a constant.

4) Show from first principles that $\frac{d}{dx}x^n = nx^{n-1}$.

5) Prove from first principles that $\frac{d}{dx}(f(x)g(x)) = g(x)\frac{df}{dx} + f(x)\frac{dg}{dx}$.

6) Prove from first principles that $\frac{d}{dx}f(g(x)) = g'(x)f'(g(x))$.

7) This question aims to prove L'Hopital's rule, which you should memorise.

(a) Suppose $f(x_0) = g(x_0) = 0$. By expanding $f(x)$ and $g(x)$ about $x = x_0$, prove that

$$\lim_{x \to x_0} \frac{f(x)}{g(x)} = \lim_{x \to x_0} \frac{f'(x_0)}{g'(x_0)}$$

(b) Now suppose $f(x_0) = g(x_0) = \infty$. By first writing $\frac{f(x)}{g(x)} = \frac{1/g(x)}{1/f(x)}$ and using part (a), or otherwise, prove that

$$\lim_{x \to x_0} \frac{f(x)}{g(x)} = \lim_{x \to x_0} \frac{f'(x)}{g'(x)}.$$

8) (a) Find the Taylor series for $f(x) = e^{ax}$ about $x = 1$

(b) Find the Taylor series of $\ln(1 + x)$ about $x = 0$, and use it to show that

$$\lim_{k \to \infty} k \ln\left(1 + \frac{x}{k}\right) = x.$$

Hence, deduce that

$$\lim_{k \to \infty} \left(1 + \frac{x}{k}\right)^k = e^x.$$

You should memorise this final result.

9) Calculate the Taylor expansions of the following, about $x = 0$:

(a) $(x^2 + a)^{-3/2}$

(b) $\ln(\cos x)$

(c) $e^{-1/(x-a)^2}$ where a is a constant

10) Determine the half-life of thorium-234 if a sample of mass 5 g is reduced to 4 g in one week. What amount of thorium is left after three months?

11) Solve the initial value problems:

(a) $y' + 2y = e^{-x}, y(0) = 1$

(b) $y' - y = 2xe^{2x}, y(0) = 1$

12) Let $f(x) = \sum_{k=0}^{n} \frac{x^k}{k!}$. Find $f'(x)$ in terms of $f(x)$.

13) Show that the general solution of

$$y' - y = e^{ux},$$

where $u \neq 1$, can be written in the form $y(x) = ke^x + \frac{e^{ux} - e^x}{u-1}$, where k is a constant to be determined.

By using L'hopital's rule to find the limit of $y(x)$ as $u \to 1$, find the general solution to the DE above for $u = 1$.

14) Solve

(a) $x \sin x y' + (\sin x + x \cos x) y = x e^x$

(b) $\tan x y' + y = 1$

(c) $y' = (e^y - x)^{-1}$

15) Find the general solutions of

(a) $y' = x^2(1 + y^2)$

(b) $y' = \cos^2 x \cos^2 2y$

(c) $y' = (x - y)^2$

16) Find the general solution of $(y')^2 - 9y^2 = 10$.

17) Without solving the DE explicitly, draw the solutions to the initial value problems:

(a) $y' = x^2 + y^2, y(0) = 1$

(b) $y' = (1 - y)(2 - y)$ for

(i) $y(0) = 4$

(ii) $y(0) = 1.5$

(iii) $y(0) = 0.5$

18) What is the derivative of $y = x|x|$?

19) Let f be a differentiable function. Evaluate

$$\lim_{x \to a} \frac{af(x) - xf(a)}{x - a}.$$

20) Find the derivative of $f(x) = (1 - x)(2 - x)(3 - x)...(n - x)$ at $x = 1$.

21) By using symmetry arguments, find $\frac{dy}{dx}$ for the following.

(a) $y^2 = 4ax$

(b) $\cos y = 3x$

(c) $\frac{1}{2} e^y - x = 0$

22) Which is larger, e^π or π^e?

23) What is the derivative of the following. You will need to consider different cases for x.

(a) $y = \{x\}$

(b) $y = \{2x\}$

(c) $y = \{x^2\}$

24) Evaluate the following limits. You should commit (a)-(c) to memory as they are important results.

(a) $\lim_{x \to \infty} \frac{x^n}{e^x}$ for any real number n

(b) $\lim_{x \to 0} \frac{\sin x}{x}$ using:

  (i) Taylor series

  (ii) L'hopital's rule

(c) $\lim_{x \to \infty} \frac{\ln x}{x^n}$ for any real number n

(d) $\lim_{x \to 0} x^{x^{x^x}}$

(e) $\lim_{x \to 1} \frac{1-\sqrt{x}}{1-x}$

(f) $\lim_{x \to 0} \frac{\ln(1+x)}{x}$

(g) $\lim_{x \to 0} \frac{\log_a(1+x)}{x}$

(h) $\lim_{x \to 0} \frac{a^x - 1}{x}$

(i) $\lim_{x \to 1} \frac{x^m - 1}{x-1}$

(j) $\lim_{x \to a} \frac{x^m - a^m}{x-a}$

(k) $\lim_{x \to 0} \frac{\sin^{-1} x}{x}$

(l) $\lim_{\infty} \frac{\sqrt{1+4x^6}}{2-x^3}$

(m) $\lim_{x \to 0} \frac{(1+x)^2-1}{(1+x)^3-1}$

(n) $\lim_{x \to 0} \frac{1-\cos x}{\sin x}$

(o) $\lim_{x \to \infty} \cos(\tan^{-1} x)$

(p) $\lim_{x \to \infty} (\sqrt{x+3} - \sqrt{x-1})$

(q) $\left(\frac{1}{n} \left(\frac{1}{n^2} \right)^2 + \frac{1}{n} \left(\frac{2}{n^2} \right)^2 + \ldots + \frac{1}{n} \left(\frac{n}{n^2} \right)^2 \right)$ as $n \longrightarrow \infty$

25) What is the derivative of $y = x^{x^{x^{x^{\cdots}}}}$?

26) Differentiate x^x.

27) Differentiate $y = x$ with respect to x^2.

28) What is $\tan^{-1}(x) + \tan^{-1}\left(\frac{1}{x}\right)$?

29) Let $f(x) = x + 2x^2 \sin(\frac{1}{x})$. Show that $f'(0) = 1$. You may assume $f(0) = 0$.

30) The line $y = mx$ touches the curve $y = \ln x$ at the point P, which has coordinates (a, b).

The line L, which has a positive gradient, passes through the point P and forms an angle of $60°$ with the horizontal.

What is the y-intercept of L?

31) You are given that $y(x) = \int_x^3 f(t)dt$ satisfies the differential equation

$$\frac{d^2y}{dx^2} + 3 - 4x = 0.$$

Given also that $f(0) = 1$, find $f(x)$.

32) (a) Verify that

$$y(x) = \int_x^\infty e^{-t^2}\,dt$$

satisfies the differential equation $y'' + 2xy' = 0$.

(b) By using an integrating factor on the DE, derive the general solution in integral form.

(c) Explain why the solution given in part (a) is one of the solutions found in part (b).

33) Prove the Leibniz rule (generalised product rule for the nth derivative of a product of 2 functions):

$$\frac{d^n}{dx^n}(f(x)g(x)) = \sum_{k=0}^{n} \binom{n}{k} f^{(k)}(x) g^{(n-k)}(x),$$

where $\binom{n}{k} = \frac{n!}{k!(n-k)!}$.

34) Recall $e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!}$ and $e^{ix} = \cos x + i \sin x$. Starting from this, derive the Taylor expansions of

(a) $\cos x$

(b) $\sin x$

Hints

Question 1

What values of x could cause problems? Investigate what happens there.

Question 5

Write the limit definition of $\frac{d}{dx}(f(x)g(x))$. Try adding and subtracting the same term to end up with the thing you want.

Question 6

What technique do you know to rewrite something of the form $f(x+h)$?

Question 7

For part (a): try expanding f and g about a suitable point.

For part (b): how do $1/f$ and $1/g$ behave as $x \to x_0$? After applying L'Hopital's rule, do you see something you can relate to the thing you're trying to find?

Question 14

What happens if you take the reciprocal of both sides?

Question 15

Part (c): It seems like combining the x and y terms by squaring is getting in the way of making a separable DE. Can you think of a way to get rid of the $(y-x)^2$ or swap it for something nicer?

Question 17

Part (a): What does the gradient really only depend on? Knowing this, how does the gradient behave?

Part (b): Are there any special points? It may help to sketch y against $\frac{dy}{dx}$ and go from there.

Question 18

Try splitting the function into different regions.

Question 19

Does the RHS remind you of anything? Perhaps you can do something to make it look more similar to whatever you are thinking of.

Question 21

Part (a), consider the graph of $x^2 = 4ay$. How can you relate the derivatives?

Question 22

(Go back to the question after first part of hint)

Try to construct a function that might be helpful.

There is more than one option for the function. You can try $f(x) = x^{1/x}$ or $f(x) = e^x - x^e$.

Question 23

Part (a): What happens when x is an integer? What does it mean for the derivative to exist at a point?

For (b) and (c): try to use part (a).

Question 24

Part (a): Can you write e^x in another way?

Part (c): Can you think of a useful substitution?

Part (d): Can you work out the limit of x^x as $x \to 0$?

Question 25

Can you spot any symmetry in the expression?

Question 27

You can try a substitution. Or work with the fact that $\frac{d((x^2)}{dx} = 2x$.

Question 28

You could make use of the tan addition formula if you know it. Or, you can let $f(x) = \tan^{-1}(x) + \tan^{-1}\left(\frac{1}{x}\right)$ and consider $f'(x)$.

Graphs Interview Practice Problems

Questions

1) Where in the plane is $\sin^2 x + \cos^2 y = 1$?

2) Sketch $y = x \ln x$.

3) Sketch $y = \frac{\ln(x)}{x}$ and hence find all natural solutions of the equation $a^b = b^a$.

4) Sketch $y = x^x$ and $y = x^{1/x}$.

5) Sketch $y = \frac{\sin x}{x}$ and $y = \frac{\sin x}{x-1}$.

6) Sketch $y = \cos\left(\frac{1}{x}\right)$ and $y = \sin\left(\frac{1}{x}\right)$.

7) Sketch $y = \frac{x+\sin x}{x-\sin x}$.

8) Sketch $y = \sqrt{x^3 - x}$ and $y^2 = x^3 - x$.

9) Sketch $y = \frac{x^4-7x^2+12}{x^4-4x^2+4}$.

10) Sketch $y = e^{-x^2} - e^{-3x^2}$.

11) Write $\frac{3e^x-e^{-x}}{e^x+e^{-x}}$ in the form $a + \frac{b}{e^{2x}+1}$ and hence sketch $y = \frac{3e^x-e^{-x}}{e^x+e^{-x}}$.

12) Sketch $x^{2n} + y^{2n} = 1$ for $n = 2$ and 4. Explain what happens to the graph as $n \to \infty$.

13) Sketch $y = \sqrt{1-x^2} + \sqrt{4-x^2}$.

14) Sketch

(a) $1 = |x| + |y|$

(b) $1 = |x| - |y|$

(c) $1 = |y - x|$

15) Is the series $1 + \frac{1}{2} + \frac{1}{3} + ...$ divergent? How about the series $1 + \frac{1}{4} + \frac{1}{9} + \frac{1}{16} + ...$?

Sketch on separate axes $y = \frac{1}{x}$ and $y = \frac{1}{x^2}$. By considering your sketches and by using integration, justify your claims.

16) By considering the graph of the function $f(x) = x^{-s}$, show that

$$\frac{1}{s-1} < 1 + 2^{-s} + 3^{-s} + \ldots < \frac{s}{s-1}$$

whenever $s > 1$.

oint which minimises the area of the triangle.

17) For which value of k is there only one solution to the equation $e^x = kx$?

18) If $f(x) = \cos x$, sketch:

(a) $f\left(\frac{1}{x}\right)$

(b) $f(2x) - f(x)$

(c) $f(f(x))$

19) Sketch $y = \sin\left(\frac{1}{(x+1)(x-1)}\right)$.

20) Give one example of the second derivative being zero and the corresponding stationary point not being a point of inflexion.

21) Define stationary point. Does $|x|$ have a stationary point? Sketch $x|x|$. Does $x|x|$ have a stationary point?

22) Consider $x^3 + ax^2 + bx + c = 0$. Under what conditions is x a rational root?

23) Sketch the graph of $y = \frac{x^2+1}{x+1}$.

24) Let L_1 and L_2 be two lines in the plane, with equations $y = m_1x + c_1$ and $y = m_2x + c_2$ respectively. Suppose that they intersect at an acute angle θ. Show that

$$\tan\theta = \left|\frac{m_1 - m_2}{1 + m_1m_2}\right|.$$

25) Let M be a large real number. Explain briefly why there must be exactly one root ω of the equation $Mx = e^x$ with $\omega > 1$. Why is $\log M$ a reasonable approximation to ω? Write $\omega = \log M + y$. Can you give an approximation to y, and hence improve on $\log M$ as an approximation to ω?

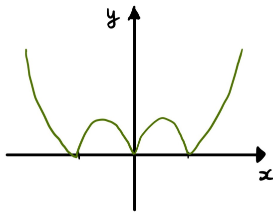
26) Sketch $y = \frac{1}{x^2+a^2}$ for $a < 0$, $a = 0$ and $a > 0$.

27) Sketch $y = \frac{1}{x}\sin\left(\frac{1}{x}\right)$.

28) Sketch $y = \tan^{-1}(x)$ and $y = \tan^{-1}(1/x)$ on the same axes. Hence, determine the value of $\tan^{-1}(x) + \tan^{-1}(1/x)$.

29) Sketch $y = e^{\sin x}$ and $y = e^{\cos x}$ on the same axes. Explain how we can get from one graph to the other.

30) The diagram shows the graph of $y = |f(x)|$. The graph of $y = f(x)$ is a continuous curve. How many different possibilities are there for the graph of $y = f(x)$?



31) Suppose f is a function that has an inverse. What property must the graph of $y = f(x)$ have if $f(x) = f^{-1}(x)$ for all x? Draw the graph of one such function f.

32) For which constants a, b, c and d does the function $f(x) = \frac{ax+b}{cx+d}$ have an inverse?

33) The graph of $y = x^2 + 2x - 2$ is reflected over the line $y = x$. How many points of intersections does the original graph have with its reflection?

34) What is the range of the function $f(x) = \frac{x^2-2x+3}{x^2+2x-3}$?

35) Suppose that $f(x)$ is non-zero and monotonically increasing. Must $1/f(x)$ be monotonically decreasing?

36) Without using a calculator, order the following from greatest to least:

$$10 - 3\sqrt{11}, \qquad 7 - 4\sqrt{3}, \qquad 5\sqrt{41} - 32, \qquad 9 - 4\sqrt{5}.$$

37) Find $f^{-1}(x)$ if $f(x) = x|x| + 2$.

38) What is the domain and range of the functions $f(x) = \ln x$, $f(f(x))$ and $f(f(f(x)))$? What about $f^n(x)$?

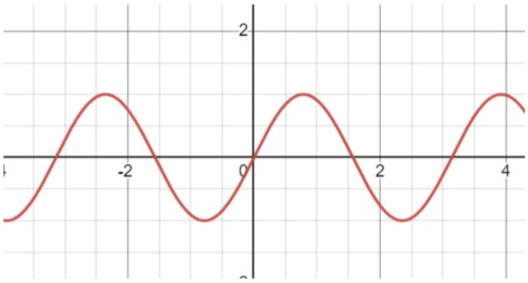
39) Show that if four distinct points of the curve $y = 2x^4 + 7x^3 + 3x - 5$ are collinear then their average x-coordinate is some constant k. Find k.

40) By sketching appropriate graphs, find all solutions to the equation $x - 1 = (e-1)\log x$. Hence sketch the graph of $f(x) = e^x - x^e$. (Here $\log x$ denotes the logarithm to base e - you may be more used to the notation $\ln x$.)

41) Which is larger as n gets large, $f(n) = 2^{2^{2^n}}$ or $g(n) = 100^{100^n}$?

42) Sketch the graph of $\sqrt{y^2-1} = \frac{1}{2}(e^x - e^{-x})$ and also of $\sqrt{y^2+1} = \frac{1}{2}(e^x + e^{-x})$. In each case find an explicit formula for x in terms of y.

43) Which of the graphs is $\sin(2x)$, $e^{-x}\sin x$, $\sin\sqrt{x}$, $\sin(1/x)$: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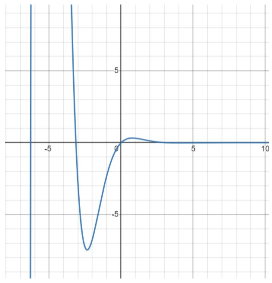
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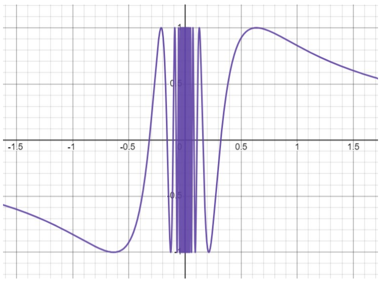
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25

Number Theory Interview Practice Questions

1) Show that $2^{19} + 5^{40}$ is not prime. Show also that $2^{91} - 1$ is not prime.

2) If n^2 is a multiple of 3, must n be a multiple of 3?

3) Suppose that we have some positive integers (not necessarily distinct) whose sum is 100. How large can their product be?

4) Find the number of integer solutions to the equation $|x| + |y| \leq 100$.

5) Are there any integer solutions to the equation $x^2 + y^2 = 3z^2$ where x, y, z are co-prime? Are there any integer solutions at all?

6) (a) Sketch $x^2 - ny^2 = 0$ where n is a natural number.

(b) Find all natural solution pairs (x, y) in the case $n = 9$.

(c) Find all natural solution pairs (x, y) in the case $n = 10$.

7) How many natural number solutions are there to the equation $\frac{1}{a} + \frac{1}{b} + \frac{1}{c} = 1$ where $a < b < c$?

8) (a) Find all positive integer solutions (x, y) to $x^2 + y^2 = 2015$.

(b) Will the equation $x^2 + 33y^2 = 555, 555, 555$ have any positive integer solutions?

9) If n, x, y, z are all positive integers, find all solutions of the equation $n^x + n^y = n^z$.

10) Given n consecutive positive integers, show that $n!$ is a factor of their product.

11) Prove that the only solution to the equation $x^2 + y^2 + z^2 = 2xyz$ for integers x, y, z is $x = y = z = 0$.

12) If x and y are positive integers, find all solutions of the equation $2xy - 4x^2 + 12x - 5y = 11$.

13) A right angled triangle has all of its sides an integer length. If the length of the perimeter equals the area, find all such triangles.

14) Prove that $n^2(n^2 - 1)(n^2 - 4)$ is divisible by 360 whenever n is a natural number.

15) Find the last two digits of 99^n.

16) Which will be larger as $n \to \infty$; $2^{2^{2^n}}$ or 100^{100^n}?

17) (a) Prove that $n^2 - 1$ is divisible by 8 when n is odd.

(b) Prove that $n^5 - n$ is divisible by 6 whenever n is a natural number.

(c) Prove that $n^5 - n$ is divisible by 30 whenever n is a natural number.

18) Construct a counter example to the statement: When written in decimal notation, every square number has at most 1000 digits that are not 0 or 1.

19) Prove that $4^n - 1$ is divisible by 3 whenever n is a natural number.

20) A natural number from 1 to $1,000,000$ is selected at random. What is the probability that its cube ends in 11?

21) Prove that there are infinitely many primes.

22) Prove that there are infinitely many primes of the form $4n + 3$.

23) Is $\log_2 3$ rational? Prove it.

24) Prove that $14^n + 11$ is never prime.

25) Let n be a natural number. Suppose $a^n - 1$ is prime. Show that $a = 2$ and n must be prime (Mersenne Primes). Comment on primes of the form $2^n + 1$ (Fermat numbers).

26) Find all prime numbers p such that $2p - 1$ and $2p + 1$ are also prime.

27) Given that $\sum_{r=1}^{\infty} \frac{1}{r^2} = \frac{\pi^2}{6}$, find the exact value of $\sum_{r=1}^{\infty} \frac{1}{(2r-1)^2}$.

28) Prove that if a, b, c are all odd then the quadratic equation $ax^2 + bx + c = 0$ cannot have rational roots.

29) If a natural number n has N digits, how many digits can n^2 have? What about n^n? How would you write a formula for the number of digits of n?

30) Is $\tan(1°)$ irrational? What about $\cos(1°)$?

31) Find

$$\lim_{n\to\infty}\left(\frac{1}{n} + \frac{(n-1)^2}{n^3} + \frac{(n-2)^2}{n^3} + \ldots + \frac{1}{n^3}\right).$$

32) Evaluate

$$\prod_{n=2}^{\infty}\left(1 - \frac{1}{n^2}\right).$$

33) Conjecture and prove a formula for $1 \times 1! + 2 \times 2! + 3 \times 3! + \ldots + n \times n!$.

34) Is 1234567891011 a square number? Is 24681012141618202224?

35) How many solutions are there to the equation

$$(x^2 - 5x + 5)^{(x^2 - 11x + 30)} = 1?$$

36) What is the last digit of 2^{2018}?

37) Find all the numbers x for which:

(i) $4 - x < 3 - 2x$

(ii) $|x - 3| = 8$.

Prove that if $|x - x_0| < \frac{}{2}$ and $|y - y_0| < \frac{}{2}$ then

$$|(x + y) - (x_0 + y_0)| < \qquad \text{and} \qquad |(x - y) - (x_0 - y_0)| < .$$

38) If the ratio of consecutive Fibonacci numbers approaches a limit, what must this limit be?

39) What regular polygons can fill the plane?

40) Prove that the following are irrational:

(a) $\sqrt{2}$

(b) $\log_2 3$

(c) $\sqrt{2} + \sqrt{3}$.

41) How many zeros are at the end of 25!? What about 80!?

42) Prove that $n^5 - n$ is divisible by 30.

43) How many digits does 100! have? Explain a method to find the number of digits of N! and explain whether your estimate provides a lower or upper bound.

44) The numbers 3, 5, 7 are all prime. Does it ever happen again that three numbers n, $n + 2$, $n + 4$ are all prime?

45) Can 10000003 be a square number? What about the sum of two squares?

46) Prove if the sum of the digits of a number is divisible by 3, then the number is divisible by 3. What about if we replaced 3 by 9?

47) Prove that 12 divides $n^4 - n^2$.

48) Find all primes p such that $2p + 1 = m^3$ where m is a positive integer.

Probability Interview Practice Questions

1) Two players, A and B, play a game where they take turns throwing a fair coin, and player A goes first. The first person to get a head wins. What is the probability that A wins? What about if the coin was biased and the probability of getting a head at every toss is p?

2) You play a game with a coin, not necessarily fair. If you throw heads, you get £1 and continue. If you throw tails, you get nothing and the game ends. What is the expected value (i.e. average) of your earnings when playing this game?

3) If a round table has n people sitting around it, what is the probability that person A is sitting exactly k seats away from person B?

4) A thin rod is broken into three pieces. What is the probability that a triangle can be formed from the three pieces? What about if we want the triangle to be acute (e.g. all the internal angles are less than 90 degrees)?

5) A natural number from 1 to 1000000 is selected at random. What is the probability its cube ends in 11?

6) A regular fair dice is rolled twelve times. What is the probability of getting 2 of each number?

7) A fair ten-sided dice is rolled four times. What is the probability your sequence of rolls is increasing?

8) If I am in a room with 5 people and guess all their birthdays randomly, what is the probability of getting exactly one correct?

9) Consider a regular polygon with an odd number of sides. If I pick 3 vertices at random and form a triangle, what is the probability that the centre of the polygon is inside the triangle?

10) There is a pile of 129 coins on a table, all unbiased except one which has heads on both sides. Ahmed chooses a coin at random and tosses it eight times. The coin comes up heads every time. What is the probability that it will come up heads the ninth time as well?

11) Twenty balls are placed in an urn. Five are red, five green, give yellow and five blue. Three balls are drawn from the urn at random without replacement. Write down expressions for the following probabilities (you need not calculate their numerical values):

a) exactly one of the balls is red

b) the three drawn balls have different colors

c) the number of blue balls drawn is strictly greater than the number of yellow balls drawn.

12) Consider a game played with a fair coin. I toss the coin until either there are 2 heads (HH) or 2 tails (TT) in a row. I win the game if I get HH and you win if I get TT. Find the probability that I win the game. Is the answer surprising? What if the game was changed so that you win if I get HT instead? What about TH?

13) In a tennis tournament, there are $2n$ participants. In the first round of the tournament, each player plays exactly once, so there are n games. Show that the pairings for the first round can be arranged in exactly $\frac{(2n-1)!}{2^{n-1}(n-1)!}$ ways.

14) A hand of thirteen playing cards is dealt from a standard pack of fifty-two. Write down expressions (in terms of binomial coefficients) for the following probabilities:

a) the hand contains exactly one king

b) the hand contains at least two queens

c) the hand contains the same number of kings as queens.

15) Six identical-looking coins are in a box, of which five are unbiased whilst the sixth one comes up heads with probability $\frac{3}{4}$. Three coins are chosen from the box at random and removed. One of those three is chosen at random and tossed three times, coming up heads each time. Given this information

a) what is the probability that the final coin selected was the biased coin?

b) what is the probability that the biased coin is amongst the three coins removed from the bag?

16) A fair coin is tossed repeatedly. Let t be the time at which we first see three consecutive heads (thus flips $t-2, t-2, t$ are all heads) and let s be the time at which we first see four consecutive heads. What is the probability that $s = t+1$? And what is the probability that $s = t+9$?

17) The random variable X has probability density function

$$f(x) = cx(1-x) \quad 0 \leqslant x \leqslant 1,$$

and 0 otherwise. Find the value of c such that this is a valid probability density function. Then, find the expected value and variance of X.

18) You throw $6n$ die at random. Show that the probability that each number appears exactly n times is

$$\frac{(6n)!}{(n!)^6}\left(\frac{1}{6}\right)^6.$$

Use Stirling's formula (stated below) to show that this probability behaves as a power law $cn^{-\frac{5}{2}}$ for some constant c. Determine this constant.

19) A biased coin has probability of landing heads equal to p. If the coin is tossed n times and let X be the number of times it lands heads in total, find $\mathbb{P}(X = k)$. Suppose now that the value of p is unknown. However, it's observed that k heads are obtained after tossing the coin n times. What is the value of p which makes this event most likely? That is, what value of p maximises $\mathbb{P}(X = k)$?

20) If we flip a coin and generate a sequence of length n, what is the probability that the number of heads is even?

21) If we flip a coin and generate a sequence of length n, what is the probability we do not see two heads in a row? Set up a recursion equation and then solve it to find u_n.

22) Tomorrow there will either be rain or snow, but not both. The probability of rain is $\frac{2}{5}$ and the probability of snow is $\frac{3}{5}$. If it rains, the probability that I will be late for my lecture is $\frac{1}{5}$, whilst the corresponding probability in the event of snow is $\frac{3}{5}$. What is the probability I will be late?

23) A rare but potentially fatal disease has an incidence of 1 in 10^5. There is a diagnostic test, but it's imperfect. if you have the disease, the test is positive with probability $\frac{9}{10}$. If you do not, the test is positive with probability $\frac{1}{20}$. Your test result is positive. What is the probability you have the disease?

24) Find the probability of getting at least k heads in n tosses of a fair coin. Express your answer as a sum.

25) We distribute r distinguishable balls into n cells at random, multiple occupancy is allowed. Find the probability that the first cell contains k balls.

26) Find the probability that two given hands in bridge contain k aces between them.

27) Find the probability that a hand in bridge contains 6 spades, 3 hearts, 2 diamonds and 2 clubs.

28) Consider the experiment of tossing a fair coin 7 times. Find the probability of getting a prime number of heads given that heads occurs on at least 6 of the tosses.

29) If X is a discrete random variable having the geometric distribution with parameter p, find $P(X > k)$.

30) You are presented with two urns. Urn 1 has 3 white and 4 black balls. Urn 2 has 2 white and 6 black balls.

a) You pick a ball randomly from urn 1 and place it in urn 2. Next you pick a ball randomly from urn 2. What is the probability that the ball is black?

b) You pick an urn at random with equal chances. Then, you pick a ball at random from the chosen urn. Given that the ball is black, what is the probability you picked urn 1?

31) A biased coin shows heads with probability $p = 1 - q$ whenever it's tossed. Let u_n be the probability that, in n tosses, no two heads occur consecutively. Show that for $n \geqslant 1$

$$u_{n+2} = qu_{n+1} + pqu_n.$$

Solve this to find u_n in the case $p = \frac{2}{3}$.

32) A fair die is thrown m times. Show that the probability of getting an even number of sixes is $\frac{1}{2}\left(1 + (\frac{2}{3})^n\right)$.

33) Independent trials are performed, each with probability p of success. Let P_n denote the probability that n trials result in an even number of successes. Show that

$$P_n = \frac{1}{2}\left(1 + (1 - 2p)^n\right).$$

34) Two people toss a fair coin n times. Find the probability that they throw an equal number of heads.

35) Let u_n be the probability that n tosses of a fair coin contain no run of 4 heads. Find a recurrence relation for u_n and use it to show $u_8 = \frac{208}{256}$.

36) There are n socks in a drawer, three of which are red and the rest are black. Ali chooses his socks by selecting two at random from the drawer, and puts them on. He is three times more likely to wear socks of different colors than to wear matching red socks. Find n.

37) The random variable X is distributed geometrically with parameter p. Find the expected value (i.e. mean) and variance of X.

38) The random variable X has binomial distributions with parameter n and p. Find the expected value (i.e. mean) and variance of X.

39) A coin is tossed repeatedly, with probability of getting heads $p = 1 - q$. Find the expected length of the initial run (this is a run of heads if the first toss gives heads, and of tails otherwise).

40) The random variable N takes non-negative integer values. Show that, provided the series exists,

$$\mathbb{E}(N) = \sum_{k=0}^{\infty} \mathbb{P}(X > k)$$

41) A fair die having two faces coloured blue, two red and two green, is thrown repeatedly. Find the probability that not all colours occur in the first k throws. Deduce that if N is the random variable which takes the value n if all three colours occur in the first n throws, but only two occur in the first $n-1$ throws (i.e. it requires the nth throw to finally hit all three colours), then the expected value of N is $\frac{11}{2}$.

42) The probability of obtaining a head when a biased coin is tossed is p. The coin is toss repeatedly until n heads occur in a row. Let X be the total number of tosses required for this to happen. Find the expected value of X.

43) A slot machine operates so that at the first turn, the probability for the player to win is $\frac{1}{2}$. Thereafter, the probability for the player to win is $\frac{1}{2}$ if he lost at the previous turn, but is $p < \frac{1}{2}$ if he won at the previous turn. Let u_n be the probability the player wins at the nth turn. Show that

$$u_n + (\frac{1}{2} - p)u_{n-1} = \frac{1}{2},$$

and solve this equation to find that (note that this equation holds if $n = 1$ if we set $u_0 = 0$)

$$u_n = \frac{1 + (-1)^{n-1}(\frac{1}{2} - p)^n}{3 - 2p}.$$

44) Ali collects figures from cornflakes packets. Each packet contains one of n distinct figures. Each type of figure is equally likely. Show that the expected number of packets Ali needs to buy to collect a complete is

$$n \sum_{i=1}^{n} \frac{1}{i}.$$

45) Two identical decks of cards, each containing N cards, are shuffled randomly. We say that a k-matching occurs if the two decks agree in exactly k places. Show that the probability that there's a k-matching is

$$\frac{1}{k!}\left(1 - \frac{1}{1!} + \frac{1}{2!} - \frac{1}{3!} + \ldots + \frac{(-1)^{N-k}}{(N-k)!}\right).$$

Mechanics Interview Practice Questions

1) A person is on a lake, in a rowing boat full of rocks. The rocks are thrown into the lake. What happens to the water level?

2) Two men walk $\frac{2}{5}$ of the way into a rail tunnel. They hear a train and run in opposite directions of each other, both at 15mph. Luckily, one of them gets out just as the train reaches the tunnel, and the other gets out just as the train reaches the end of the tunnel. What is the speed of the train?

3) At what angle should you fire a projectile to get the largest horizontal displacement?

4) A ladder rests against a wall and a man stands on it. Label all the forces acting on the ladder and deduce four equations that relate them.

5) If you drove here at 20mph and drove back at 30mph, what was your average speed?

6) Is it possible to calculate the circumference of the Earth if a satellite is known to be moving above you around the Earth at height x metres?

7) If I place a cube in water, what shape does it make on the surface?

8) Consider two identical frictionless slopes, down which we send two identical particles. If each particle starts the same height up each slope, but one rolls whereas the other one simply translates down the slope, which particle will reach the bottom first?

9) I am an oil baron in the desert and I need to deliver oil to four different towns that happen to lie on a straight line. In order to deliver the correct amounts to each town, I must visit each town in turn, returning to my warehouse in between each visit. Where should I position my warehouse in order to drive the shortest distance possible? Roads are no problem since I have a friend who is a sheikh and will build as many roads as I like for free.

10) A body with mass m is falling towards earth with speed v. It has a drag force equal to kv. Set up a differential equation and solve for v.

11) If I throw a ball in a train, what does its flight look like to someone also on the train? And what about the person on the platform as the train passes?

12) If you're standing on a pair of scales in a lift, what happens to the reading when you start moving up a floor? Why?

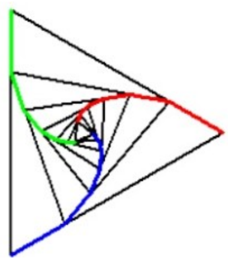
13) How long does a mirror have to be for you to see your whole body?

14) A particle of mass m has potential energy

$$V(x) = \frac{K}{mx^2} - \frac{G}{x}$$

where x is a distance, and K and G are constants. Sketch the potential energy $V(x)$ as a function of x. Find the extrema of $V(x)$, if there are any, and in that case indicate whether they are maxima or minima. What types of motion are possible for this particle?

15) Suppose that A, B and C are three points in a plane, such that $AB = AC = BC = 1$. At each point in time, A is moving toward B, B is moving toward C, and C is moving toward A, all with speed $v = 50$. Shown below is what the movement would look like.



The red curve represents the path from point A, the green curve the path from point B, and the blue curve the path from point C. At what time t will all the points reach the centre of the triangle?

16) A packing case is held on the side of a hill and given a kick down the hill. The hill makes an angle of θ to the horizontal, and the coefficient of friction between the packing case and the ground is □. What relationship between □ and θ guarantees that the packing case eventually comes to rest? Let gravitational acceleration be g. If the relationship above is satisfied, what must the initial speed of the packing case be to ensure that the distance it goes before stopping is d?

17) You are standing at the point $(0, -2)$ and you shine a torch at the curve $y = x^3 - x$. Where should you shine your torch, so that the reflection of light comes back to you?

18) A beam of light falls vertically at the point $x = x_0$ on an arbitrary curve $y = f(x)$. Find the gradient of the reflected beam of light.

19) A bead falls from rest from the local maximum of $y = x^3 - x$ to the local minimum. Find the speed of the bead as it passes through the local minimum.

20) Write down the total energy of a closed system which has a spring on a horizontal ground, in terms of the displacement of equilibrium position x. Hence or otherwise, prove the total energy of the system is conserved.

21) One end of a rod of uniform density is attached to the ceiling in such a way that the rod can swing about freely with no resistance. The other end of the rod is held still so that it touches the ceiling as well. Then the second end is released. If the length of the rod is l metres and gravitational acceleration is g metres per second, how fast is the unattached end of the rod moving when the rod is first vertical.

22) Consider a mass m at position $x(t)$ on a rough horizontal table attached to the origin by a spring with constant k (restoring force $-kx$) and with a dry friction force f

$$\begin{cases} f = F & \text{if } \dot{x} < 0 \\ -F \leq f \leq F & \text{if } \dot{x} = 0 \\ f = -F & \text{if } \dot{x} > 0. \end{cases}$$

What is the range of x where the mass can rest? Show that if the mass moves, the maximum excursion decreases by $2F/k$ per half cycle. Discuss the motion.

23) A cylindrical spaceship of mass M and cross-sectional area A is coasting at constant velocity when it suddenly encounters a dust cloud. The captain is dismayed to find that the dust sticks to the spaceship. If the density of dust is ρ, how far does the ship travel before its velocity is reduced by half?

24) What is the minimum angle to the vertical at which the ball can be thrown so that its distance to the origin is always increasing (i.e. the distance d is always increasing with time)?

25) A ball is projected horizontally with speed U m/s. Find the speed of the ball at time $t = T$ seconds later, assuming there is no air resistance.

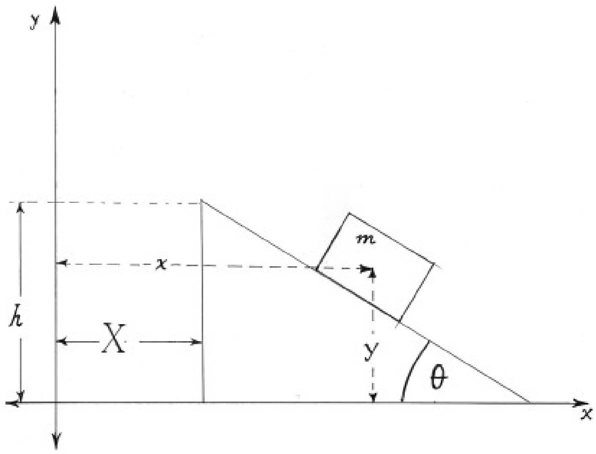
26) A cannon on horizontal ground, at point C, is used to target a point T, 25m behind a narrow wall. Unfortunately the cannon is damaged and can only fire at a 45° angle and at one speed. So, the only way to aim the cannon is by moving it towards and away from the target. The gunners aren't sure if they can actually hit the target.

(a) If the cannonball leaves the cannon at $u = 35$ m/s, at what distance, d , must the cannon be placed in front of the wall in order to hit the target, if the wall is ignored and the target is at the same height as the cannon?

(b) The wall is 15.0m high. Does the cannonball actually go over the wall and hit the target? If so, by how much?

27) What is the minimum angle to the vertical at which the ball can be thrown so that its distance to the origin is always increasing (i.e. the distance d is always increasing with time)?

28) A block moves on top of a wedge of angle θ, along the side of the wedge tilted with respect to the horizontal, as shown in the figure. The wedge, in turn, moves on a horizontal table. Show that the components of the accelerations of the block and the wedge satisfy $\ddot{x} - \ddot{X} = -\ddot{y}\cot\theta$.



29) The planar polar coordinates (r, θ), are related to Cartesian coordinates (x, y) by the relations $x = r\cos\theta, y = r\sin\theta$, while the relation between the Cartesian unit vectors $\hat{x}$ and $\hat{y}$ and the polar unit vectors $\hat{r}$ and $\hat{\theta}$ are

$$\hat{r} = \cos\theta\hat{x} + \sin\theta\hat{y}$$
$$\hat{\theta} = -\sin\theta\hat{x} + \cos\theta\hat{y}$$

In polar coordinates the vector position of a particle m is given by

$$r = r\hat{r}.$$

(a) Show that the velocity $\mathbf{v}$ of the particle m is given by

$$v = \dot{r}\hat{r} + r\dot{\theta}\hat{\theta}$$

Where the dot indicates a derivative respect of time, for example $\dot{r} = dr/dt$.

(b) A bead moves with constant speed $|\mathbf{v}| = u$ along the spoke of a wheel. It starts from the centre of the wheel at $t = 0$. The angular position of the spoke is given by $\theta = \omega t$, where ω is a constant. Find the velocity and the acceleration of the bead.

(You may use in your calculation that the acceleration is

$$a = (\ddot{r} - r\dot{\theta}^2)\hat{r} + (r\ddot{\theta} + 2\dot{r}\dot{\theta})\hat{\theta}.)$$

(c) Make a sketch of the trajectory of the bead on the (x, y)-plane, and indicate the direction of the velocity on the path of the bead at three different points of your choice.

30) The position x of a particle moving down a mountain side making a positive angle α to the horizontal satisfies the equation

$$\frac{d}{dt}\left(x\frac{dx}{dt}\right) = gx\sin\alpha,$$

where g is the acceleration due to gravity.

By multiplying the equation by $x(dx/dt)$, obtain the first integral

$$x^2\dot{x}^2 = \frac{2g}{3}x^3\sin\alpha + C,$$

where C is an arbitrary constant of integration and the dot denotes differentiation with respect to time.

Sketch the positive quadrant of the $(x, \dot{x})$ phase plane. Show that all solutions approach the trajectory

$$\dot{x} = \left(\frac{2g\sin\alpha}{3}\right)^{1/2} x^{1/2}.$$

Hence show that, independent of initial conditions, the particle ultimately has constant acceleration of magnitude a given by

$$a = \frac{1}{3}g\sin\alpha.$$