Interview questions

Topic 1: Complex numbers

1. Cam Robinson (2015)

   Why $e^{ia} = \cos a + i \sin a$

2. Cam Magdalene (2019)

   Explain complex number

Topic 2: Differentiation

3. Cam Fitzwilliam (2015)

   Differentiate $y = x^x$

   Differentiate $y = x^{(\ln x)x^x}$ using different method from what you used for the previous.

Solution: $\ln y = x \ln x$, differentiate both sides, $\frac{1}{y} y' = \ln x + 1$,so ,

$y' = x^x(\ln x + 1)$

$\ln y = x^x (\ln x)^2$, differentiate both sides, we have ,

$\frac{1}{y} y' = x^x(\ln x + 1)(\ln x)^2 + 2x^{x-1} \ln x$, then , $y' = [x^x(\ln x + 1)(\ln x)^2 + 2x^{x-1} \ln x] x^{(\ln x)x^x}$

4. Ox St. Hugh's (2014)

   a. $y = 2x + 1$. Find its derivative

   b. Solve the differential equation: $\frac{dy}{dx} = -2xy$, y(0)=0

5. Ox unknow (2014)

   $u = \frac{x^2+1}{x^2-1}$. Express x in terms of u. Find $\frac{dx}{du}$ and $\frac{du}{dx}$

Solution:

$u = 1 + \frac{2}{x^2 - 1}$, then $\frac{du}{dx} = -2 \times \frac{1}{(x^2-1)^2} \times 2x = -\frac{4x}{(x^2-1)^2}$.

Next ,solve for x, $x = \begin{cases} -\sqrt{1 + \frac{2}{u-1}}, x < 0 \\ \sqrt{1 + \frac{2}{u-1}}, x > 0 \end{cases}$

so $\dfrac{dx}{du}=\begin{cases}\dfrac{1}{(u-1)\sqrt{(u-1)^2+2(u-1)}}, x<0\\ -\dfrac{1}{(u-1)\sqrt{(u-1)^2+2(u-1)}}, x>0\end{cases}$

6. Ox Somerville (2015)
   Solve ODE: $y'' - 4y' + 4y = e^{3t}$

7. Ox Herford (2015)
   $\dfrac{dy}{dx}x^2 \sin x$
   $\dfrac{dy}{dx}a^x$

8. Oxford (2015)

   $y = x^3 + x^2$. Find the equation of the tangent to the curve at the x-intercept.

Solution:

$y = x^2(x+1)$, so the x-intercepts are (0,0), (-1,0).

Since $y' = 3x^2 + 2x$, $y'(0) = 0, y'(-1) = 1$, so the equation to the curve at (0,0) is y=0, at(-1,0) is y=x+1.

9. Cam Sidney Sussex (2016)
   $$\cfrac{1}{1+\cfrac{1}{1+\cfrac{1}{\ddots\,\cfrac{1}{1+x}}}}$$
   There are n lines in total. Differentiate it.

10. Cam Peterhouse (2016)
   $$f(x) = \begin{cases} x^2 \sin\dfrac{1}{x}, \ x \neq 0 \\ 0, \ x = 0 \end{cases}$$
   what is the value of $f'(0)$?

Solution:

$$f'(0)=\lim_{h\to 0}\frac{h^2\sin\frac{1}{h}-0}{h}$$
$$=0$$

11. Cam Peterhouse (2017)

    Differentiate $\ln x$ using first principle

Solution:

$$f'(x)=\lim_{h\to 0}\frac{\ln(x+h)-\ln x}{h}$$
$$=\lim_{h\to 0}\frac{\ln\frac{x+h}{x}}{h}$$
$$=\lim_{h\to 0}\ln\left(\frac{x+h}{x}\right)^{\frac{1}{h}}$$
$$=\lim_{h\to 0}\ln\left(\frac{x+h}{x}\right)^{\frac{x}{h}\times\frac{1}{x}}$$
$$=\ln e^{\frac{1}{x}}$$
$$=\frac{1}{x}$$

12. Ox Keble (2017)

    Differentiate $\frac{1}{1+x^2}$

13. Ox St. Peters (2017)

    $\frac{dy}{dx}x^2\sin\frac{1}{x}$

14. Ox Brasenose (2017)

    Find the derivative of $y=\sin^{-1}x$ (also written as $\arcsin x$)

    Why use $\cos x=\sqrt{1-\sin^2 x}$ not $\cos x=-\sqrt{1-\sin^2 x}$

Solution:

$y=\sin^{-1}x$ is through (x, y), or x=siny

$$\frac{d\sin^{-1}x}{x}=\frac{1}{\frac{d\sin x}{dx}|y}=\frac{1}{\cos y}=\frac{1}{\sqrt{1-x^2}}$$

Since for $y=\sin^{-1}x$, the range $-\frac{\pi}{2}\le y\le\frac{\pi}{2}$, therefore, cosy⩾ 0.

That’ s why we use $\cos x=\sqrt{1-\sin^2 x}$ not $\cos x=-\sqrt{1-\sin^2 x}$

15. Ox Oriel (2019)

Find:

a. $\frac{d}{dx}e^{-x}\sin x$

b. $\frac{d}{dx}e^{-x}\cos x$

c. $\frac{d}{dx}|e^{-x}\sin x|$

Solution:

a. $=-e^{-x}\sin x+e^{-x}\cos x$

b. $=-e^{-x}\cos x-e^{-x}\sin x$

c.

$$\frac{d}{dx}|e^{-x}\sin x|=$$

$$=\frac{d}{dx}\left(\sqrt{(e^{-x}\sin x)^2}\right)$$

$$=\frac{1}{2}\cdot\frac{1}{\sqrt{(e^{-x}\sin x)^2}}\cdot 2(e^{-x}\sin x)\cdot(-e^{-x}\sin x+e^{-x}\cos x)$$

$$=\frac{(e^{-2x}\sin x)\cdot(-\sin x+\cos x)}{|e^{-x}\sin x|}$$

Topic 3: Functions (inequalities) and limits

1. Cam (2014)

   $x^2 + y^2 + z^2 - xy - xz - yz \geq 0$

2. Ox New (2014)

   There has a new function $f_2(a)$.

   Here is an example: if $40 = 2^3 * 5$, then $f_2(40) = 3$

   a. What is $f_2(12),\ f_2(-6),\ f_2(9)$?

   b. Can you write $f_2(a^2)$ in the form of $f_2(a)$?

   c. Is there any rational roots for $x^2 = 2,\ x^3 = 2,\ x^2 = 3$?

   d. Assume a and b are integers, is it possible for $a^2 = 2b^2$? If $f_2(a) < f_2(b)$, what is $f_2(a+b)$?

   e. Assume that a, b, c are integers, is it possible that they satisfy the equation: $4c^2 = a^2 + 2b^2$?

3. Ox University (2014)

   a. Let $y(x) = f_1(x) + f_2(x)$, $f_1(x)$and $f_2(x)$ are both convex function, is $y(x)$ a convex function? Prove that

   b. Let $h(x) = \max\{f_1(x), f_2(x)\}$, $f_1(x)$and $f_2(x)$ are both convex functions, is $h(x)$ a convex function? Prove that

4. Ox St. Catherine (2014)

   $\lceil a \rceil$ gives the smallest integer that is greater or equal to a;

   for instance, $\lceil 2 \rceil = 2,\ \lceil 2.5 \rceil = 3$. For $\left\lceil \frac{x}{a} \right\rceil = \left\lceil \frac{x}{a+1} \right\rceil$, state whether it has a solution or not. If yes, how many solutions?

5. Ox unknown (2014)

   I have created a function: if $40 = 5 \times 2^3$, then $f_2(40) = 3$. How about f_2, f2(-6) and $f_2(9)$?

   Can you write $f_2(a^2)$ in the form of $f_2(a)$. Are there any rational roots for $x^2 = 2$, $x^3 = 2,\ x^2 = 3$?

   Assume a and b are integers, is it possible for $a^2 = 2b^2$? If $f_2(a) < f_2(b)$, what is $f_2(a+b)$?

   Assume a, b and c are integers, is it possible for $4c^2 = a^2 + 2b^2$

6. Ox Trinity (2015)

   $2a^2 - b^2 = 1$ (a, b are integers)

   Find the solutions.

7. Ox St. Cats (2015)

Here has a process of solving an equation:

$x-2 = \sqrt{x}$

$\Longrightarrow (x-2)^2 = x$

$\Longrightarrow x^2-5x+4 = 0$

$\Longrightarrow (x-1)(x-4) = 0$

$\Longrightarrow x = 1 \text{ or } 4$

Is this process valid?

How about its reverse?

8. Ox Queens (2015)

   Find all sets A (A is a subset of Z) where satisfied the property:

   For $x \in A$, $y \in A$, then $x - y \in A$ (*)

   a. What are the finite sets A
   b. Prove $\{0\}$ and $\emptyset$ are the only finite set satisfied (*)
   c. Is there any infinity set A other than Z where satisfied (*)?
   d. There are some $b \in Z$, such that $A_b = \{bk|k \in Z\}$ satisfied (*), and what is the relationship between a and b, where $a \in A$?
   e. Prove there are no other A satisfied (*)? [Hint: there is a real number c, where $kb < c < (k+1)b$, prove that cannot be the element of A.]

9. Ox (2015)

   Solve $x^2 - 4x + 1 < |4x+1|$

Answer: 0<x<8

$$\begin{cases} 4x+1 \geq 0. \\ x^2-4x+1 < 4x+1 \end{cases} \quad \text{or} \quad \begin{cases} 4x+1 \leq 0 \\ x^2-4x+1 < -4x-1 \end{cases}$$

$$\therefore \begin{cases} x \geq -\frac{1}{4} \\ x^2-8x < 0. \end{cases} \quad \text{or} \quad \begin{cases} x \leq -\frac{1}{4} \\ x^2+2 < 0. \end{cases} \rightarrow \text{no real solutions}$$

$$\begin{cases} x \geq -\frac{1}{4} \\ 0 < x < 8 \end{cases} \quad \text{so the solution is } 0 < x < 8$$

10. Ox (2015)

    $ax^2 + 3x + 1 \leq x^2 - x$ The inequality has at least one solution, what is the value of a.

Solution1: $(a-1)x^2+4x+1\le 0$.

$a=1$ ✓ or $\begin{cases} a-1>0 \\ \Delta=16-4(a-1)\ge 0 \end{cases}$ or $\begin{cases} a-1<0 \end{cases}$

$a=1$ or $a\le 5$ or $a<1$

so $a\le 5$

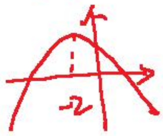
Solution 2:

if $x=0$, then $1\le 0 \to$ contradiction. so $x\ne 0$, $x^2>0$

$a+\frac{3}{x}+\frac{1}{x^2}\le 1-\frac{1}{x}$

$a\le -\frac{1}{x^2}-\frac{4}{x}+1$ Let $\frac{1}{x}=k$, $\therefore k\ne 0$

$\le -k^2-4k+1$, $k\ne 0$



≤ 5

11. Cam Sidney Sussex (2016)

    Solve the inequality $25^{2x}-5\times 5^x+6\ge 0$

12. Cam Christ's (2016)

    Polynomial 所有系数为 1，0，-1，问 degree 为 2016 的这样 polynomial 所有的解为整数的个数为多少？

13. Cam Peterhouse (2016)

    Prove that $\int_0^1 f(x)dx=\lim_{n\to\infty}\sum_{n=1}^{m}\frac{1}{n}f\left(\frac{m}{n}\right)$

    What is $\lim_{n\to\infty}\left(\frac{1}{n+1}+\frac{1}{n+2}+\dots+\frac{1}{n+1}\right)$

14. Ox Worcester (2016)

    $F(n)\ge 0$ $n\ge 1$ $n\in\mathbb{Z}$

    a. $F(mn)=F(m)+F(n)$
    b. $F(10)=1$

c. $F(n) = 0$ if n ends in 3

Find $\mathrm{F}(17)$ and possible of $\mathrm{F}(500)$?

15. Cam Pembroke (2017)

Solve $\sin x + 1 \leq \cos x$

16. Cam Pembroke (2017)

$|x-9| + |x-3| + |x-10|$最小值，hint：考虑数轴上到点的距离

17. Cam Sussex (2017)

$\mathrm{x} = \mathrm{y}^2 - 3\mathrm{y} + 2$ 写出 $\mathrm{y}(\mathrm{x})$

$\mathrm{x}^2\mathrm{y}^2 - 3xy + 2 = 0$ 的 $\mathrm{y}(x)$

$25^{\mathrm{x}} - 25 \times 5^{x-1} + 6 \geq 0$ 求解

18. Ox Herford (2017)

a. Show $\frac{1}{n+1} \leq \frac{1}{n^2+1} + \frac{1}{n^2+2} + \cdots + \frac{1}{n^2+n} \leq \frac{1}{n}$

Show $\lim_{n\to\infty} \frac{1}{n^2+1} + \frac{1}{n^2+2} + \cdots + \frac{1}{n^2+n} = 0$

b. Show $\frac{\sqrt{n}}{\sqrt{n+1}} \leq \frac{\sqrt{n}}{\sqrt{n^2+1}} + \frac{\sqrt{n}}{\sqrt{n^2+2}} + \cdots + \frac{\sqrt{n}}{\sqrt{n^2+n}} \leq \frac{n}{\sqrt{n^2+1}}$

What is $\lim_{n\to\infty} \frac{\sqrt{n}}{\sqrt{n^2+1}} + \frac{\sqrt{n}}{\sqrt{n^2+2}} + \cdots + \frac{\sqrt{n}}{\sqrt{n^2+n}}$?

19. Ox Brasenose (2017)

Find the range of $\frac{a}{b+c} + \frac{b}{a+c} + \frac{c}{a+b}$

20. Ox Oriel (2017)

a. WTP: $1 + x + x^2 + \cdots + x^n = \frac{1-x^{n+1}}{1-x}$

b. WTP: $1 + x + x^2 + \cdots + x^n + \cdots = \frac{1}{1-x}$

c. WTP: $1 - x^2 + x^4 - x^6 + \cdots = \frac{1}{1+x^2}$

d. WTP:$\arctan x = x - \frac{x^3}{3} + \frac{x^5}{5} - \frac{x^7}{7} + \cdots$

e. Find expression for $x + 2x^2 + 3x^3 + \cdots$

f. Find expression for $x + 4x^2 + 9x^3 + 16x^4 + \cdots$

g. Hence, get $\frac{1}{2} + \frac{2}{4} + \frac{3}{8} + \cdots + \frac{n}{2^n} + \cdots$ and $\frac{1}{2} + \frac{4}{4} + \frac{9}{8} + \cdots \frac{n^2}{2^n} + \cdots$

h. According to $(e^x)^1 = e^x$, show that $e^x = 1 + x + \frac{x^2}{2!} + \cdots + \frac{x^n}{n!} + \cdots$

i. According to $e^{ix}=\cos x+i\sin x$, show that $\sin x=x-\frac{x^3}{3!}+\frac{x^5}{5!}-\cdots$ and $\cos x=1-\frac{x^2}{2!}+\frac{x^4}{4!}-\cdots$

j. Hence show that $\frac{\sin x}{x}=(1-\frac{x^2}{\pi^2})(1-\frac{x^2}{4\pi^2})(1-\frac{x^2}{9\pi^2})\cdots$

21. Ox Wadham (2017)

a. Show that $\frac{x}{y}+\frac{y}{x}\geq 2$

b. Find all x, y where $x,y\in\mathbb{R}$ in which $\frac{x}{y}+\frac{y}{x}=integer$

22. Ox Jesus (2018)

Function of $\mathrm{y}=\ln\left(\ln\left(\ln(\mathrm{x})\right)\right)$

Function of $\mathrm{x}^2+\mathrm{ny}^2=1$.

23. Cam Churchill (2018)

比较$\mathrm{sech}\,\mathrm{x}\sin\mathrm{x}$, $\mathrm{esin}\,x^{-|\mathrm{x}|}$, $\cos 2\mathrm{x}\,\mathrm{sech}\,\mathrm{x}$大小

24. Ox Lincoln (2018)

$\mathrm{x}^2-\mathrm{y}^2=1$ 绕 x 轴或 y 轴旋转，找曲面的表达式 in terms of x, y, z

找一个有界函数同时导函数无界

25. Ox Lincoln (2018)

$\left(1+\frac{r}{2}\right)^{\left(\frac{1}{2}\right)}$的近似（$\frac{r}{2}$趋近于 0）

26. Ox St Annes (2019)

$\ln\ln\cdots\ln x$（N 个 ln）when it is negative ， what's the range of x

27. Ox St Annes (2019)

$x^2-ny^2=1$

When x， y is whole number

a. If n=16, what is solution

b. when n=5, why no whole number solution

28. Ox Mansfield (2019)

$f(x)=\begin{cases}2x,\ 0\le x<\frac{1}{2}\\ x,\ \frac{1}{2}\le x<1\end{cases}$，若 plug in x，n 次迭代后得到 period，则 x 为多少

29. Ox St. Hugh's (2019)

    Solve $\sqrt{3-x}-\sqrt{1+x}>\frac{1}{2}$

Topic 4: Graph Sketching

1. Cam (2014)
   Sketch $y^3 = x^2$

2. Cam (2014)
   Sketch the following functions:
   a. $|x|+|y|\le 1$
   b. $|x-1|+|y+1|\le 1$
   c. $|x-1|-|y-1|\le 1$
   d. $z = ra$
   e. $y^3 = x^2$

3. Cam (2014)

   Sketch $y=\frac{2x}{x^2+1}$

Solution: this is a rational function.

Steps:

1. **asymptotes:** The horizontal asymptote is $\lim_{x\to\infty}\frac{2x}{x^2+1}=0$, no vertical asymptotes.

2. The x-intercepts and y-intercepts: $f(0)=0$, let f(x)=0, x=0. so the x-intercept and y-intercept are the same point: (0,0).

3. Sign chart: when x>0, f(x)>0; when x<0, f(x)<0.

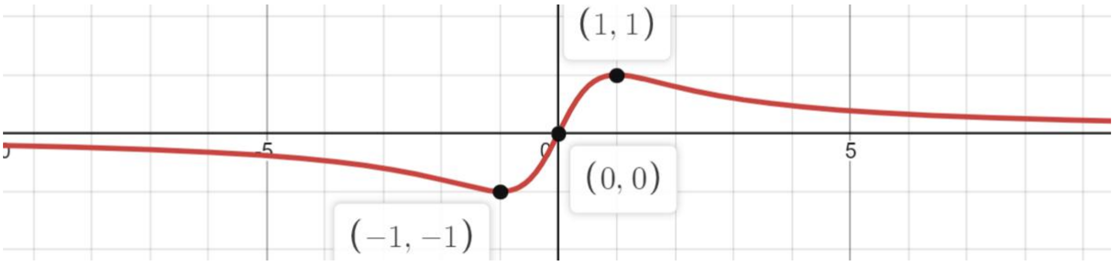
4. **Intersection with asymptotes:** let f(x)=0, x=0, so the graph crosses the horizontal asymptote at (0,0). Since f(x) is an odd function,

its graph is symmetrical about the origin.

5. **Any stationary points:** $f'(x)=\dfrac{2(x^2+1)-2x(2x)}{(x^2+1)^2}=\dfrac{-2x^2+2}{(x^2+1)^2}=0$, $x=\pm 1$.

so the stationary points are(-1,-1) and (1,1).

The graph is shown below:



4. Cam (2014)

   Sketch the following functions:

   a. $y = x(x + 1)(x - 2)^4$

   b. $y^2 = x(x + 1)(x - 2)^4$

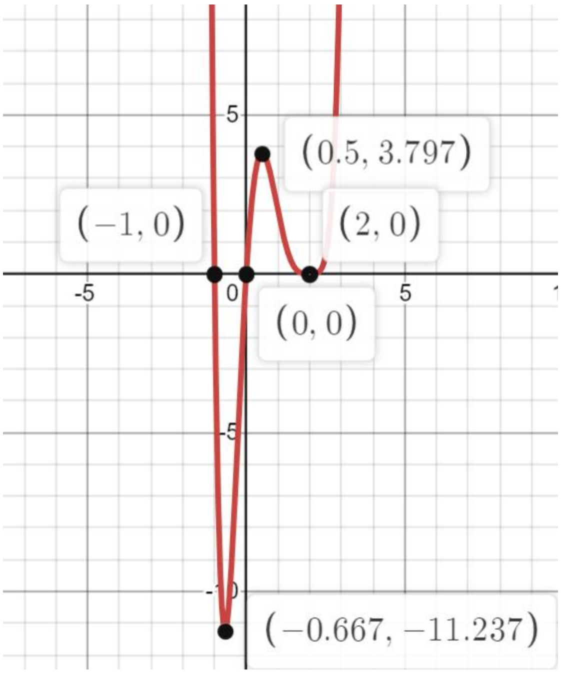
   Solution:

a.For the first graph,it is a polynomial, and we could use the same method as discussed previously.

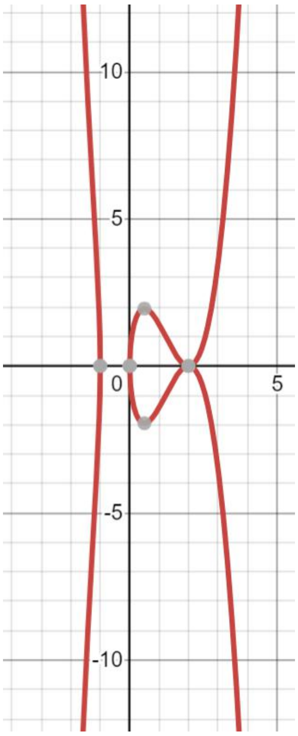
Steps: ①the graph crosses the x-intercepts (0,0),(-1,0), and touches (2,0).

②When $x \to +\infty, h(x) \to +\infty$; when $x \to -\infty, h(x) \to +\infty$

③Finally , find the stationary points: Let $h'(x)=(2x+1)(x-2)^4+(x^2+x)\cdot 4(x-2)^3=(x-2)^3(6x^2+x-2)=0$,so, $x=2$ or $x=0.5,-0.667$, so (2,0), (-0.667,-11.237), (0.5,3.797) are stationary points.



b. $y^2 = x(x+1)(x-2)^4$, the function is symmetrical about x-axis, and its domain is **x≥0 or x≤-1(**by letting x(x+1)⩾0**).** The graph could be plotted by taking square root of y-coordinate of each point on the graph of $y = x(x+1)(x-2)^4$ while keeping the x-coordinate unchanged. Then reflect the resulting graph in x-axis, as shown below.



5. Ox University (2014)

   Draw an arbitrary convex function $f(x)$, connect two arbitrary points with a line.

   Find the equation of the line $g(x)$.

From this, define a convex function

6. Ox Balliol (2014)

   Sketch $2\mathrm{x}\sin\frac{\pi}{x}$

7. Ox Brasenose (2014)
   Draw graphs for $\cos x$, $\cos 2x$, $\cos(\sin x)$ and $\mathrm{e}^{\cos(\sin x)}$. Prove that the latter two have the same maximal and minimal value.

8. Ox St. Hugh's (2014)

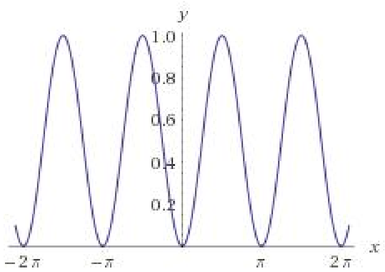
   Draw $\mathrm{y} = \mathrm{e}^{-\mathrm{x}^2}$

9. Ox St. Catherine (2014)

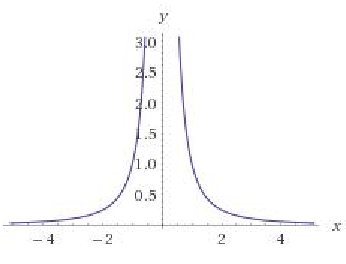
   Sketch $\frac{\sin x}{x}$

   Are there any value of y for which $y > f(x)$for any x?
   Are there any value of z $(z > x)$ for which $f(z) > x$ for any x?

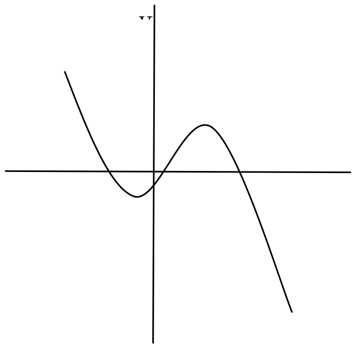
10. Ox St. Catherine (2014)
    The followings are sketches of the derivative of $f(x)$, sketch $f(x)$for each case.



(x from −6.6 to 6.6)

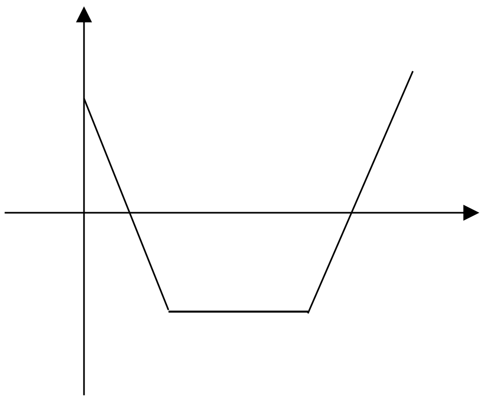


(x from −5.1 to 5.1)



11. Ox Somerville (2014)

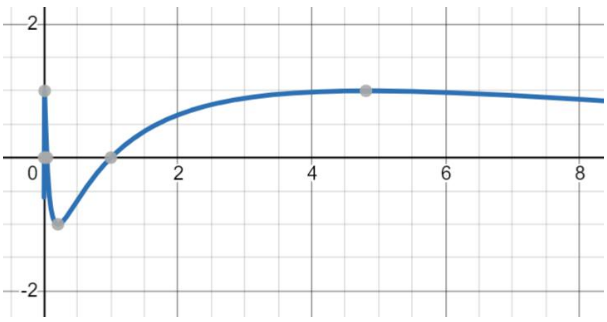
    Graph of $f'(x)$: (马萱煜 2014)

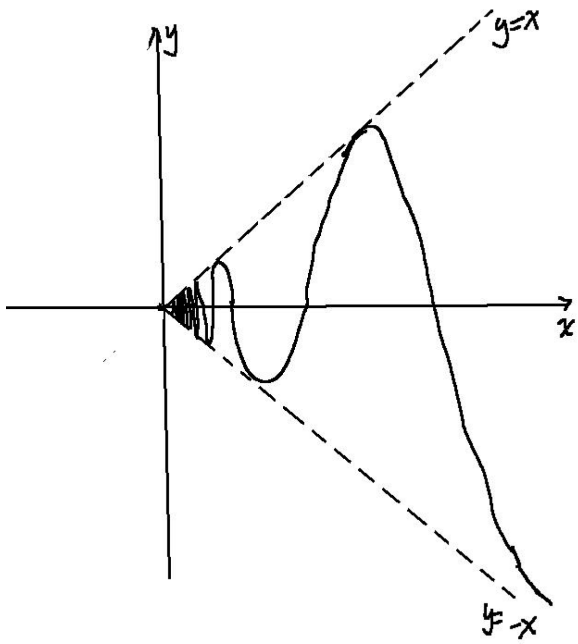


12. Ox unknow (2014)

    Sketch $\mathrm{y} = \sin(\ln x)$

    Sketch $\mathrm{y} = x\sin(\ln x)$





13. Cam (2015)

$x(t) = t - \sin t$, $y(t) = 1 - \cos t$, Sketch $y(x)$

14. Cam Fitzwilliam (2015)

Sketch $y = 1 - \frac{1}{(1+x)^2}$

Solution: after simplification: $y = 1 - \frac{1}{(1+x)^2} = \frac{x(x+2)}{(x+1)^2}$.

Steps:

1. Asymptotes: the vertical asymptote is x=-1, and the horizontal asymptote is $y = \lim_{x\to\infty} 1 - \frac{1}{(1+x)^2} = 1$

2. The x-intercepts, y-intercepts: the x-intercepts are(0,0),(-2,0), and the y-intercept is(0,0,)

3.

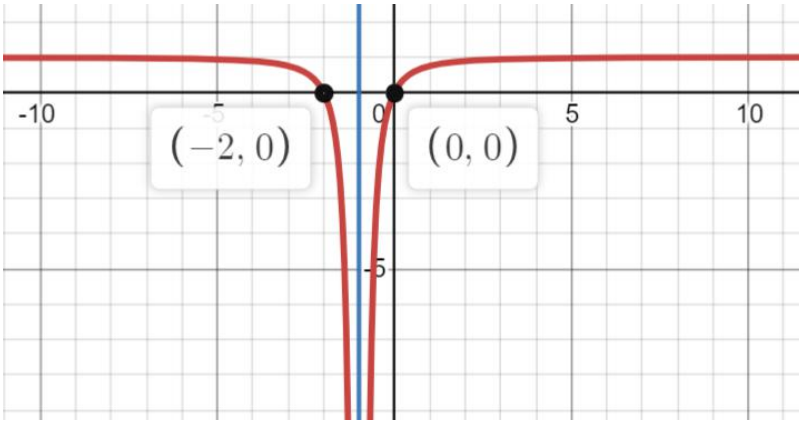
x<-2	-2<x<-1	-1<x<0	x>0

+	-	-	+

4. Intersection with the horizontal asymptote: let f(x)=1, no real solutions, so no intersections. The function is neither an odd nor an even function(actually , it's symmetrical about line x=-1).

5. Any stationary points: let f'(x)=0, no real solutions. So there're no any stationary points.

The graph is shown below:



15. Ox St. Cats (2015)

Sketch $\frac{1}{\sin(x^2-2)}$

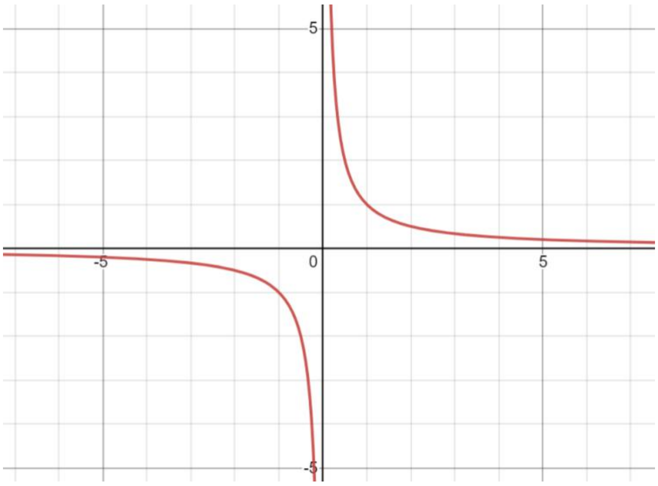
16. Ox Herford (2015)

Sketch $\frac{1}{x}$ and $\frac{1}{x^2}$

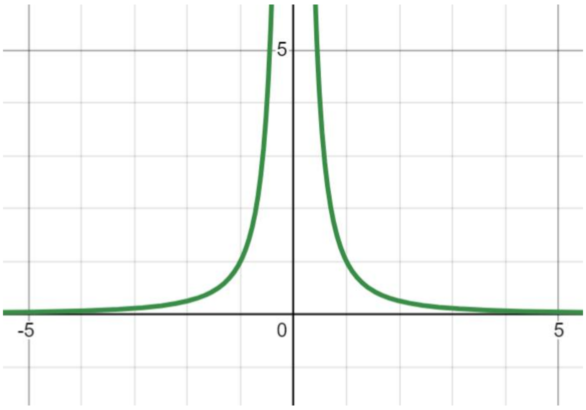
Sketch $\frac{1}{1-x^2}$

Solution:

$$y = \frac{1}{x}$$



$y = \frac{1}{x^2}$



The third function $y = \frac{1}{(1-x)(1+x)}$ is a rational function that can be solved using the basic steps discussed previously.

Steps:

1. **Asymptotes:** the vertical asymptotes are x=1 and -1, horizontal asymptote is y=0.

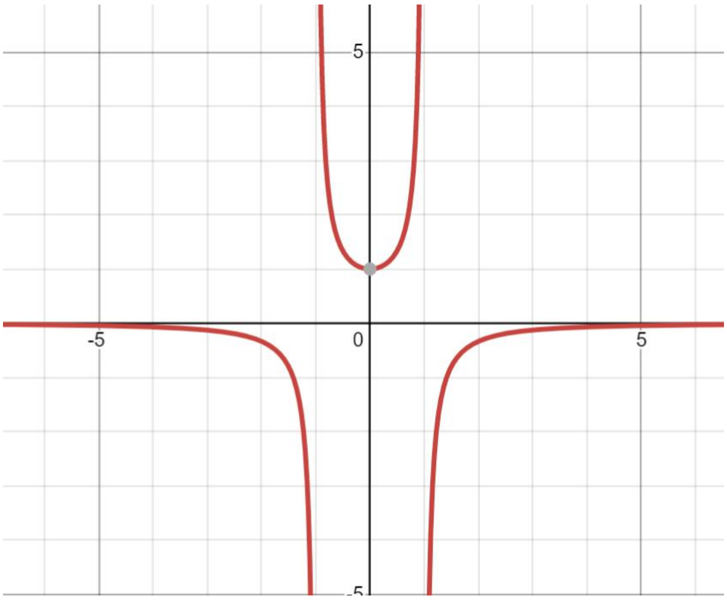
2. X-intercepts, y-intercepts: no x-intercepts; y-intercept is (0,1).

3. Sign chart:

x<-1	-1<x<1	1<x
-	+	-

4. Intersection with horizontal asymptote: f(x)=0, no real solutions, so no intersection with horizontal asymptote. Since f(-x)=f(x), an even function, so the graph is symmetrical about y axis.

The graph is shown below:



17. Ox (2015)

    Plot $y = 2x^2 + \frac{4}{x}$

18. Cam (2016)

    A parametric equation satisfying:

    a. $x(t) = t - sint$

    b. $y(t) = 1 - cost$

    Sketch $y(x)$

19. Cam Churchill (2016)

    a. Sketch the graph $y = \sin x$

    b. Sketch the graph $y = \sin\frac{1}{x}$

    c. Sketch the graph $y = x^2\sin\frac{1}{x}$

20. Ox Oriel (2016)

Sketch $y = xe^{-\frac{x^4}{4}}$

Sketch $y = m^2xe^{-\frac{m^4x^4}{4}}$, with different values of m

21. Ox St. Peters (2016)

    Draw $\frac{\ln x}{x}$

    Try to slove $a^b = b^a$ if $a < b$, $a\epsilon(1, e)$

22. Ox Worcester (2016)
    a. Sketch $y = \ln x$
    b. Sketch $y = \frac{\ln x}{1-x}$
    c. Sketch $\left||x| - 2\right|$
    d. For what values does $\int_x^{x+1} \left||t| - 2\right| dt$ takes the smallest value

23. Cam Pembroke (2017)
    a. Sketch $y = (4x^4 - 13x^2 + 3)e^{-x^2}$
    b. Sketch $\mathrm{Im}(Z^4) > 0$

24. Cam Sidney Sussex （2017）
    a. Sketch $\begin{cases} x(t) = t \\ y(t) = t^2 \end{cases}$
    b. Sketch $\begin{cases} x(t) = \cos t \\ y(t) = \sin t \end{cases}$

25. Cam 线下上海 (2017)

    $x(t) = t - sint$

    $y(t) = 1 - cost$

    Sketch y(x)

26. Cam Sussex (2017)

    画 $y = \frac{1}{x}$, $y = \frac{1}{\sqrt{x}}$,$[0,1]$ 求面积

    $y = x^{-x}$ 在$[0,1]$上什么时候面积 converge

27. Cam Magdalen (2017)

    $f > g$, $f' < g'$ 画一个符合的函数图像

    再写适合的 f,g 表达式

    $f > g > h$, $f' < g' < h'$ 图像+表达式

如果有无数个这样接下去的函数，可能么，怎么得到

28. Ox Exeter (2017)

$\mathrm{f(x) = xe^{-x}}$

画图，哪段是 increasing function? 哪段是 decreasing? Max/min point

求$\mathrm{f}_2(x)$图像与 $\mathrm{f(x)}$关于 $\mathrm{x = 1}$ 对称。写表达式

29. Ox Herford (2017)

Sketch $\left|\frac{e^x-1}{\sin x}\right|$

30. Ox Herford (2017)

Sketch

   a. $x + y = 0$
   b. $x^2 - 2x + y^2 - 4y + 4 = 0$
   c. What is the shortest distance between graph a and b

31. Ox Herford (2017)

Sketch $y = \ln x$ and find the equation of a tangent to $y = \ln x$ passing through the origin.

32. Ox Oriel (2017)

$F(x) = \int_1^x f(x)\,dx$, draw the graph of F(x)

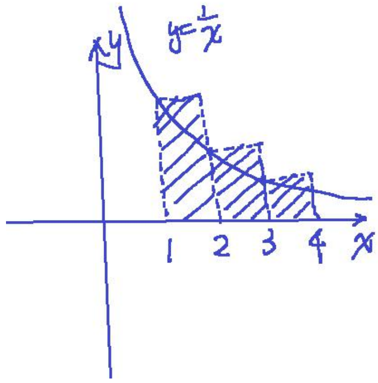
What is F(x) if $f(x) = \frac{1}{x}$

Then prove $\sum_{k=1}^{\infty} \frac{1}{k}$ is infinity

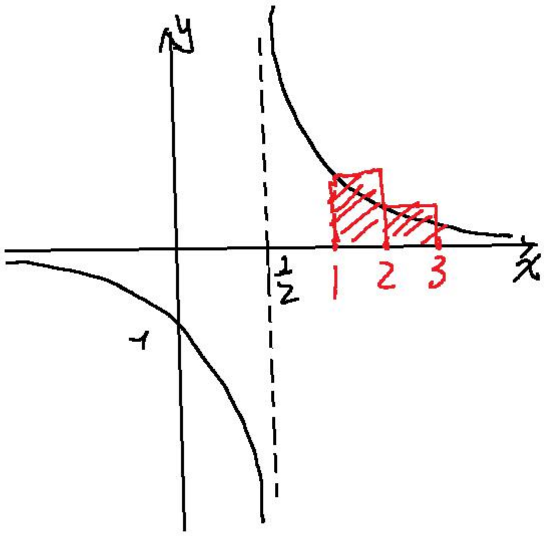
Image whether $\sum_{k=1}^{\infty} \frac{1}{2k-1}$ is infinity and prove it. If not , find its limit.

Solution:

$$F(x) = \int_1^x \frac{1}{x} dx = \ln x$$



$\sum_{k=1}^{\infty}\frac{1}{k}=\frac{1}{1}\times1+\frac{1}{2}\times1+\frac{1}{3}\times1+...>\int_1^{\infty}\frac{1}{x}dx=\infty$, so it's infinity.



$$\sum_{k=1}^{\infty}\frac{1}{2k-1}=\frac{1}{2\times1-1}\times1+\frac{1}{2\times2-1}\times1+\frac{1}{2\times3-1}\times1+...>\int_1^{\infty}\frac{1}{2x-1}dx=[1/2\ln(2x-1)]_1^{\infty}=\infty$$

It's infinite.

33. Ox St. Peters (2017)

    How many solutions for x in the equation $x=e^x$?

34. Ox St. Cats (2017)

    Sketch $\frac{\sqrt{x}}{\sin\sqrt{x}}$

35. Ox Keble (2017)

    a. Sketch $\frac{1}{1+x^2}$

b. Find the equation of the line that passes through (1,0) and is the tangent to another point of the curve.
c. Show that this line and the curve do not has other intersection points (.e except (1,0) and the tangent point)
d. By considering areas, show that $\pi > 3$

36. Ox Jesus (2018)
Graph about $1 + \mathrm{x}^2 + \mathrm{x}^4 + \dots + \mathrm{x}^{2\mathrm{n}}$ for different values of n

37. Cam Murray Edwards (2019)
Sketch y^2=x^3-2x^2+x
Sketch x^2+y^2=2xy

38. Cam Lucy Cavendish (2019)

画出$e^x - \frac{1}{x}$的图像

39. Cam Magdalene (2019)

Sketch $\frac{\ln x}{x}$

40. Ox St. Annes (2019)
画图给了 四条曲线 让你根据交点 交点的顺序 曲线间的关系 变成三直一曲

41. Ox St. Annes (2019)
Sketch $x^2 - ny^2 = 1$

42. Ox St. Annes (2019)

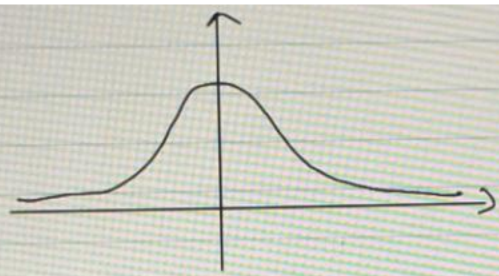
a. Sketch $\frac{1}{1+x}$

b. Sketch $\sin\frac{1}{x}$

43. Ox Jesus (2019)
Draw x^2, x^3, x^4
Analyze the graph

44. Ox Jesus (2019)

7



1) find the function looks like this.

2) what we need to change to make the graph's gradient sharper but the point on y-axis stays the same

3) let $\int_{-\infty}^{\infty} e^{-x^2} = A$

what change needs to make to make the area equals to 1 but the maximun point stays the same.

45. Ox Oriel (2019)

Sketch $y = |e^{-x}\sin x|$

Topic 5: Matrix

1. Cam Sidney Sussex (2016)

Find e^{A} where $\mathrm{A} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$

2. Ox Exeter (2017)

写 $\mathrm{A} = \begin{pmatrix} x \\ y \end{pmatrix}$, $\mathrm{B} = \begin{pmatrix} c \\ d \end{pmatrix}$

表示 length：$|A| = \sqrt{x^2 + y^2}$

把 2 次方，$\sqrt{\ }$ 换成 $\mathrm{p} \Rightarrow |A|p = (x^p + y^p)^{\frac{1}{p}}$

问 $\mathrm{p} = 3$（自选 1 个数）length 是不是 good?

＊如果是奇数，$(x^p + y^p)^{\frac{1}{p}}$可能为负，length 不能为负，所以要变成$(|x^p| + |y^p|)^{\frac{1}{p}}$

Topic 6: Probability & Combination

1. Ox (2014)

   7 accidents happen in a week, what is the probability for one accident per day?

   What happen when n accidents happen in a week where n tends to infinity?

2. Ox New (2014)

   There are 13 socks in a wardrobe, 8 are red and 5 are blue. A blind came in and pick the socks, if he take one sock at each time, what is the probability of the man get a pair of sock (two in same color) in the first two round?

3. Ox Oriel (2014)

   There are 8 red socks and 5 blue socks in the wardrobe. A blindfolded man picks socks twice. What is the probability of the man getting a pair of socks.

4. Ox Oriel (2014)

   Find the number of combination of x and y when integers x, y and z satisfy the following conditions:

   $1 \leq z \leq n+1$

   $1 \leq x \leq z$

   $1 \leq y \leq z$

   a. If z=k+1, k is an integer, how many ways of writing (x,y,z)?
   b. If x = y, find again the number of combination.

      Answer: ${}^{n+1}_{2}C$
   c. How many combinations are there if x > y.

      Answer: ${}^{n+1}_{3}C$
   d. Prove that: $1^2 + 2^2 + \ldots\ldots + n^2 = {}^{n+1}C_2 + 2\,{}^{n+1}C_3$
   e. What does this imply?
   f. What is $1^3 + 2^3 + \ldots\ldots + n^3$ (Hint: int the form of last term)?

5. Ox Mansfield (2014)

	H	D
H	0.5(V-C)	V
D	0	V/2

   D and D share V when they encounter. When D meet H, H gets V while D gets nothing. When H meet H, they have equal chance of gaining V and loosing C. Given that chance of H in P_2 is q, D in P_2 is q-Q. Find q in terms of V and C, explain why q takes the value.

6. Ox Mansfield (2014)
   Pick a pair of number from integers 0-6. How many combinations are there? What about from 0-n?

7. Ox unknow (2014)
   Pick one number from 1 to 10. What is the expected outcome? What would the expected outcome be after picking one number out?

8. Ox unknow (2014)
   Throw a dice, find the probability of first time outcome not being 6 while the second time outcome being 6. Find the probability of the sixth outcome not being 6 while the six outcomes after that are all 6. Find the probability that 6 out of 12 outcomes are 6?

9. Cam Fitzwilliam (2015)
   Dice question: A and B ply against each other. The first to get 6 points wins. A throws first. Calculate the probability that A will win.

10. Ox (2015)
    2 persons eat 8 bars of chocolates in turn. 1 of the chocolates is toxic. Which is more risky, start eating first or later?

11. Ox (2015)
    A person throw a dice twice, what is the probability that the two results add up to 4

12. Ox (2015)
    A class has 14 students, what is the probability for each students' birthdays to be on different days?

13. Ox (2015)
    10 persons sitting around a table. Each one of them is given a card with a number ranging from 1 to 10. Define N as the sum of numbers hold by a person A and his neighbors B and C.
    Prove that there exist a value of N that $N \geq 17$
    Find the probability of N=6

14. Ox (2015)
    Find the probability of 5 people having different birthdays.
    What is the probability of 60 people having different birthdays?

15. Cam Christ's (2016)

有 AB 两个 word 只含 HT，开始扔硬币，扔出的结果比先得到那个 word，同时得到短的赢。
AB 等长胜算一样吗？
A 比 B 长，胜算一定小吗？

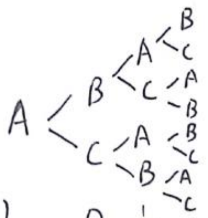
16. Cam (2016)
2 个人吃 8 块巧克力，其中一块有毒，先吃和后吃，哪个吃到有毒的概率小？

17. Cam Peterhouse(2016)
3*6 的座位，10 个女生，7 个男生做，要求同一排同一列不全是男生或者女生，问有几种做法

18. Cam Churchill (2016)
We intend to put positive integers into an $\infty * \infty$ size chessboard, with the property that each number in the square is no less than the average of the surrounding four squares. A trivial solution is to keep all the numbers equal. Show that it is the only possible case.

19. Ox Lincoln (2016)

2.



$P(A)$ $\quad 0, \frac{1}{2}, \frac{1}{4}, \underset{\frac{3}{8}}{(1-\frac{1}{4})\times\frac{1}{2}}, \frac{5}{16}$

1

$$\frac{3}{8} = \frac{1}{2}\times\frac{1}{2} + \frac{1}{2}\times\frac{1}{4}$$

$$\frac{5}{16} = \frac{1}{2}\times\frac{3}{8} + \frac{1}{2}\times\frac{1}{4}$$

$$P_n = \frac{1}{2}(P_{n-1} + P_{n-2})$$

20. Cam Pembroke (2017)
有一排灯泡一共 4 个，26 个颜色可选，问对称的可能性

21. Cam Pembroke (2017)

Find the probability of $\frac{1}{x}$ starting with 1 in its decimal expression given that $x \in (0,1)$.

(for example: $\frac{1}{\frac{2}{3}} = 1.5, \cdots, \frac{1}{\frac{2}{21}} = 10.5, \cdots$)

22. Cam Sussex (2017)

a. 四个点，共 2，3，k 种颜色，求多少种染色方式
b. 5 个点？
c. 7 个点？
d. 通项式？有没有反例？
e. 一个点集，$f(k)$种染色方式，多了一个点会怎样？

23. Ox Keble(2017)

① 6 right
- 7 right
  - 8 right p^2
  - 8 wrong $p(1-p)$ ×
- 7 wrong
  - 8 right $p(1-p)$ ×
  - 8 wrong $(1-p)^2$ ×

Not necessary cause the probability of passing actually drop.

②. 10 题只能错一题

$$\underbrace{\overset{p^4}{\text{全对}} \quad \overset{\binom{4}{1}p^3(1-p)}{\text{错 1}}}_{p^3(p+4-4p)} \quad \overset{\binom{4}{2}p^2(1-p)^2}{\text{错 2}} \quad \overset{\binom{4}{3}p(1-p)^3}{\text{错 3}} \quad \overset{(1-p)^4}{\text{错 4}}$$

when $p^3(4-3p)>p$

$p^2(4-3p)>1$ $0.76<p<1$ worth

otherwise no

24. Ox Keble(2017)

There are 10 multiple choice questions. Candidates should answer these questions in order, which means if a candidate does not answer one of the questions, he or she wil not be allowed to answer all questions after that. A correct answer will score a mark and a wrong answer will lose a mark. The probability of answer a question correctly is p, where 0<p<1. The passing score is 7. Now a candidate has answered the firtst 6 questions correctly so that she has already got 6 mark.

a. Is it necessary for her to attempt question 8 if she is not going yo answer yhr the last 2 questions?
b. Is it necessary for her to attempt question 10?

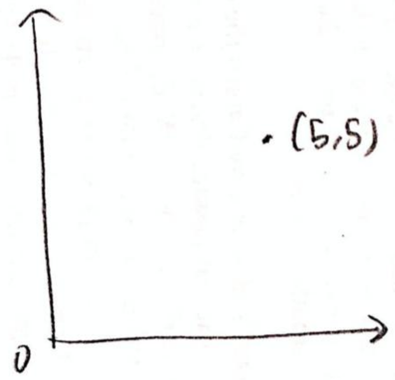
25. Cam Magdalene (2019)

N read 2n blue, draw 2 times, if you get same color then win. What is the probability of winning?

26. Cam Magdalene (2019)

13 张扑克牌（A-K），随机叠加排列。假设第一张为 n，则倒序排列前 n 张牌。如果 A 在第一张，则 stuck（无法继续改变排列）。问是否一定会 stuck？

27. Cam Magdalene (2019)



an ant can only go up or right → # to go from (0,0) to (5,5).

28. Ox St. Hilda's (2019)

有无数条平行的线，每条线之间的间隔都是是 r。有一根长为 l 的针，l 比 r 短。把这根针丢到平面上，求针与线相交的概率有多少。

a. 先假设针与线垂直 $p = \frac{l}{r}$
b. 再考虑针与线成某角度的情况（考虑所有 0 到 pi 的情况就是 integration
c. 然后得出 p 的公式后，设计一个实验来求 pi
d. 如果不是针，是一个直径为 l 的圆，相交的概率（还是$p = \frac{l}{r}$）

29. Ox University (2019)

A~K 13 张扑克牌，随机叠加/排列，假设第一张牌为 N，则倒序排列前 N 张牌。如果 A 在第一张，则 stuck。是否一定会 stuck?

Topic 6: Integration

1. Ox New (2014)

$$\int x^2 \cos x \, dx$$

2. Ox Mansfield (2014)

Integrate $(\sin x)^6 \cos x$, $(\sin x)^6 (\cos x)^3$ and $(\sin x)^{\mathrm{m}} (\cos x)^{\mathrm{n}}$.

Determine under what conditions can the third integration be done in similar ways as the first two.

Solution:

$$\int (\sin x)^6 \cos x dx = \int (\sin x)^6 d \sin x = \frac{(\sin x)^7}{7} + C$$

$$\int (\sin x)^6 \cos^3 x dx = \int (\sin x)^6 (1-\sin^2 x) d\sin x = \frac{(\sin x)^7}{7} - \frac{\sin^9 x}{9} + C$$

When n is odd, the third can be done in a similar way.

3. Ox St. Hugh's (2014)

   $y(x) = 1 + x\int_0^1 y(t)dt$

   Write an expression for y(x)

   Answer: Let y=kx+1, so kx+1=1+x$\int_0^1 (kt+1)dt$=1+x(1/2k+1), so

   k=1/2k+1, k=2. therefore, $y = 2x + 1$

4. Ox oriel (2014)

   $\int x^2 \cos x \, dx$

5. Ox unknow (2014)

   Evaluate $\int_{-1}^{1} x^2 - x \, dx$ and $\int_{-1}^{1} x^2 - x + 2x \, dx$ and find the difference between the area represented by the integrals

6. Ox Herford (2015)

   $\int 2x(1-x^2)^{-\frac{1}{2}} dx$

   $\int \frac{2x}{(1+x^2)^{10}} dx$

7. Ox (2015)

   $\int_0^{1/2} x\sqrt{1-x^2} \, dx$

8. Ox (2015)

   $\int_a^{a^2} e^{x^2} dx = 0$ Find a.

9. Cam Sidney Sussex (2016)

Draw $y=\frac{1}{x}, y=\frac{1}{\sqrt{x}}$

Integrate from 0 to 1

Integrate $\frac{1}{x^a}$. Find the range fo a when the integration is finite.

10. Ox Brasenose (2017)

$$\int \arcsin x \, dx$$

$$\int \frac{1}{\arcsin x} dx$$

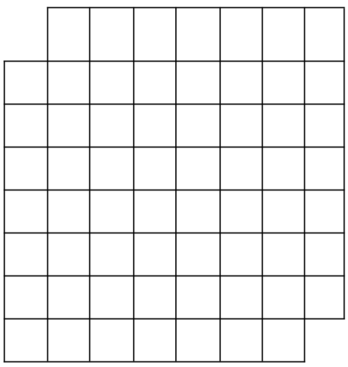
11. Cam Lucy Cavendish (2019)

求 f（t）的反函数从 f（a）到 f（b）的积分。

12. Ox St. Annes (2019)
积分 0-正无穷 sin1/x convergence

Topic 7: Logic Puzzle

1. Ox New (2014)
We have couple dominoes ,each has one side of 1 and one side of 2.
    a. What kind of squares can we made by the dominoes? Is it possible to make a square with odd sides? Prove it.
    b. The dominoes can be in either vertical or horizontal, now we note that the number of vertical dominoes as v, and the number of horizontal dominoes as h. What can you find about v+h? Prove it. Is it possible for both v and h are odds? Prove it.
    c. Now suppose that I have a chessboard (a 8*8 square), and I cut two 1*1squares at the opposite angle (shown as below), can you use the dominoes to cover it? Prove it.



2. Ox Queens (2014 & 2015)
   10 bucket contains coins. One of them contains fake coins. Genuine coins weigh 10g while the fake ones weigh 9g. How many times do you need to determine which bucket contains fake coins?

3. Ox St. Cats (2015)

   八个水桶秤几次可以得到最重的，再秤几次得到最轻的。

4. Ox St. Cats (2015)
   机器人跑 m*n 个格子最短路径怎么跑

5. Ox St. Cats (2015)
   State the following statements are true or false:
   a. For all positive real number y, there is a positive real number x that $1/x<y$.
   b. There exists a positive real number x, where $1/x>y$ for all positive real number .
   c. For all positive real number y, there is a positive real number x such that x>z, where $-y < \frac{1}{x} < y$
   d. For all positive real number y, there is a positive real number x such that x>z, where $x^2 \sin x > y$.
   e. There exists a positive real number x such that x>z, where x2sinx>y for all positive real number y.

6. Ox St. Cats (2015)
   Here has a grid of 2*n, and I have some 1*2 dominoes. Now you need to use the dominoes to cover the grid in either horizontal or vertical. How many ways you can put the dominoes?

					……		
					……		

7. Ox St. Cats (2015)
   Here has a truth table:

A	B	A=>B
True	True	True
True	False	False
False	True	False

False	False	True

Statement "A=>B" can be written as "If A, then B", which is called A imply B

State the following statements are true or false by using the truth table:

a. 2 is prime => 2 is odd
b. 3 is odd => 3 is prime
c. 2 is odd => 2 is not prime
d. If a=b, then a^2=b^2
e. If n is a perfect square, then n is not prime

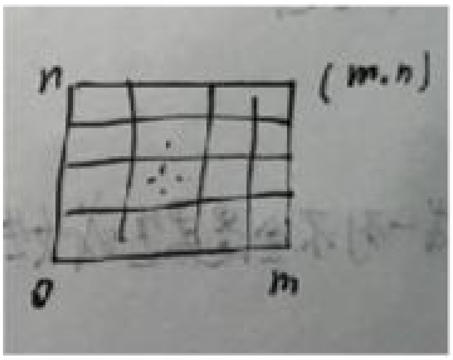
8. Ox Queens (2015)

There are many points with red and green, prove there exists two points with same colour have distance of 1

9. Cam Christ's (2016)

一个 2016*2016 表格里所有项为 1 和-1，问所有 2*2 squares 和都为 0 的排列数

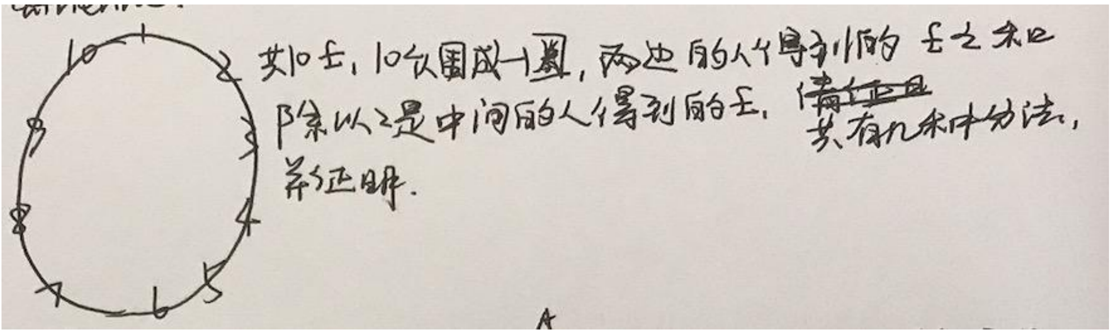
10. Cam Peterhouse (2016)



老鼠爬奶酪问题:

a. 一点到另一点 n 种爬法?
b. A 点先到 B 点再到 C 点?
c. A 点不经过 C 点到 C?
d. A 不经过 B，C 到 D?

11. Ox St. Cats (2017)



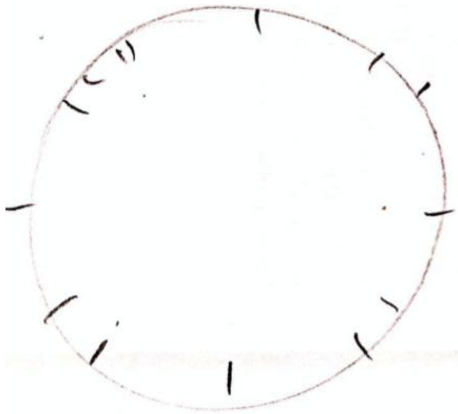
12. Ox Queens (2017)
    a. There are n people in a party. Explain why there is at least two of them that know exactly the same number of people in the party. (Hint: use pigeonhole)
    b. There are 2n+1 people, assume that if you pick any n there is somebody sles that knows them all. Explain why 2n+a people know each other. (Hint: prove by induction that there are n+1 people that they all know each other)
13. Ox Queens (2015)
    There are infinite points in a plane that are either red or green. The distance between two red points (for example x) is denoted by a red number x. the distance between two green points is denoted by a green number. distance between one green and one red point is an unknown number (can be red, can be green, can be both or nothing). Show that at least one of the following is true:
    a. All numbers are red
    b. All numbers are green

    (Hint: try to show that if m is not red then every number smaller than m is green)

14. Cam Churchill (2018)
    一个杯子先放入两个球，再拿出一个，如此循环，问无数次循环后杯子里球的个数

15. Ox Lincoln (2018)
    有两堆笔，每次可以拿一堆笔里任意数量的笔，你和对手轮流交换拿，拿到最后一支笔的算赢，用二元数组表示两堆笔，现在有(5,7)，怎么保证你赢？

16. Cam Magdalene (2019)



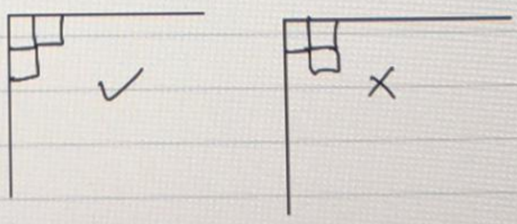
a clock with each position either red or blue, prove must have one symmetrical axis that divides them into 3 red 3 blue on each side

4. N red 2N blue, draw 2 times, if you get same colour → win, what is the probability of winning.

17. Ox Jesus (2019)

You have a infinity chessboard
and you can put n boxes
but boxes should pushed to the wall

for example



draw for n = 4, 5
find the rule for drawing symmetric graph
find the box relative to number n.

18. Ox Keble (2019)

Explain the following is true or false

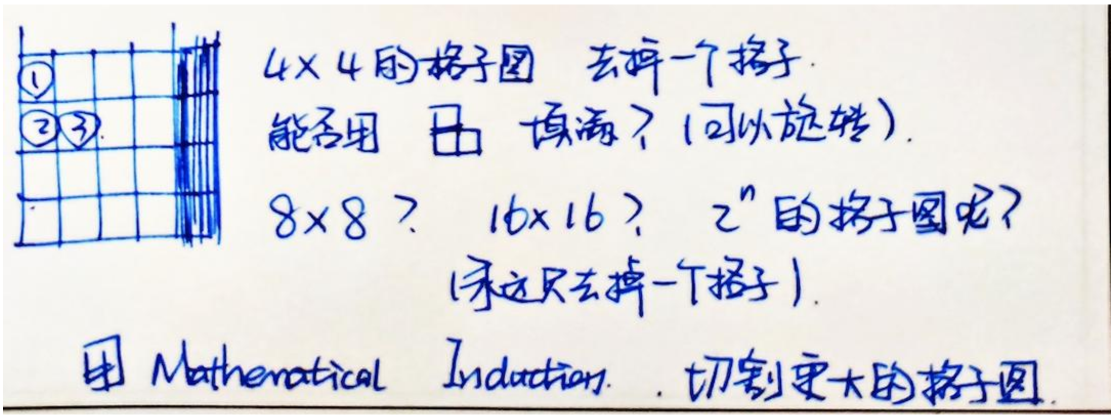
a. $a^{\log b} = b^{\log a}$
b. $\sin(cosx) = \cos(sinx)$
c. $x^4 + 3 + x^{-4} = 5$

d. 存在$p(x)$so that $|p(x)| - \cos x < 10^{-6}$，$p(x)$是polynomial

19. Ox St. Annes (2019)
    九个颜色给九宫格涂颜色 多少种
    可以旋转多少种
    可以翻折多少种
    里面有小人可以向四个方向旋转多少种

20. Ox St. Hugh's (2019)
    If we can use n r*2r rectangles to fill in a 1*1 square, what values can n be? R can be any number , and rectangles can be in different sizes.

21. Ox Somerville (2019)



22. Ox Somerville (2019)
    假设定义 n-square：
    n 维空间里边长为 1 的 n 维多面体 （2-square：正方形，3-square：立方体）
    a. N-square 有多少顶点
    b. 定义 one-walk：每次走一格，同一方向上不能后退，从$(0,0,\cdots,0)$到$(1,1,\cdots,1)$有多少种 one-walk？
    c. 定义 two-walk：每次走 2 格（2 个方向），同一方向上不能后退，从$(0,0,\cdots,0)$到$(1,1,\cdots,1)$有多少种 two-walk？

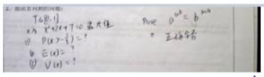
23. IC Maths & CS (2019)
    100kg of potato has 99% of water, after the sunlight exposure, the content of the water has dropped to 98%. What is the weight of potato now?

Topic 8: Number Theory

1. Cam (2014)
   $T \in [0,1]$,x 为$x^2 + 2x + T = 0$ 最大值
   a. $\mathbb{P}\left(x > \frac{1}{3}\right)$

b. $\mathbb{E}(x)$

c. $Var(x)$



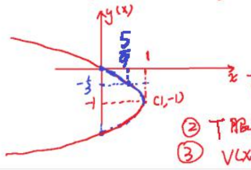
考点: 连续型随机变量

$x^2+2x+T=0$

$x=\frac{-2+\sqrt{4-4T}}{2} \le \frac{-2+2\sqrt{1-0}}{2}=0$

$\therefore x_{max}=0$

① $x=-1+\sqrt{1-T}$, $(x+1)^2=1-T$, $\therefore (x+1)^2=-(T-1)$



$p(x>-\frac{1}{3})=\frac{5}{9}$

② T服从均匀分布 $E(T)=\frac{1}{2}$, $E(X)=-1+\sqrt{1-\frac{1}{2}}=-1+\frac{\sqrt{2}}{2}$

③ $V(X)=E(X^2)-(E(X))^2$

$x^2+2x+1=1-T$, $\therefore E(X^2)+2(-1+\frac{\sqrt{2}}{2})+1=1-\frac{1}{2}$ $\therefore E(X^2)=\frac{3}{2}-\sqrt{2}$

$\therefore V(X)=\frac{3}{2}-\sqrt{2}-(-1+\frac{\sqrt{2}}{2})^2=0$

2. Cam (2014)

Prove $a^{\ln b}=b^{\ln a}$是否正确

$\int_0^1 x^2dx-\frac{1}{4}=\frac{1}{3}-\frac{1}{4}=\frac{1}{12}$

prove: $a^{\ln b}=b^{\ln a}$

prove: since $\ln b\cdot\ln a=\ln a\cdot\ln b$

$\Rightarrow \ln a^{\ln b}=\ln b^{\ln a}$

$\Rightarrow a^{\ln b}=b^{\ln a}$

3.

Prove $x^2+y^2+z^2\ge xy+yz+zx$

$(x-y)^2+(y-z)^2+(z-x)^2\ge 0$

$\therefore x^2+y^2+z^2\ge xy+yz+zx$

4. Cam (2014)

$x^3+lx^2+mx+n=0\ (l,m,n\in\mathbb{Z})$

prove if $x\notin\mathbb{Z}$, then $x\notin\mathbb{Q}$

Note: Q is the set of all rational numbers.

This is equivalent to proving if x is rational, then x is an integer.

考点: contrapositive, divisibility

prove by contradiction

if x is rational, but $x \notin \mathbb{Z}$

so $x = \frac{p}{q}$, $(p,q)=1$, and $q \neq 1$.

$$\Rightarrow \frac{p^3}{q^3} + \frac{lp^2}{q^2} + \frac{mp}{q} + n = 0$$

$$p^3 + lp^2q + mpq^2 + nq^3 = 0$$

$\therefore p^3 \equiv 0 \pmod{q}$ since $(p,q)=1$, $q \neq 1$

this is absurd.

so if $x \in \mathbb{Q}$, then $x \in \mathbb{Z}$

$\Leftrightarrow x \notin \mathbb{Z}$ then $x \notin \mathbb{Q}$

5. Cam (2014)

Harbor constant: $H(t) = \frac{R'(t)}{R(t)}$

a. 证明$t < \frac{1}{H(t)}$

b. 写出关于$R(t)$的方程式，

c. 如果$H(t) = \frac{R'(t)}{R(t)} = \frac{a}{t}$,求出 a 的范围满足$\begin{cases} R(t) = 0 \\ R'(t) > 0 \\ R'(t) < 0 \end{cases}$

6. Ox Mansfield (2014)

$f(x+y) = f(x)f(y)$, prove that $f(0) = 1$. Prove that there exists k such that for any value of n, $f(n) = kn$.

6. Ox Mansfield (2014)

$f(x+y)=f(x)f(y)$, prove that $f(0)=1$. Prove that there doesn't?? exists k such that for any value of n, $f(n)=kn$.

Let $x=y=0$. $f(0)=f^2(0) \Rightarrow f(0)(f(0)-1)=0$, $f(0)=0$ or $f(0)=1$

Let $x=0$ $f(y)=f(0)f(y)$, $f(0)\neq 1$, otherwise, $f(y)=0$ for all y

so $f(0)=1$.

Let $x=-y$ $f(0)=1=f(-y)f(y)$

suppose there exists k such that for any value of n, $f(n)=kn$.

then $f(-n)=-kn$, $\because f(n)f(-n)=1 \Rightarrow -k^2n^2=1$, which is impossible

so there doesn't exist k such that for any value of n, $f(n)=kn$

7. Ox Brasenose (2014)

a. Show that if n^2 is odd, then n must be odd.

b. If $x^n+1=0$, does n have to be odd?

c. Fractionise (x^n+1) when n is odd.

d. $(2n+1)$ is prime, does n have to be odd?

e. Following c), if $m=2n$, does m have to be odd?

f. $2^{2m}+1$, $m=2k$, does k have to be odd?

考点：odd and prime numbers

---

a. If n is even, then n^2 is even, which contradicts the given condition so n must be odd.

b $x^n=-1=e^{i(2k+1)\pi}$ $\therefore x=e^{i\frac{(2k+1)\pi}{n}}$, So when $x=e^{i\frac{\pi}{2}}$,

n could be 2, so n doesn't have to be odd.

c. $x^n+1=(x+1)(x^{n-1}-x^{n-2}+x^{n-3}-x^{n-4}+\dots+x^2-x+1)$

d. No. If $n=2$, $2n+1=5$ is prime. but n is even

e. the problem is weird

f. ×

8. Ox Mansfield (2014)

   a, b, c, d are integers. Subtract c from d and take the modulus of the result. Repeat this for all numbers, why will all number become 0 at the end. 3.

9. Ox Brasenose (2014)

   Simplify the following expression if a and b are non-zero real numbers: $\frac{1}{a-b+\sqrt{a^2+b^2}}+\frac{1}{a+b+\sqrt{a^2+b^2}}$

   Prove also that the $a+\sqrt{a^2+b^2}$cancelled is not equal to zero.

$$\frac{1}{a-b+\sqrt{a^2+b^2}}+\frac{1}{a+b+\sqrt{a^2+b^2}}$$

$$=\frac{a-b-\sqrt{a^2+b^2}}{-2ab}+\frac{a+b-\sqrt{a^2+b^2}}{2ab}$$

$$=\frac{2b}{2ab}=\frac{1}{a}$$

if $a+\sqrt{a^2+b^2}=0$, then $\frac{1}{a-b+\sqrt{a^2+b^2}}+\frac{1}{a+b+\sqrt{a^2+b^2}}=-\frac{1}{b}+\frac{1}{b}=0$

so $\frac{1}{a}=0$, which is absurd for $a\neq 0$.

so $a+\sqrt{a^2+b^2}\neq 0$

10. Ox St. Hugh's (2014)

   $a<100$ and a is a positive integer, find a where $100 \mid a(a-1)$

$a(a-1)\equiv 0 \pmod{100}$

so, $\begin{cases} a(a-1)\equiv 0 \pmod{4} \\ a(a-1)\equiv 0 \pmod{25} \end{cases}$

since $(a,a-1)=1$, $\Rightarrow \begin{cases} 4\mid a \text{ or } 4\mid a-1 \\ 25\mid a \text{ or } 25\mid a-1 \end{cases}$

$\therefore$ ① $\begin{cases} 4\mid a \\ 25\mid a \end{cases} \Rightarrow 100\mid a$, $a<100$, positive integer a doesn't exist

② $\begin{cases} 4 \mid a \\ 25 \mid a-1, \end{cases} \Rightarrow \begin{cases} a \equiv 0 \pmod 4 \\ a-1 \equiv 0 \pmod{25} \end{cases} \Rightarrow a = 4k,\ 4k-1 \equiv 0 \pmod{25},\ 4k \equiv 1 \pmod{25}$

$k = 19 + 25m$

So $a = 4k = 76 + 100m.$

$a = 76$

③ $\begin{cases} 4 \mid a-1 \\ 25 \mid a \end{cases} \Rightarrow \begin{cases} a-1 \equiv 0 \pmod 4 \\ a \equiv 0 \pmod{25} \end{cases} \Rightarrow a = 25k, \Rightarrow 25k \equiv 1 \pmod 4$

$\Rightarrow 24k + k \equiv 1 \pmod 4$

$k \equiv 1 \pmod 4$

$\therefore k = 1 + 4m, \Rightarrow a = 25k = 25 + 100m$

$a = 25$

④ $\begin{cases} a-1 \equiv 0 \pmod 4 & ① \\ a-1 \equiv 0 \pmod{25} & ② \end{cases} \Rightarrow$ from ①, $a = 4k+1$. substitute into ②

$4k \equiv 0 \pmod{25}$

$k = 25m$

$a = 4k+1 = 1 + 100m$

$a = 1$

$\boxed{\text{Therefore, } a = 1, 25, 76}$

11. Ox St. Hugh's (2014) (**问题**)

    If you were given 7 different integers, the sum of which is 100. Verify that the sum of 4 of them is no less than 50. Verify that the sum of three of them is no less than 50.

12. Ox St. Catherine (2014)

    Is it possible to write 2015 into two squared numbers?

    What about three squared numbers?

13. Ox New (2014)

    What is golden number?

    Why rectangle in golden ratio is seemingly perfect?

14. Ox New (2014)

    Prove that the following equation has no non-zero integer root: $96837x^2 + 63165xy + 31785y^2 = 0$

15. Ox New (2014)

Given a list of prime numbers:2, 3, 5, 7, 11, 13. Are any of these 1 smaller than a square number? Are any of these 1 smaller than a cubic number? Can $\mathrm{a}^{\mathrm{k}-1}$ be a prime number?
Are any of these 1 greater than a cubic number? Are any of these 1 greater than a 4th-power number?

16. Ox Somerville (2014)
    $\mathrm{x}^2 + b + c = 0$, if k is a root, k^{-1} is also a root, prove either $c = 1$ or $c =- 1$ and $b = 0$.

17. Ox Somerville (2014)
    n is integer, when n is not a prime, is it true that n divides $(n-1)!$

18. Ox unknown (2014)
    Example: the number 961 can be manipulated in the following way: $9 + 6 + 1 = 16$, $1 + 6 = 7$. Numbers that can be manipulated in similar way into 4 or 7 are unlucky numbers. Is 13 unlucky? Is 7876777 unlucky?

19. Ox Somerville (2015)
    Find the 3rd largest number and the 7th largest number from an array of number.
    How could you increase the time complexity (how could you save time) and what is it?

20. Ox Somerville (2015)
    How many 0s are there at the end of 37!

21. Cam Fitzwilliam (2015)
    Why is $\frac{6}{11}$ greater than $\frac{1}{2}$?

22. Cam Robinson (2015)
    What is a prime number? Can any prime number be expressed as $\mathrm{x}^2 - 1$?

考点: prime numbers, factoring

A prime number doesn't have any other factors other than 1 and itself.

if $p = x^2-1 = (x-1)(x+1)$

then $x-1=1$ or $x+1=1$

$x=2$ or $x=0$

so $p=3$ or -1 (reject)

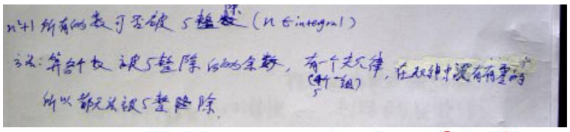
so it's only 3 that can be expressed as x^2-1 (2^2-1)

23. Ox St. Cats (2015)

    Here has an equation: $x^2+bx+c=0$, where b and c are both odd. Prove that there has no integer solutions.

24. Ox (2015)

    Can n^2+1 be divided by 5, where n is an integer


n^2+1 所有的数可否被5整除 ($n \in$ integral)

方法：算各个数被5整除后的余数，有一个规律（5个一组），在双种中没有有整除的

所以都无法被5整除

应该是求 n^2+1 中所有可被5整除的数

$\boxed{n = 5k+2 \text{ or } 5k+3}$

考点：modular arithmetic

若 $n^2+1 \equiv 0 \pmod 5$

$n^2 \equiv 4 \pmod 5$

if $n \equiv 0$, $n^2 \equiv 0$ ✗

$n \equiv 1$ $\quad n^2 \equiv 1$ ✗

$n \equiv 2$ $\quad n^2 \equiv 4$ ✓

$n \equiv 3$ $\quad n^2 \equiv 9 \equiv 4$ ✓

$n \equiv 4$ $\quad n^2 \equiv 16 \equiv 1$ ✗

so when $n \equiv 2, 3 \pmod 5$

$5 \mid (n^2+1)$

25. Ox (2015)

    Multiply the odd numbers between 1 and 1000000. Find the last digit of the summarized number

考点：任何奇数与5相乘，个位仍为5

$1 \times 3 \times 5 \times 7 \times 9 \times \cdots 999999$

ends with 5

26. Cam (2016)

    30!中有多少个 0

100!中有多少个 0

1000!中有多少个 0

考点：公式 n!结尾0的个数: $\left\lfloor \frac{n}{5} \right\rfloor + \left\lfloor \frac{n}{25} \right\rfloor + \left\lfloor \frac{n}{125} \right\rfloor + \cdots$

$$\left\lfloor \frac{30}{5} \right\rfloor + \left\lfloor \frac{30}{25} \right\rfloor + \left\lfloor \frac{30}{125} \right\rfloor + \cdots = 6 + 1 = 7$$

$$\left\lfloor \frac{100}{5} \right\rfloor + \left\lfloor \frac{100}{25} \right\rfloor + \left\lfloor \frac{100}{125} \right\rfloor + \cdots = 20 + 4 = 24$$

$$\left\lfloor \frac{1000}{5} \right\rfloor + \left\lfloor \frac{1000}{25} \right\rfloor + \left\lfloor \frac{1000}{125} \right\rfloor + \left\lfloor \frac{1000}{625} \right\rfloor = 200 + 40 + 8 + 1 = 249$$

27. Cam Sidney Sussex (2016)

A number $n = a \times b$. Find the number of pairs in this form

考点："number of factors" theorem

Since for every factor a of n, the other factor b can also uniquely determined,

the # of pairs in this form $= \frac{\text{the number of factors}}{2}$

or $\frac{\text{the \# of factors} + 1}{2}$ if n is a square number

① if n is not a square

$n = p_1^{k_1} p_2^{k_2} p_3^{k_3} \cdots$

# of pairs in this form $= \frac{(k_1+1)(k_2+1)(k_3+1)\cdots}{2}$

② if n is a square number.

$n = p_1^{k_1} p_2^{k_2} p_3^{k_3} \cdots$

# of pairs in this form $= \frac{(k_1+1)(k_2+1)(k_3+1)\cdots + 1}{2}$

28. Cam Christ's (2016)

$b \geq 2$, p is prime, $(b, p) = 1$, 是否存在$b^n \equiv -1 (mod\ p)$

modular arithmetic

not necessarily 不一定总存在

举例：

$$\begin{cases} 3^1 \equiv 3 \pmod 5 \\ \boxed{3^2 \equiv -1 \pmod 5} \text{ 存在} \\ 3^3 \equiv 2 \pmod 5 \\ 3^4 \equiv 1 \pmod 5 \end{cases}$$

$3^5 \equiv 3 \pmod 5$

1个循环

$$\boxed{\begin{matrix} 4 \equiv 4 \pmod 7 \\ 4^2 \equiv 2 \pmod 7 \\ 4^3 \equiv 1 \pmod 7 \end{matrix}}$$

$4^4 \equiv 4 \pmod 7$

$4^5 \equiv 2 \pmod 7$

1个循环，不存在 $b^m \equiv -1 \pmod p$

29. Cam Peterhouse(2016)

已知$x+y$为有理数，$xy=1$，证明x^n+y^n（$n \in \mathbb{N}^*$）为有理数

$\boxed{\text{考点：prove by mathematical induction (stronger form)}}$

base case: when $n=1$, $x+y$ is rational.

when $n=2$, $x^2+y^2=(x+y)^2-2xy=(x+y)^2-2$ is also rational

assume when $n=1,2,3,\dots k$, x^n+y^n is also rational

Let's prove when $n=k+1$, $x^{k+1}+y^{k+1}$ is rational

$$x^{k+1}+y^{k+1}=(x^k+y^k)(x+y)-x^ky-xy^k$$

$$=(x^k+y^k)(x+y)-xy(x^{k-1}+y^{k-1})$$

$$=(x^k+y^k)(x+y)-(x^{k-1}+y^{k-1})$$

Since x^k+y^k, $x+y$ and $x^{k-1}+y^{k-1}$ are rational

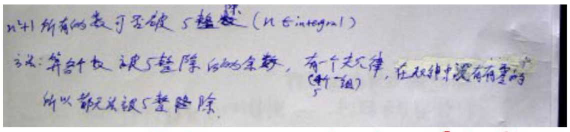
$x^{k+1}+y^{k+1}$ is also rational

So when $n=k+1$, x^n+y^n is also rational

It's true x^n+y^n is rational for all positive integers n.

30. Cam (2016)

$n^2 + 1$ 所有的数可否被 5 整除?


n^2+1 所有的数可否被 5 整除 ($n \in$ integral)

方法：算各个数被5整除后的余数，有一个规律（5个一组），在规律中没有有整除的

所以都无法被5整除

应该是求 n 中所有可被5整除的数

$\boxed{n = 5k+2 \text{ or } 5k+3}$

考点: modular arithmetic

若 $n^2+1 \equiv 0 \pmod 5$

$n^2 \equiv 4 \pmod 5$

if $n \equiv 0$, $n^2 \equiv 0$ ✗

$n \equiv 1$ $\quad n^2 \equiv 1$ ✗

$n \equiv 2$ $\quad n^2 \equiv 4$ ✓

$n \equiv 3$ $\quad n^2 \equiv 9 \equiv 4$ ✓

$n \equiv 4$ $\quad n^2 \equiv 16 \equiv 1$ ✗

so when $n \equiv 2, 3 \pmod 5$

$5 \mid (n^2+1)$

31. Ox Oriel (2016)
    a. Find integer numbers satisfying $a^2 + b^2 = c^2$
    b. Find rational solutions for $a^2 + b^2 = 1$ from the answer above
    c. If there are integer solutions for $a^2 + b^2 = c^2$, are there always rational solutions for $a^2 + b^2 = 1$?
    d. If there are always rational solutions for $a^2 + b^2 = 1$, are there always integer solutions for $a^2 + b^2 = c^2$?
    e. Draw graphs of $a^2 + b^2 = 1$
    f. If there is a point $(-1,0)$ on the circle, draw a line with gradient r. the line cuts the circle at another point, the coordinate of that point are rational numbers, is r a rational number? How about vice versa?

32. Ox Oriel (2016)
    a. Find solutions for $x^n + 1 = 0$, where n is an odd number. Is there any solutions if n is even?
    b. Factorize $x^n + 1$, where n is an odd number. What is the second last term?
    c. If a number in the form $2^n + 1$ is a prime number, show that n musr be in the form of 2^k

33. Ox John's (2016)

Euler foruma: $V - E + F = 2$

What is the condition for $F > V$?

Define $F_n = no. of faces with n vertices$

Deduce that

a. $F = F_3 + F_4 + \cdots F_n$

b. $E = \frac{F_3 \times 3 + F_4 \times 4 + \cdots F_n \times n}{2}$

让你用 F，E，V 证明$F > V$满足什么条件时，$F \geq V + 1$

34. Ox Oriel (2016)

    Prove Pythagoras' Theorem

35. Ox Brasenose (2017)

    a. x, y, z, t are positive integers. Solve: $3|xyzt + xy + xz + zy + 1 = 40(yzt + y + z)$

    b. if n is odd, how many real roots does $n^2 + 1$ have?

    c. Prove $x = -1$ is the only real root

36. Ox Herford (2017)

    Let number a_k, a_{k-1}, $\cdots$, a_1, a_0 be defined as $n = a_0 * 3^0 + a_1 * 3^1 + \cdots + a_k * 3^k$.

    a. What is 101 in the form of n?

    b. What happen to n when it's multiplied by 3? By 9?

    考点：3 进制

a.

$$\begin{array}{l} 3\,\underline{|101} \\ 3\,\underline{|33} \cdots 2 \\ 3\,\underline{|11} \cdots 0 \\ 3\,\underline{|3} \cdots 2 \\ 3\,\underline{|1} \cdots 0 \\ \quad 0 \cdots 1 \end{array}$$

$$(10202)_3 = 2\cdot 3^0 + 0\cdot 3^1 + 2\cdot 3^2 + 0\cdot 3^3 + 1\cdot 3^4$$

b.when it's multiplied by 3, the base-3 representation of n becomes 102020； when it's multiplied by 9, the base-3 representation of n becomes 1020200

37. Cam Magdalen (2017)

    Proof $\frac{n^2-1}{8}$ is integer when n is odd.

    考点: modular arithmetic

方法一：$\frac{n^2-1}{8}$ since $n=2k+1$

$$\frac{n^2-1}{8}=\frac{(2k+1)^2-1}{8}=\frac{4k^2+4k}{8}=\frac{k^2+k}{2}=\frac{k(k+1)}{2}\in\mathbb{Z}$$ since

$k(k+1)$ is an even number.

方法二：since n is an odd number, $n^2\equiv 1 \pmod 8$

$n^2-1\equiv 0 \pmod 8$, so n^2-1 is a multiple of 8

$\frac{n^2-1}{8}$ is an integer.

38. Ox St. Peters (2017)

Prove by contradiction that there are infinite number of primes.

prove: assume the set of primes is finite: $\{p_1, p_2, p_3, \cdots, p_n\}$

then I create a new number $N=p_1p_2p_3\cdots p_n$

① if $N+1$ is prime, then there's another prime number outside the set, which contradicts the assumption

② if $N+1$ is composite, since $(N, N+1)=1$, and N has all prime factors including $p_1, p_2, \cdots p_n$, which means $N+1$ has factors outside the set, contradicting the assumption that the set includes all primes.

39. Cam Churchill (2018)

a. $5n+1$ 被 7 整除，求 n 的表达式

b. 5^n被 7 整除，求 n 的表达式

a. $5n+1 \equiv 0 \pmod 7$

$5n \equiv 6 \pmod 7$

$\therefore n = 4 + 7k$

고: $5n + 7k = 6$

$7 = 1 \times 5 + 2$

$5 = 2 \times 2 + 1$

$2 = 2 \times 1$

$\therefore 5 - 2 \times 2 = 1$

$5 - 2 \times (7-5) = 1$

$\therefore 5 \times 3 - 2 \times 7 = 1$

$\therefore 5 \times (18 - 12k) = 6$

$\therefore n = 18 + 7k$

$\therefore n = 4 + 7$

b. $5^n \equiv 0 \pmod 7$

since $5 \equiv 5 \pmod 7$

$5^2 \equiv 4 \pmod 7$

$5^3 \equiv 6 \pmod 7$

$5^4 \equiv 2 \pmod 7$

$5^5 \equiv 3 \pmod 7$

$5^6 \equiv 1 \pmod 7$

$5^7 \equiv 5 \pmod 7$

$5^8 \equiv 4 \pmod 7$

There's no n.

40. Cam Magdalene (2019)

Prove $2^{2^n} - 1$ has at least one prime factor

$$(2^{2^n}-1)$$
$$=(2^{2^{n-1}}+1)(2^{2^{n-1}}-1)$$
$$=(2^{2^{n-1}}+1)(2^{2^{n-2}}+1)(2^{2^{n-2}}-1)$$
$$=(2^{2^{n-1}}+1)(2^{2^{n-2}}+1)(2^{2^{n-3}}+1)\cdots(2+1)(2-1)$$

since the factorization has a 3,

$2^{2^n}-1$ has at least one prime factor.

41. Cam Magdalene (2019)

$\sqrt{2019}+\sqrt{2022}$ and $\sqrt{2021}+\sqrt{2020}$ which is bigger?

考察: 简便运算技巧。

$$(\sqrt{2019}+\sqrt{2022})^2=4041+2\sqrt{2019\cdot 2022}$$
$$(\sqrt{2021}+\sqrt{2020})^2=4041+2\sqrt{2021\cdot 2020}$$
$$\because\ 2019\cdot 2022=(2020-1)(2021+1)=2020\cdot 2021-2<2020\cdot 2021$$
$$\therefore\ \sqrt{2019}+\sqrt{2022}<\sqrt{2021}+\sqrt{2020}$$

方法二: 构造函数

$$f(x)=\sqrt{x+1}-\sqrt{x}$$
$$f'(x)=\frac{1}{2\sqrt{x+1}}-\frac{1}{2\sqrt{x}}=\frac{\sqrt{x}-\sqrt{x+1}}{2\sqrt{x}\sqrt{x+1}}<0\ \therefore f(x)\text{ is}$$

a decreasing function

$$\therefore f(2021)<f(2019)\ \therefore \sqrt{2022}-\sqrt{2021}<\sqrt{2020}-\sqrt{2019}$$
$$\sqrt{2019}+\sqrt{2022}<\sqrt{2021}+\sqrt{2020}$$

42. Cam Magdalene (2019)

If $x > 0$, which rational number a satisfies $\frac{x^a+(x+1)^a}{2} > \left(x+\frac{1}{2}\right)^a$

If $x > 0$, which rational number a satisfy $\frac{x^a+(x+1)^a}{2} > \left(x+\frac{1}{2}\right)^a$

$\Leftrightarrow \quad x^a + (x+1)^a > 2(x+\frac{1}{2})^a$

$\Leftrightarrow \quad x^a - (x+\frac{1}{2})^a > (x+\frac{1}{2})^a - (x+1)^a$

Let $f(x) = x^a - (x+\frac{1}{2})^a$

$\therefore \ f(x) > f(x+\frac{1}{2})$

$\therefore \ f(x)$ is decreasing $\Rightarrow f'(x) < 0$

$\therefore \ f'(x) = ax^{a-1} - a(x+\frac{1}{2})^{a-1}$

$= a(x^{a-1} - (x+\frac{1}{2})^{a-1}) < 0$

i) if $a<0$

$\therefore \ a-1<0$, $1-a>1$.

$\therefore \ x^{a-1} - (x+\frac{1}{2})^{a-1}$

$= \frac{1}{x^{1-a}} - \frac{1}{(x+\frac{1}{2})^{1-a}}$

$\because \left(\frac{x+\frac{1}{2}}{x}\right)^{1-a} = \left(1+\frac{1}{2x}\right)^{1-a} > 1$

$\therefore \ (x+\frac{1}{2})^{1-a} > x^{1-a} > 0$

$\therefore \ \frac{1}{(x+\frac{1}{2})^{1-a}} < \frac{1}{x^{1-a}} \quad \therefore f'(x)<0$

So $a<0$ is ok

ii) if $0<a<1$

$a-1<0$, $0<1-a<1$

$\therefore \ \frac{1}{x^{1-a}} - \frac{1}{(x+\frac{1}{2})^{1-a}} > 0$

$\therefore \ f'(x) > 0$ rejected

iii) if $a>1$. $a-1>0$

$\therefore \ x^{a-1} - (x+\frac{1}{2})^{a-1} < 0$

$\therefore \ f(x) < 0$ ok

$\therefore \ \boxed{a<0 \text{ or } a>1}$

"Jensen Inequality"

Let $f(x) = x^a$.

So $f(x)$ must be a convex function

$\Rightarrow f''(x) > 0$. So $f''(x) = a(a-1)x^a > 0$

$\therefore \ a(a-1) > 0$

$\therefore \ a>1$ or $a<0$

43. Ox Keble (2019)

x, y, z are prime

solve $x + y^2 = 4z^2$

考点: use factoring to solve diophantine equations.

$x=(2z)^2-y^2$

$x=(2z+y)(2z-y)$ 3, 5, 7, 9, 11, 13, 17

$\therefore \begin{cases} 2z+y=x \\ 2z-y=1 \end{cases} \Rightarrow \begin{cases} z=\frac{x+1}{4} \\ y=\frac{x-1}{2} \text{ always an integer} \end{cases}$

$x+1 \equiv 0 \pmod 4$, $x \equiv -1 \pmod 4$, $x=4m-1$

So the solution is $\begin{cases} x=3, 7, 11, 19, 23, 31, 43, 47, \cdots \\ y=1, 3, 5, 9, 11, 15, 21, 23, \cdots \\ z=1, 2, 3, 5, 6, 8, 11, 12, \cdots \end{cases}$

44. Ox Somerville (2019)

$360|n^6-5n^4+4n^2$?

考点: 要证明一个数被一个合数整除，只需要证明这个数能被这个合数的各个互质的因子整除。其实就是 Chinese remainder theorem 的特例。

$360|n^6-5n^4+4n^2$?

we just need to prove $8|n^6-5n^4+4n^2$

$9|n^6-5n^4+4n^2$

$5|n^6-5n^4+4n^2$

$f(n)=n^6-5n^4+4n^2$

$=n^2(n^4-5n^2+4)$

$=n^2(n^2-4)(n^2-1)$

$=n^2(n+2)(n-2)(n+1)(n-1)$

$=\underbrace{(n-2)(n-1)n}\,\underbrace{n(n+1)(n+2)}_{\text{three consecutive integers}}$

① Since the product of three consecutive integers is always a multiple of 3, and $f(n)$ is the product of two groups of three consecutive integers. $9|f(n)$

② $8 \mid$ the product of two consecutive even integers, Let's prove

if $n=2k$ is even, then $n(n+2) = 2k \cdot (2k+2)$

$= 4\underbrace{k(k+1)}_{even}$

So $8 \mid n(n+2)$ when n is even

(i) therefore, in $f(n)$, if n is odd, $(n-1)(n+1)$ is the product of 2 consecutive even integers, so

$8 \mid f(n)$

(ii) if n is even, $4 \mid n^2$, $2 \mid n+2$, $\Rightarrow 8 \mid f(n)$

so $8 \mid f(n)$

③ $5 \mid$ the product of 5 consecutive integers.

Let's prove.

if $n \equiv 0 \pmod 5$ $\quad 5 \mid (n-2)(n-1)n(n+1)(n+2)$

if $n \equiv 1 \pmod 5$, then $n-1 \equiv 0 \pmod 5 \Rightarrow 5 \mid (n-2)(n-1)n(n+1)(n+2)$

if $n \equiv 2 \pmod 5$ then $n-2 \equiv 0 \pmod 5 \Rightarrow 5 \mid (n-2)(n-1)n(n+1)(n+2)$

if $n \equiv 3 \pmod 5$ then $n \equiv 3 \equiv -2 \pmod 5 \Rightarrow n+2 \equiv 0 \pmod 5 \Rightarrow 5 \mid (n-2)(n-1)n(n+1)(n+2)$

if $n \equiv 4 \equiv -1 \pmod 5$ then $(n+1) \equiv 0 \pmod 5 \Rightarrow 5 \mid (n-2)(n-1)n(n+1)(n+2)$

so $5 \mid f(n)$

therefore, $360 \mid f(n)$

45.IC Math &CS (2019)

Can 10003 be constructed as the sum of two squares?

Suppose $10003 = x^2 + y^2$

since $x^2 \equiv 0, 1 \pmod 4$

$y^2 \equiv 0, 1 \pmod 4$

$\Rightarrow x^2 + y^2 \equiv 0, 1, 2 \pmod 4$

but $10003 \equiv 3 \pmod 4$

therefore, 10003 can't be constructed as the sum of 2 squares.

46. IC Math &CS (2019)

How many d and e (0 to 9) can let 35d4e be divisible by 12?

考点：被1个合数整除，则拆成互质的因子

$3 \mid 35d4e \Rightarrow 3 \mid 12+d+e, \Rightarrow 3 \mid d+e$

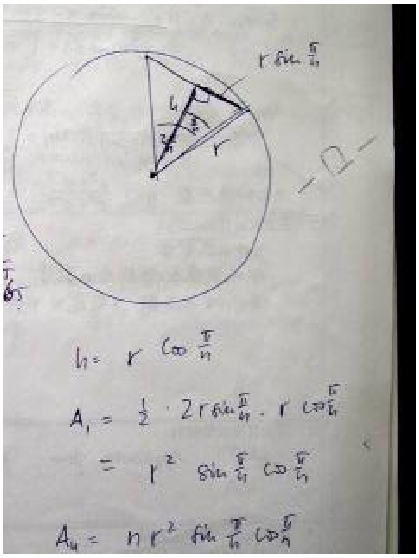
$4 \mid 35d4e \Rightarrow 4 \mid 4\times 10+e, \Rightarrow 4 \mid e$

$\therefore$ $e = 0, 4, 8,$

$d = 0, 3, 6, 9 \;\vdots\; 2, 5, 8 \;\vdots\; 1, 4, 7,$

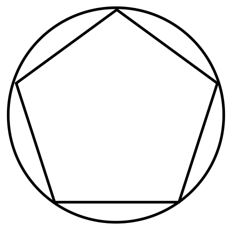
Topic 9: Geometry

1. Cam (2014)



2. Ox Balliol (2014，2015)

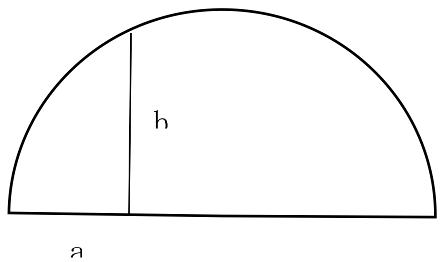
Radius of the circle is 1, evaluate the perimeter of the pentagon.



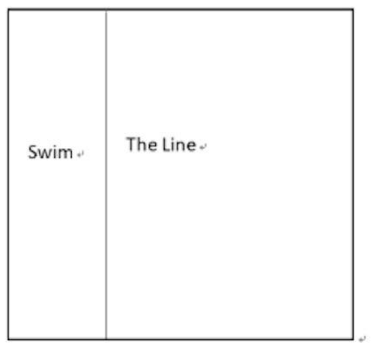
Evaluate the perimeter of n-side polynomial.

3. Ox Brasenose (2014)
   Three circles are tangential to each other, find the area bounded by the circumference of the circle. Find the maximal area of a circle that can be inserted in the shaded area.

4. Ox Keble (2014)
   For a semi-circle with radius 1, be is always perpendicular to a. Prove that (b2+a2/a is constant)



5. Ox St. Catherine (2014)
   In a race, I start from the bottom left corner at (0,0) and needs to get to the upper right corner at (10,10). I need to swim before run. I run 5 times faster than I swim. I need no shoes for swimming but I do need them for running. The shoes are on the line shown below. Which point is the best position for the shoes so that I can get to (10,10) fastest?



6. Ox. New (2014)
   An isosceles triangle has two sides of 1cm, what is the minimum area of the triangle.
   How many diagonals in a pentagon?
   What about an n-side polygon?

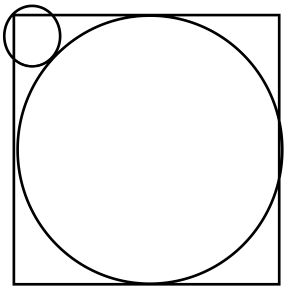
7. Ox. New (2014)
在一条河盛水，有 3L 和 5L 的容器，怎样盛出 4L？1L？

8. Ox. New (2014)
披萨饼，切 4 刀，最多能切几块？ n 刀呢？

9. Ox Mansfield (2014)
A Venn diagram contain n elements, how many parts are there?

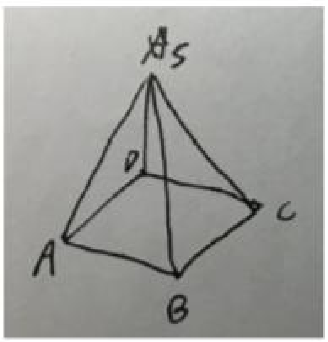
10. Ox (2015)
A n-sided isogon is internally tangent to a circle. Calculate the area of the isogon.

11. Ox St. Cats (2015)
Here has a square and an inscribed circle, then we get a smaller circle tangent to the large one and passing through the corner of the square. Now we know that the circumference of the small circle is 1, what is the circumference of the large circle?

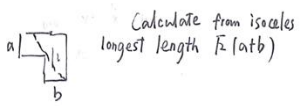


12. Cam Peterhouse(2016)

求面SAB与面SBC的二面角?

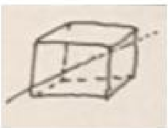


13. Ox Worcester (2016)

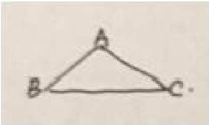


14. Ox Lincoln (2016)

1. Here is a cube. Find the axis of symmetry and degree of rotation.

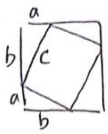


2. Here is a triangle. A bee is at A initially. There is same chance to go either 2 points. What's the probability after nth jump for the bee to get back to A?



15. Ox Oriel (2016)

1.



$(a+b)^2-2ab=c^2$

$a^2+b^2=c^2$

2. $3^2+4^2=5^2$

3. if $a^2+b^2=c^2$ $(a,b,c\in\mathbb{Z})$

$(\frac{a}{c})^2+(\frac{b}{c})^2=1$

4 ✓

5. a, b may not be expressed as same denominator.

6

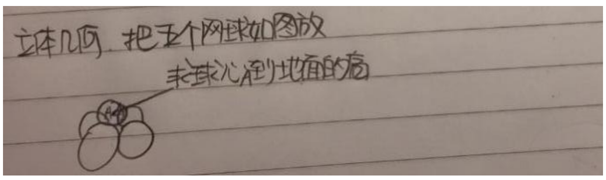
7. Another point (m, n) $m,n\in\mathbb{Q}$

$r=\frac{n-0}{m+1}=\frac{n}{m+1}$ is a rational number by finding LCM ~~common~~ of denominator.

Not necessary for vice versa

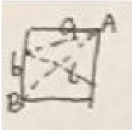
16. Ox St. John's (2016)
    What is polygon? Example
    What is a polyhedral? Example

17. Cam Magdalen (2017)



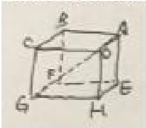
18. Ox St. Peters (2016)

一个长方形，点B和A重合折，求折痕l的长度



一个4 × 4 × 4 cube. 找对称轴和对称轴对应的order of rotation, 对于AG的order of rotation?

Then, prove that, given five 1 × 1 × 1 cube, six 4 × 2 × 1, six 3 × 2 × 2, 让用这些拼出一个5 × 5 × 5 cube



19. Ox Herford (2017)

A square paper is to be cut into n pieces of squares

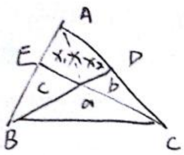
   a. Can it be cut into 8 pieces of squares?
   b. Can it be cut into 7 pieces of squares?
   c. Which n can't be reached?

30. Ox St. Peters (2017)

There are 3 people in town A and 5 people in town B. Where is the best place for them to meet together? That means the total distance travelled by all people is the shortest.

20. Ox Queens (2017)

2017 Queens.



Find X

$\frac{x_1}{c} = \frac{x_2+b}{a} \Rightarrow x_1 = \frac{c(x_2+b)}{a}$

$\frac{x_2}{b} = \frac{x_1+c}{a}$

$\frac{x_2}{b} = \frac{\frac{c(x_2+b)}{a}+c}{a}$

$a^2x_2 = bc(x_2+b)+abc$

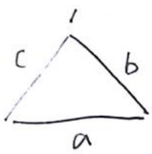
$x_2 = \frac{b^2c+abc}{a^2-bc}$

$x_1 = \frac{c}{a}\left(\frac{b^2c+abc}{a^2-bc}+b\right)$

$= \frac{c}{a}\left(\frac{a^2b+abc}{a^2-bc}\right) = \frac{abc+bc^2}{a^2-bc}$

$x = x_1+x_2 = \frac{b^2c+bc^2+2abc}{a^2-bc}$

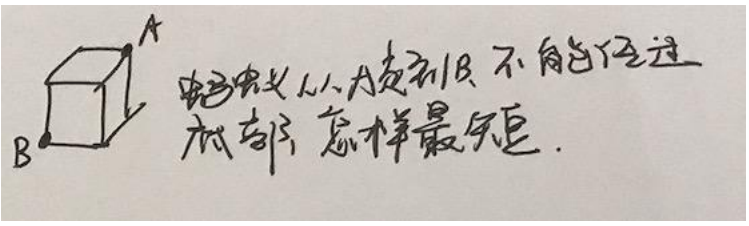
2.



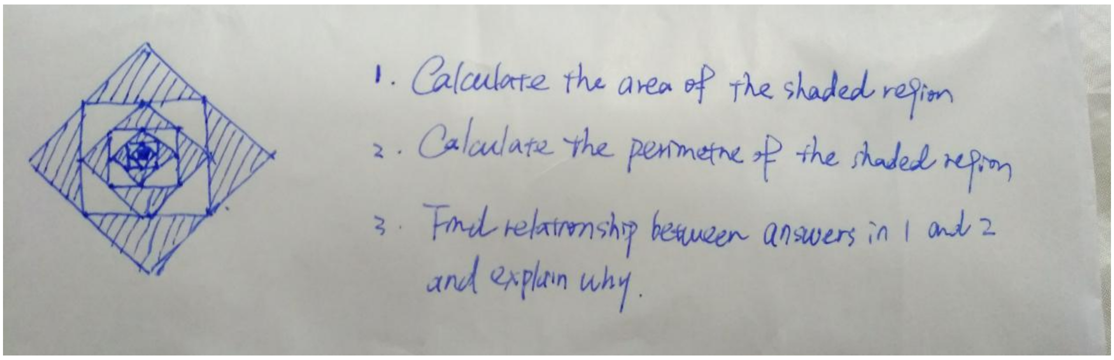
Boundry of $\frac{c}{a+b}+\frac{a}{b+c}+\frac{b}{a+c}$ $2 \geq \quad \geq \frac{3}{2}$

Same as Brasenose

21. Ox St. Cats (2017)



22. Ox Oriel (2017)

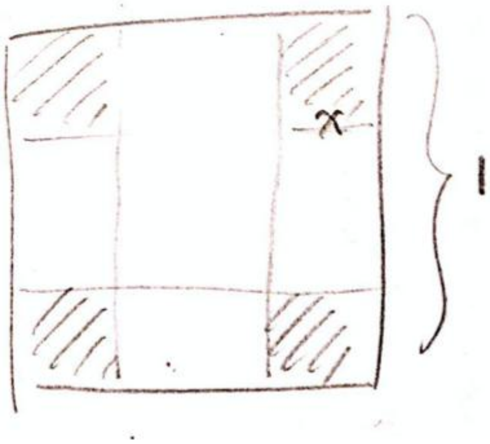


23. Ox St. John’s (2018)
证明三角形两边大于第三边
证明四边形对角线只和小于对边之和
一个面上 n 个点是否总能从一个点出发然后回来连接起来所有点一次不 crossing
是可以，为什么？
从因为对角线小于对边然后距离可以缩短，当做成最短 circuit 就不会有 crossing
为什么总有最短 circuit？
因为是有限的个数的点

24. Ox University (2018)
正方形里最大的半圆 radius 可以是多少？画出 radius 范围的图像（横竖两种），并求出在长方形里一边变化的不同情况。

25. Cam Murray Edwards(2019)
There is a square ABCD (in clockwise order), find the set of points X inside the square such that the total distance from A to X and from C to X is equal to that from B to X and from D to X.

26. Cam Magdalene (2019)



fold into a box
→ find max volume.

27.

Topic 10 Sequence

1. Cam Peterhouse (2016 &2017)

   Prove an infinite real sequence has either a non-increasing or non-decreasing subsequence?

   for an infinite sequence, $x_1, x_2, x_3, x_4, \cdots x_k, x_{k+1}, \cdots$

   either $x_k \le x_{k+1}$ or $x_k \ge x_{k+1}$

   So the subsequence x_k x_{k+1} is non-decreasing or non-increasing

2. Ox Worcester (2016)

   $(1+x)^{2n} = (1+x)^n(1+x)^n$

   i) $\sum_{k=0}^{n} \binom{n}{k}^2 = \sum_{k=0}^{n} \binom{n}{k} \cdot \binom{n}{n-k}$ = RHS coefficient before x^n
   $= \binom{2n}{n}$

   (ii). $\sum_{k=0}^{n} (-1)^k \binom{n}{k}^2$

   $\Rightarrow (1-x)^n(1+x)^n = (1-x^2)^n$

   when n is odd, $=0$ since all the term on $(1-x^2)^n$ has even power.

   when n is even $= \binom{n}{\frac{n}{2}} \cdot (-1)^{\frac{n}{2}}$

   (iii) determine $\sum_{k=1}^{n} k \binom{n}{k}$

   $(1+x)^n = 1 + \binom{n}{1}x + \binom{n}{2}x^2 \cdots \binom{n}{n}x^n$

   $((1+x)^n)' = 1 \cdot \binom{n}{1} + 2\binom{n}{2}x + \cdots n\binom{n}{n}x^{n-1}$

   when $x=1$ RHS $= \sum_{k=1}^{n} k\binom{n}{k}$

3. Cam Pembroke (2017)

   What is the value of $\binom{2017}{1} + \binom{2017}{3} + \cdots + \binom{2017}{2017}$

   考点: binomial theorem

$$\binom{2017}{1}+\binom{2017}{3}+\binom{2017}{5}+\binom{2017}{7}+\cdots+\binom{2017}{2017}$$

$$=\binom{2017}{2016}+\binom{2017}{2014}+\binom{2017}{2012}+\binom{2017}{2010}+\cdots+\binom{2017}{0}$$

$$\because \binom{2017}{0}+\binom{2017}{1}+\binom{2017}{2}+\cdots+\binom{2017}{2017}=2^{2017}$$

$$\therefore \binom{2017}{1}+\binom{2017}{3}+\binom{2017}{5}+\cdots+\binom{2017}{2017}=2^{2016}$$

另解:

考察 pascal's triangle

$$C_{n-1}^{k-1}+C_{n-1}^{k}=C_{n}^{k}$$

$$=C_{2016}^{0}+C_{2016}^{1}+C_{2016}^{2}+C_{2016}^{3}+\cdots+C_{2016}^{2014}+C_{2016}^{2015}+C_{2016}^{2016}$$

$$=2^{2016}$$

4. Cam (2017)

   $1=1$

   $3+5=8$

   $7+9+11=27$

   $13+15+17+19=64$

   求 general law, prove it [《the Stanford maths problem book》原题]

5. Cam Sidney Sussex (2017)

   写出斐波那契数列前 10 项

   写出S_n，找规律

   证明$S_k=a_{k+2}-1$

$a_1=1,\ a_2=1,\ a_3=2,\ a_4=3,\ a_5=5$

$a_6=8,\ a_7=13,\ a_8=21,\ a_9=34,\ a_{10}=55$

方法：use mathematical induction to prove.

when $k=1$, $S_1=1$, $a_3-1=2-1=1$, so $S_1=a_3-1$ is true

assume for $n=k$, it's true that $S_k=a_{k+2}-1$

when $n=k+1$, $S_{k+1}=S_k+a_{k+1}$

$=a_{k+2}-1+a_{k+1}$

$=a_{k+3}-1$

so when $n=k+1$ the statement is also true.

therefore, by mathematical induction,

$S_k=a_{k+2}-1$ is true for all positive integers k.

6. Ox Keble (2017)

   The triangle number is defined as $t(n)=1+2+3+\cdots+n$. Show that:

   a. $t(n)+t(n+1)$ is a perfect square
   b. If 8k+1 is a perfect number, then k is a triangular number.
   c. If k is a triangular number, so is ⋯

7. Ox St. Annes (2019)

   Pascal triangle （explain）为什么横着的和是 2 的次方

   （explain）为什么斜着的和是斐波那契

   （explain）为什么在一行里一项正一项负 加起来的和是 0

Topic 11 Trigometry

1. Ox Keble (2014)

   Prove $(\cos a+1)(\sin a+1)<3$

2. Cam Lucy Cavendish (2019)

   证明$\sin x$不是多项式

Topic 12 Sets

1. Ox Trinity (2017)

   If $f(p+q)=f(p)+f(q)$, $f(tp)=tf(p)$ is true, we call it a linear image (transform)

a. If $f\big((1,0)\big) = (1,1)$, $f\big((0,1)\big) = (1,0)$, what is $f((1,1))$? What is $f((2,2))$?

If $f\big(g(p)\big) = p$ and $g\big(f(p)\big) = p$, then we call g is the inverse image of f

b. If $f\big((a,b)\big) = (a+b,a)$, what is the inverse of f?
c. Try $f((0,0))$, $f((1,0))$, $f((2,0))$, what conclusion you can get geometrically?
d. Show that all inversible linear image with
   i. Linear combination form
   ii. Line to a line

   Can all transfer
e. Show that when we input two parallel lines, the images we get are also parallel
f. Show that when we input a triangle, we also get a triangle as image.

If for all $p, q \in S$, the segment component between p and q also $\in S$, S is called a convex set.

g. Is a straight line convex? Is a triangle convex? Given an example of a non-convex set.
h. Show that is S is convex, the linear image of Sis also convex.

2. Ox St. Cats (2017)
   1 个 set 里面有 2017 个数，使得这 2017 个数的 sum 等于 product，有 finitw 还是 infinite 的情况？证明

3. Ox Pembroke (2017)
   将$\{1,2,3,\cdots,2017\}$这个 set 里任意取几个数组成一个新的集合，含偶数个数的 set 多还是含奇数个数的 set 多，证明

Topic 13 Physics related

1. Ox New (2014)
   Assume that I am on a train that is travelling in a steady speed. If I jump up, where will I fall back to?

2. Ox Balliol (2014)
   Write down an expression for Θ(t)
   Two pendulums, one with frequency that is half of another. When will two pendulum coincide and move towards the same direction. $t = 4\pi$

3. Cam Peterhouse (2017)
4. 一个球，从离地面 h 的距离掉下来，然后弹下去的速度v，但起来的速度v_r,求 total distance travelled.