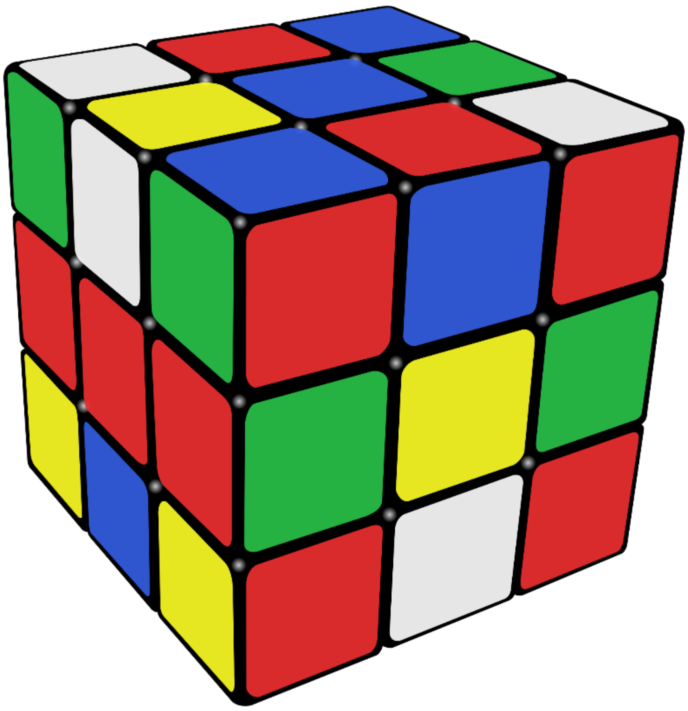
TBO's

Problem Solving Booklet



Outline Solutions

1. The first part is simply 6! as there are no repeated element amongst the digits 1 to 6, the second part is answered by blocking 1, 2 and 3 as a single element and the third part is simply half of the arrangements.
The last part is more difficult and care is needed to ensure certain arrangements are not counted twice.
The sixth digit can be any of 0 to 9 but the cases where the extra digit is 0-5 must be considered carefully.

2. The size of the power set is simply a power of two (the power being the number of elements) as each element is either included or excluded within each subset. The number of subsets containing a given element is the previous power of two as we can imagine forming every possible subset without the given element and then include it in each (alternatively you could sum the binomial coefficients as you are choosing 0, 1, 2, ..., n-1 other elements to form a subset with the given element).

3. Care is needed when n is even and k is half of n because in this case there is only one seat that B can take which satisfies the condition giving a probability of $\frac{1}{n-1}$. In all other cases there are two seats B can take hence the probability is $\frac{2}{n-1}$.

4. Clearly the maximum will be achieved when no three lines intersect at a single point and no two lines are parallel. The nth line drawn adds n regions and we start with one region so the maximum is $1+\frac{1}{2}n(n+1)$. If we consider circles instead it seems that the number of regions is a power of two but this eventually breaks down with four circles meaning that in order to construct a Venn diagram consisting of four of more sets then circles are not up to the task. With one circle we have two regions and at best each new circle drawn intersects each previously drawn circle twice. This means each new circle adds 2, 4, 6, ... regions and hence the total number of regions is given by the quadratic formula n^2-n+2.

5. You can imagine forming a square by lifting a line segment up in to the air, you can imagine creating a cube by lifting a square in to the air and thus you try to imagine lifting a cube in to the fourth dimension to form a tesseract. It should be clear the number of vertices of an n dimensional hypercube will be 2^n as we double each time a new dimension is added. The number of edges will be $n2^{n-1}$ and this result is easily obtained if you consider the number of edges which coincide at a given vertex (it is just the number of dimensions and each edge has two end points). The coordinates of the vertices of the tesseract will be all coordinates consisting of zeroes and ones e.g. (0,0,0,0), (0,0,0,1), through to (1,1,1,1) and the longest diagonal will have length 2 because it would go between the origin and (1,1,1,1).

6. There must be n-1 matches played as at the end of every match one team is sent out of the competition and of course there is only one winner at the end.

7. For the tennis pairings consider player 1, they can be paired with 2n-1 other people. Next relabel the remaining players and consider player 1 again, this new player 1 can be paired with 2n-3 other people. Continue this line of thought and it is clear the number of pairings is the product all odd numbers from 1 to 2n-1. This is not something that can be evaluated quickly and some messing around with factorials and powers of two will lead to the more succinct expression $\frac{(2n)!}{2^n n!}$.

8. A tessellation of the plane using only a regular polygon can only occur if the interior angle of the regular polygon used is a factor of 360 (consider a single vertex where multiple polygons meet, the total interior angles around the point must equal 360) thus the only possibilities are triangles, squares and hexagons. There are only five platonic solids: consider a single vertex of our platonic solid, at least 3 faces meet at this point and the sum of interior angles present must be less than 360 otherwise the shape would flatten out and cease to be a solid. Thus the only platonic solids that can exist are made up of 3, 4 or 5 triangles, 3 squares or 3 pentagons meeting at each vertex. Consider the platonic solids made up of triangles, each face will have three vertices and as already stated 3, 4 or 5 faces will meet at each vertex. We can double count the edges (actually twice the edges) of our platonic solid as follows: 2E=3F=pV where p takes the values 3, 4 or 5. Substituting this in to Euler's formula will yield the desired result.

9. Consider the total of all remaining numbers of the blackboard, it can be shown that the parity (odd or evenness) of this total is invariant under the proposed operation and as it begins as an even quantity then it is impossible to achieve the desired scenario.

10. There are only 2 possible configurations, three blue faces meeting at a point and three blue faces wrapping around the cube joined edge to edge.

11. The first term counts the number of subsets of size r that contain the element 1 and the second counts the number not containing 1, together they must count the total number of subsets of size r. For the second part the first term counts the number of subsets of size r that contain 1 and 2, the second term counts the number of subsets of size r that contain either 1 or 2 but not both and the final term counts the number of subsets of size r that contain neither 1 or 2. In total these must count the number of subsets of size r.

12. Divide the square in to nine smaller squares and apply the pigeon hole principle.

13. Consider the smallest (or equal smallest) integer on the board and call it z, it is surrounded by four other positive integers a, b, c, d and it is their mean. By our choice of z none of a, b, c, d can be smaller than it and thus none can be larger due to the mean condition hence they must all be equal to z.

14. It is easiest to think about the second pack as fixed with a particular order then to look at how many different arrangements (shufflings) of the first pack lead to a success in each part. Of course to have all 52 pairs matching we can only have one arrangement for the first pack (the identical arrangement to pack two) so the probability of this occurring is $\frac{1}{52!}$. 51 matching pairs is impossible because as soon as one pair fails to match so must another. 50 matching pairs is possible and this would result from the two packs initially matching but then two cards being switched around in the first pack. There are 52 choose 2 pairs that can be switched hence the probability of 50 pairs matching is $\frac{1326}{52!}$. For the last two parts it makes things easier to consider the term derangement, this is a permutation of {1, 2, 3, ..., n} in which no element is in its usual place. An example would be {3, 1, 2} whereas {3, 2, 1} would not be as 2 is in its usual place. To have k pairs matching we must have both packs initially matching and then in pack one we must 'derange' k cards. In order to find the relevant probability we need to count derangements which is not an easy task in itself. Let us denote the number of derangements of {1, 2, …, n} as d_n. The probability of k pairs matching is then $\frac{d_{n-k}\binom{52}{k}}{52!}$.

15. Initially it seems the number of regions follows the powers of 2 but the persistent among us will realise the sequence goes 2, 4, 8, 16, 31. With no points we have one region and it is easily seen as points are added each chord drawn between them creates a new region. There are also regions created when chords intersect within the circle. Counting these events is fairly straightforward: there are n choose 2 chords (as each chord has two points on the circumference as its endpoints) and there are n choose 4 internal intersections (as each intersection can be viewed as the intersection of the diagonals of a quadrilateral with its vertices being points on the circumference). Hence there are $1 + \binom{n}{2} + \binom{n}{4}$ regions created by n points on the circumference.

16. Consider a given configuration of n straight lines drawn between the red and blue points, if there is no intersection between the lines we are done and if there is then we must have a quadrilateral with vertices red, blue, blue, red (taken in a clockwise/anticlockwise sense) and the intersection is the intersection of its diagonals. We then switch the lines to go from being the diagonals of the quadrilateral to two of its sides (note that the total length of these new lines is less than the previous by the triangle inequality). Each time we perform one of these swaps, the sum of the lengths of the segments decreases. This means that we will never arrive back at the same set of lines we had. There are n! different ways we could have drawn the segments to join the points together, so we will eventually be unable to perform any more swaps, and will be left at a configuration with no crossings.

17. Consider the cells labelled 1 and n^2, there must exist a path between them made up of no more than n adjacent squares (inclusive of 1 and n^2). If each pair of adjacent squares had values that differed by at most n then the difference between the first and last would be, at most, n(n-1) however the actual difference is n^2-1 which is greater for all n>1. Thus some pair of adjacent squares along this chain has difference at least n+1.

18. If the points were not a single straight line, you could draw different triangles with the points as vertices. Consider the smallest such triangle, that is the triangle with the least area. At least two of the vertices of this triangle have the same colour, this means between them is a point of the different colour. Join this point with the third vertex and you end up with two triangles, each having less area than the previously chosen 'triangle with the least area', hence we reach a contradiction. Thus such a triangle could not have existed in the first place and the points must all lie on a single straight line.

19. Consider the smallest distance between any pair of students. Since this is the smallest distance, the closest student to each of these is the other, so these students throw their ball to each other.

20. Every multiple of nine has a digit sum which is divisible by nine hence the longest chain possible is eight (as any list containing nine consecutive integers must contain a multiple of nine).

21. It is easily seen that the T-tetrominoes can tile a 4x4 grid and if a and b are even then copies of the 4x4 grid can tile the larger 4cx4d grid (where 2c=a and 2d=b). In the case where a and b are odd it is impossible to tile such a grid with T-tetrominoes. This is because an odd number of tiles would be needed and if we imagine colouring the grid like a chessboard we can see each T-tetromino would cover an odd number of white squares (1 or 3 depending on orientation) but there are 2ab white squares in total so such a tiling is not possible.

22. It is possible to answer this question rather elegantly using base three arithmetic but here we will follow a solution relying on the binomial expansion. If we count the number of arrangements with k red counters we would have $\binom{64}{k}2^{64-k}$ and if we evaluate the difference between those where k is even and those where k is odd we should notice that we have the binomial expansion of $(2-1)^{64}=1$. Thus there is one more arrangement containing an even number of red counters.

23. We can quickly check that a cube cannot satisfy the conditions set, for surface area and perimeter to be equal the dimensions of the cube must be 2 and this does not give the correct result for volume. If we equate surface area and perimeter and rearrange we can show that at least one of the dimensions must be larger than two and less than 2. If we equate volume and surface area and rearrange we can show that each dimension must be larger than 2 which yields a contradiction. Hence the volume, surface area and perimeter of a cuboid cannot all be equal.

24. The best approach here is start at the finish and label each square L or W depending on whether it leaves the opponent with a winning or losing move. Quickly a pattern will emerge and the extension questions relies on working modulo 2 and modulo 3 when changing the size of the grid to mxn.

25. Player one has a winning strategy unless there are $n=2^m-1$ sweets in the pile to start. The winning strategy is to try to leave 2^k-1 sweets at the end of your go and this is only possible for one player.

26. Assume that two distinct lattice points lie on a particular circle in the family and by considering arguments of irrationality show they must in fact be the same point. As the circle becomes (continuously) larger it meets one lattice point at a time on its boundary and each lattice point is a fixed distance away from the centre hence all points must be contained in the family of circles given.

27. Let e_n denote the expected value of the number of loops from this process starting with n ropes. We have the following recurrence relation $e_n = e_{n-1} + \frac{1}{2n-1}$. This can be derived as follows: if the process starts with n ropes, after one loose end is selected there are 2n-1 loose ends remaining, giving a probability of $\frac{1}{2n-1}$ that the other end of the same rope will be selected as the second choice. If this occurs, there is one loop already formed and n-1 loose ropes left, so the expected value for the number of loops formed in this case is $1+e_{n-1}$. There is a probability of $\frac{2n-2}{2n-1}$ that an end of a different rope will be chosen, leaving n-1 ropes (1 longer one and n-2 short ones). In this second case the expected value of the number of loops is e_{n-1}. Combining the two cases gives the above result. The recurrence relation can be used to find the required expectation of $\frac{6508}{3465}$.

28. After several trials with small n it is clear the result to be conjectured is that the sum equals n. In fact it does and an induction is easily performed to confirm the initial guess. For the inductive step we realise all subsets in the case where n=k appear when n=k+1 but additional subsets also occur, these consist of the former with the added element k+1 and the set containing just the element k+1. The result follows easily from these observations.

29. Assume the rod has length one and that the pieces have length x, y and 1-x-y. Consider the possible values x and y can take: x, y>0 and x+y<1. If we plot this region in the xy plane we have a triangle of area 0.5. If the rod is broken arbitrarily in to three pieces then all points within the triangle are equally likely. For a triangle to be formed from the three pieces then their lengths must obey the triangle inequality thus each of the three lengths must be less than 0.5. If we plot the three inequalities x<0.5, y<0.5 and 1-x-y<0.5 we get another triangle of area 0.125. Thus the probability a triangle can be formed is $\frac{0.125}{0.5} = 0.25$.

30. The simplest way to justify this claim is to consider the quantity $\binom{m+n-1}{n}$ which we know takes on an integer value for all m, n>0. If we consider using the formula for the binomial coefficient it becomes clear that this quantity is precisely the quotient of n consecutive integers and n! which proves the result.

31. The three lengths a, ar, ar^2 must obey the triangle inequality and as such three inequalities can be formed, two of which yield the required result (the other happens to always be true).

32. Begin by sketching |x|+|y|=100, this is a diamond with vertices at (-100,0), (0,100), (100,0) and (0,-100). We want to count all lattice points on and within this graph as each corresponds to an integer solution to the original equation. If we start at the top vertex (0,100) then there is one lattice point, moving downwards the next line of lattice points has 3 on or within the graph, the next line has 5, ..., and the x-axis has 201 lattice points on or within the graph. Each line above the x-axis is repeated below and so we have 2(1+3+5+...+199)+201=20201 integer solutions to the original equation.

33. Consider the square of an integer mod 3 (that is its remainder when divided by 3) the only possibilities are 0 or 1 but in mod 3 our equation reads $x^2+y^2=0$ which is only possible if x^2 and y^2 are both multiples of three (and hence so are x and y as three is prime). This means there are no solutions to the equation in which x, y and z are coprime. Next assume x and y are both multiples of three, quickly we can see z must also be a multiple of three. Now if all three variables are a multiple of three we can see another solution will be given by $(\frac{x}{3}, \frac{y}{3}, \frac{z}{3})$ but we can repeat this procedure over and over to yield more solutions which are decreasing in magnitude, of course this cannot happen indefinitely as there is a smallest natural number hence there must not have been any solutions to begin with (this method of proof is known as infinite descent). Thus there are no integer solutions.

34. If we rearrange the given equation it is clear the graph will be two lines which intersect at the origin and have gradients $\pm\frac{1}{\sqrt{n}}$. In the case n=9 we have infinitely many pairs of natural number solutions, namely any pair with $x = \pm 3y$. In the case n=10 we have no natural number solutions for if we did it would imply $\sqrt{10}$ were rational.

35. Quickly we can see that the smallest variable a must be less than 3 and as it cannot be 1 it must take the value 2. Then we reduce the problem to finding natural solutions for the equation $\frac{1}{b}+\frac{1}{c}=\frac{1}{2}$ where $b<c$. It can be deduced the next smallest variable b must be less than 4 and greater than 2 hence must take the value 3.
This leads immediately to the value of 6 for c and hence there is only one natural number solution which satisfies the conditions, (2, 3, 6).

36. Consider the first equation mod 4, remembering that squares mod 4 take on the values 0 or 1 only, it becomes $x^2+y^2=3$ which clearly has no solutions. Exactly the same technique can be applied to the second equation, it must be noticed that 33 is 1 mod 4 and then the equation reduces to $x^2+y^2=3$ also.

37. It should be clear as n, x, y, z are all positive integers that $z>x, y$ and from here it is a good idea to divide by n^z. This leaves two unit fractions which sum to one and thus each must be equal to $\frac{1}{2}$. From here all solutions can be obtained, there are infinitely many but they all take the form ($n=2$, x, $y=x$, $z=x+1$) as an example note $2^9+2^9=2^{10}$.

38. The equation can be written as $x^4+x^3+x^2+x^3+x^2+x+x^2+x+1=0$ and from this it can be factored as $(x^2+x+1)^2=0$ and thus has no real solutions.

39. The key thought needed for all three parts is that solutions can only be obtained if each modulus is simultaneously zero. For the first part there are no solutions, for the second we have only one solution $x=\frac{4}{3}$ and finally for the third part we have solutions at every odd multiple of π.

40. Notice that the second set of equations can be written as $a+b+c=3=\frac{ab+bc+ca}{abc}$ and as $abc=1$ this implies a third set of equations: $a+b+c=3=ab+bc+ca$. Next consider the polynomial $f(x)=(x-a)(x-b)(x-c)$ which can be expanded and simplified to $f(x)=x^3-3x^2+3x-1$ (by using the equations involving a, b, c given already) which can be further factored as $f(x)=(x-1)^3$. This polynomial has the single repeated root 1. Obviously the roots of the polynomial are a, b, c by virtue of its construction hence the only real solution to the original equations is (1, 1, 1).

41. The stated solution is obvious but let us focus on another non trivial solution. Let this solution be denoted (x, y, z) and consider the highest power of two which divides x, y and z, call it 2^k where $k\geq 0$. Let us rewrite our solution as $(a2^k, b2^k, c2^k)$ and upon substitution and division by 2^k our equation becomes $a^2+b^2+c^2=2^{k+1}abc$. By construction of a, b and c they cannot all be even and in fact it must be that only one of them is (as the RHS of the last equation is even). If we consider the last equation mod 4 it becomes clear no such solution can exist. Thus the only integer solution to the original equation is (0, 0, 0).

42. It is first worth noting that none of a, b or c can be zero otherwise their reciprocals would not be defined. The second equation implies a third equation, namely $ab+bc+ca=0$. If we consider $(a+b+c)^2=a^2+b^2+c^2+2(ab+bc+ca)$ which in combination with the given equations implies $a^2+b^2+c^2=0$. From this we deduce there are no real number solutions to the original equations as (0,0,0) is not admissible.

43. If we rearrange the given equation to make y the subject and then perform polynomial long division we deduce that $2x-5$ must equal an odd divisor of 6.
This leads to only two positive integer solutions $(x,y)\in\{(3,11),(4,9)\}$.

44. Let the two perpendicular sides have length a and b,
the area then becomes $\frac{ab}{2}$ and the perimeter is given by $a+b+\sqrt{a^2+b^2}$. The stated condition leads to, upon squaring and simplifying, $ab-4a-4b+8$ which can be partially factored as $(a-4)(b-4)=8$. As we only want positive integer solutions this leads to the two triangles with side lengths (5,12,13) and (6,8,10).

45. First note 360=1x2x3x4x5x3 and that the given expression can be factored as (n-2)(n-1)(n)(n+1)(n+2)(n). The expression is clearly the product of 5 consecutive integers (and an additional n) which will certainly be divisible by 1x2x3x4x5 and the extra factor of 3 is accounted for in one of two ways; either n=1 or 2 mod 3 in which case there will be two multiples of 3 among the 5 consecutive factors or n=0 mod 3 and the two factors of 3 are accounted for within the n^2 term.

46. By using the binomial theorem we see that $99^n=(100-1)^n=(-1)^n+100k$ where k is an integer. From this we deduce that for even n the last two digits required will be 01 and for odd n the last two digits required will be 99.

47. It should be intuitively clear that the first expression will eventually grow much larger than the second. To formalise our intuition we could utilise logarithms or simply replace 100 with the slightly larger value of $2^7=128$ and compare exponents.

48. Hopefully most will spot x=2 as a trivial solution and in fact that is the only solution. Consider dividing the given equation by 5^x, this yields the new equation $1 = (\frac{4}{5})^x + (\frac{3}{5})^x$. If x>2 then each term on the right will decrease and hence cannot sum to 1 and if x<2 each term will increase and thus can no longer sum to 1 also. Thus we deduce the only real solution to the given equation is when x=2.

49. For the first part we factor the expression as (n-1)(n+1) and note if n is odd then one of these factors is divisible by 4 and the other 2 hence the expression is divisible by 8. For the next two parts we factor the given expression as $(n-1)(n)(n+1)(n^2+1)$ and note that 30=1x2x3x5. The expression contains the product of 3 consecutive integers so is certainly divisible by 1x2x3 and the factor of 5 is accounted for in one of two ways; if n=0, 1 or 4 mod 5 then on of n-1, n or n+1 is divisible by 5 otherwise n= 2 or 3 mod 5 and in this case n^2+1 is divisible by 5.

50. The terms can be taken in pairs and factored using the difference of two squares which yields the rather simpler sum S(n)=-3+-7+-11+...+-(4n-1)=-n(2n+1). The second part is can be written as S(20)-S(10)=-610.

51. Of course infinitely many counter examples exist but as a specific example consider $x=10^n-1$ where n is a positive integer. We find that $x^2=10^{2n}-2(10^n)+1$=999...9998000...0001 where there are n digit 9's, of course if n is greater than 1001 we have a counter example to the initial statement.

52. The given expression can be factored as $(2^n-1)(2^n+1)$ and as these factors are either side of a power of two then one of them must be a multiple of 3 as required.

53. Let n be a natural number and write it as n=100q+r where q and r are integers and r<100. From the binomial expansion we can see that the last two digits of n^3 will come from the last two digits of r^3. The only way for the last digit of r^3 to be 1 is if the last digit of r is also 1. Now we just need to check the cubes of {11, 21, 31, 41, 51, 51, 61, 71, 81, 91} to see how many end in 11. It turns out only $71^3=357911$ ends with 11 and as 1 in every 100 numbers from 1 to 1000000 end in 71 the probability we are looking for is 0.01.

54. Using the change of base formula for logarithms we can write the first expression as $\log_\pi 2 + \log_\pi 5$.
This can be further simplified to $\log_\pi 10$ which is greater than two by the given inequality. The second expression can be rewritten as $\log_\pi 2 + \log_2 \pi$ and as $\pi<4$ and $\pi^2>8$ we deduce the second expression is also greater than 2.

55. There must be infinitely many prime numbers for if there was not we could devise a finite list of all known prime numbers and this process will yield a contradiction. Construct the integer k which is one more than the product of all primes on our list, now we must have that no prime on our list divides k. We must now
be in one of two cases; k may be prime in which case we have constructed a prime not on our initial list or alternatively k is composite but has a prime factor that is not on our list. Either way it is impossible to construct a finite list containing every prime hence there must be infinitely many primes.

56. This question is similar to the last but is slightly trickier and here we will construct the integer k which is one less than 4 times the product of our finite list of primes of the form 4n+3. Of course it is worth noting k is of the form 4n+3 also and clearly, as before, none of our primes on our finite list divide k. Again we have two cases; k may itself be prime in which case we have missed a prime of the form 4n+3 from our finite list of all such primes or k is composite but has prime factors that are not on our list. The second case is not quite as straight forward as before as perhaps k has prime factors of the form 4n+1 instead, this would not violate our claim that there are finitely many primes of the form 4n+3. If k has only prime factors of the form 4n+1 then it must itself be of the form 4n+1 which is a contradiction and hence it must have at least one prime factor, not on our list and of the form 4n+3. Again we deduce there must be infinitely many primes of the form 4n+3.

57. It must be that $\log_2 3$ is irrational for otherwise assuming it to be rational we reach a contradiction: if $\log_2 3 = \frac{p}{q}$ where p and q are integers then this implies $2^p = 3^q$ which is of course impossible as the LHS is even and the RHS is odd.

58. For n odd the given expression will always be a multiple of 5 and for n even it will always be a multiple of 3, to justify this claim rewrite the expression as $(15-1)^n+11$ and upon expanding with the binomial theorem it should become clear.

59. The given expression can be factored as $(a-1)(a^{n-1}+a^{n-2}+...+a+1)$ but as it must be prime that implies a-1=1 and hence a must be 2. Now if n was composite we could write 2^n-1 as $2^{pq}-1\equiv(2^p)^q-1$ but this would imply 2^p-1 was a factor and due to the primality of the initial expression p could only be 1 and hence n is prime. The second part involves noting that $x^n+1\equiv(x+1)(x^{n-1}-x^{n-2}-...-x+1)$ for n odd. This means for 2^n+1 to be prime we must have that n has no odd factors, in other words n must be a power of two. Be cautious, what we have not said is that 2^n+1 is always prime when n is a power of 2.

60. An obvious solution is p=3 and if other values of p are trialled it seems one of 2p-1 and 2p+1 are always multiples of three. Now no other value of p, except p=3, can be a multiple of three as p must be prime and from this we deduce 2p is also not a multiple of three. If we consider the three consecutive integers 2p-1, 2p and 2p+1 we have that 2p is not a multiple of three so clearly one of 2p-1 and 2p+1 must be. Thus we have found the only solution and that is p=3.

61. Begin by considering a circle of unit radius. To achieve the lower bound we consider the area of the inscribed 12 sided regular polygon and to achieve the upped bound we consider the area of the circumscribed square.

62. The Fibonacci sequence is defined by the recurrence relation $f_{n+2} = f_{n+1} + f_n$ and this will allow us to find the limit of the quotient of consecutive terms in the sequence. We are looking for $L = \lim_{n\to\infty} \frac{f_{n+2}}{f_{n+1}}$ which can be simplified (after using the recurrence relation) to $L = 1 + \frac{1}{L}$. The positive solution to this equation is in fact the golden ratio.

63. A nice trick will simply this problem immediately, notice that cos(90-x)=sin(x) which means cos(89) can be replaced with sin(1) and then we can further simplify $\cos^2(1)+\sin^2(1)$ to 1. Continuing this process we will end up with a final answer of $44+\cos^2(45)=44.5$.

64. If n has N digits then $10^{N-1}<n<10^N$ which implies $10^{2N-2}<n^2<10^{2N}$ and thus n^2 may have 2N-1 or 2N digits. To find the number of digits of any given n we can take its logarithm base 10 and round down to the nearest integer.

65. Consider the discriminant b^2-4ac mod 8, as b is odd then $b^2\equiv1$ mod 8 and as a and c are both odd also this means $4ac\equiv4$ mod 8. This give the discriminant the value 5 mod 8 but all square numbers are 0, 1 or 4 mod 8 so we have that the discriminant cannot be a perfect square. If a quadratic is to have rational roots its discriminant must be a perfect square (think about it for a moment) thus the result is proven, if a, b and c are all odd then the given quadratic equation has no rational roots.

66. The quantity tan(1) is not rational and this can be shown by repeated application of addition formulae tan(A+B). If we first assume tan(1) is rational then tan(2)≡tan(1+1) must also be rational and we keep going until we reach the conclusion tan(30) must also be rational. Of course it is well known that tan(30) is irrational hence we have arrived at a contradiction and we must have that tan(1) was irrational in the first place. A similar argument can be followed to deduce that cos(1) is irrational also but in this case the following formula proves useful; cos(n+1)+cos(n-1)=2cosncos(1), remembering that cos(0)=1.

67. The key to finding the desired sum is utilising the following identity $\sum_1^\infty \frac{1}{r^2} = \sum_1^\infty \frac{1}{(2r-1)^2} + \sum_1^\infty \frac{1}{(2r)^2}$ from which it soon becomes clear the second term on the RHS of the above summation is simply 0.25 of the LHS so the desired sum is $\frac{3}{4}\left(\frac{\pi^2}{6}\right) = \frac{\pi^2}{8}$.

68. If we write the expression within the limit as one fraction we see we have the sum of the squares of the integers 1 to n over n^3 which can be rewritten as $\frac{1}{6}(1)(2+\frac{1}{n})(1+\frac{1}{n})$. It is clear as n grows large that the expression will tend to $\frac{1}{3}$ which is the limit we desire.

69. If we write the expression inside the product as a single fraction it becomes $\frac{n^2-1}{n^2} \equiv \frac{(n-1)(n+1)}{n^2}$ and as we evaluate the product we see a telescoping effect leaving only a numerator of 1 and denominator of 2 behind. Thus the rather complicated looking infinite product has value 0.5.

70. After a few evaluations for low values of n it is easily seen that the expression seems to give the value (n+1)!-1 and a simple induction will confirm this guess.

71. For the first number is should be noted that as its last two digits are 11 then it is three more than a multiple of 8 and unfortunately no square number is three more than a multiple of eight hence it cannot be a perfect square. For the second it should be noted its digits add to 156 which is a multiple of three but not nine and hence (by well-known laws of divisibility) the same is true of the original number. This means it cannot be a perfect square as any square number which is a multiple of three must also be a multiple of nine.

72. As f(x,y)=k(f(x,y)) we can apply this equality twice in a row to obtain f(x,y)=k^2(f(x,y)) and as f(x,y) attains non zero values for some x,y then this implies k=1 or -1. An example where k=1 would be f(x,y)=x+y and an example where k=-1 would be f(x,y)=x-y.

73. The cubic function will have an inverse for all x if it has no maximum or minimum and this will occur if its first derivative cannot be equal to zero for any x. On differentiating the function it is clear this will be when $a \geq 0$.

74. The first inequality must be dealt with carefully, we must consider the cases when x<0 and x>0 separately. When x<0 we can deduce that x<f(x)<0 and when x>0 we can deduce that 0<f(x<x. Now the integral we want to bound can be split into two parts as follows; $I = \int_{-1}^{0} f(x)dx + \int_0^1 f(x)dx$.
In order to achieve a minimum value we can replace f(x) with x in the first part and f(x) with 0 in the second leaving us with $I = \int_{-1}^{0} xdx$ which of course evaluates to our lower bound of $-\frac{1}{2}$. In order to achieve the maximum we replace f(x) with 0 in the first part and f(x) with x in the second leaving us with $I = \int_0^1 xdx$ which of course evaluates to our upper bound of $\frac{1}{2}$. For the second part as x^2 is always positive we deduce that 0<f(x)<x^2 which leads to a lower bound of 0 and upper bound of $\frac{1}{3}$.

75. The notation may look off putting but if we start with small n we can get a feel for what the question is about. Clearly $f_0 \equiv 1$ as $\cos(0)=1$ and $f_1 \equiv x$ as $\cos(\theta) \equiv \cos(\theta)$ and the double angle formula for cosine tells us that $f_2 \equiv 2x^2-1$ as $\cos(2\theta) \equiv 2\cos^2(\theta)-1$. Now to find $\cos(n\theta)$ we will need to utilise the formula $\cos(n\theta+1)+\cos(n\theta-1) \equiv 2\cos(n\theta)\cos(\theta)$ and if we let $\cos(\theta)=x$ and $\cos(n\theta)=f_n(x)$ we reach the desired result. To solve the given equation we rewrite it as $\cos(2\theta)+\cos(3\theta)=0$ and make use of factor formulae it write it as $2\cos(2.5\theta)\cos(0.5\theta)=0$ and from here we just need to find three values of θ which yield a unique value for $\cos(\theta)$ and hence x.

76. For a cubic to have both a local minimum and maximum its first derivative must be equal to zero for two different values of x. If we consider the discriminant of the quadratic obtained by differentiating y we reach the conclusion $b^2-3ac>0$ if the cubic is to have two turning points. Note at this point the x coordinate of the point half way between the two turning points will be $\frac{-b}{3a}$ (this is easily derived by considering the quadratic expression giving the first derivative, the line of symmetry of any quadratic is always half way between the two roots and has equation x=-(half of the x coefficient) in the case of any monic quadratic). A point of inflection occurs on the cubic when the second derivative is zero and by differentiating twice we find that a point of inflection occurs at $x = \frac{-b}{3a}$ thus yielding the required result.

77. If we let y=0 in the given identity we can deduce that f(0)=0. If we let y=-x we can deduce that $f(x) \equiv 0$.

78. For f(x) we have domain x>0 and the range is f(x) can take any real number value, for ff(x) we have domain x>1 (or else the outermost logarithm would be undefined) and range unchanged and for fff(x) we have domain x>e and range again unchanged. For $f^n(x)$ we must have domain $x > {}^{n-1}e$ where the notation ${}^{n}a$ means *n* copies of *a* combined by exponentiation, right-to-left.

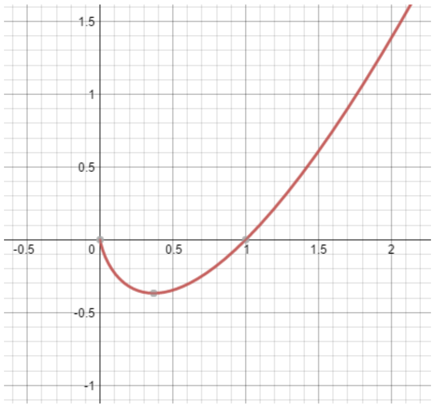
79. Let the continued fraction we wish to find have the value x, it should be clear from the iterative process defining x that $x = 1 + \frac{1}{x}$. From this we can solve for x and it turns out that x the golden ratio (as x is clearly positive we ignore the other root).

80. Upon rearrangement the inequality becomes $e^x(a + \sin(x)) \leq e^y(a + \sin(y)$ and if we want this to be true whenever $x \leq y$ then $f(x) = e^x(a + \sin(x))$ must be an increasing function. Thus we must have that its derivative is always greater than or equal to zero. Upon differentiating and using some trigonometric formulae we find $f'(x) = a + \sqrt{2}\sin(x + \frac{\pi}{4})$ and thus we must have $a \geq \sqrt{2}$ for the original inequality to hold whenever $x \leq y$.

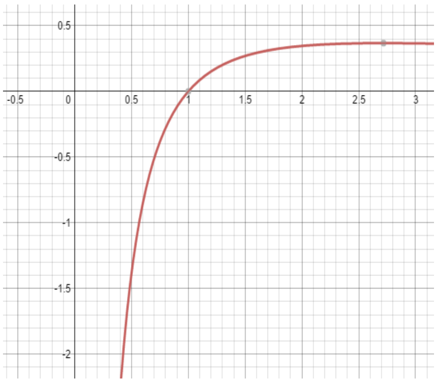
81. To start if we let y=0 then we deduce that f(0)=1. If x=y=1 then we see $f(2)=(f(1))^2$ and if we continue in this manner and let x=2 and y=1 we see that $f(3)=(f(1))^3$. It is not difficult to see that this pattern continues and for n an integer we have that $f(n)=(f(1))^n$. To see that the same result holds for n a rational number we simply write $1 = \frac{q}{q} = \frac{1}{q} + \ldots + \frac{1}{q}$ which leads us to deduce $f(1) = (f\left(\frac{1}{q}\right))^q$ and hence $f\left(\frac{1}{q}\right) = (f(1))^{\frac{1}{q}}$. In turn if we write $\frac{p}{q} = \frac{1}{q} + \ldots + \frac{1}{q}$ we see that $f\left(\frac{p}{q}\right) = (f(1))^{\frac{p}{q}}$ as required. To show that f(n)>0 for all n we write $n = \frac{n}{2} + \frac{n}{2}$ which then implies $f(n) = f\left(\frac{n}{2}\right) f\left(\frac{n}{2}\right) = (f\left(\frac{n}{2}\right))^2$ which is clearly always positive.

82. We first replace $\cos^2(y)$ with $1-\sin^2(y)$ and this reduces the equation to $\sin^2(x)=\sin^2(y)$, so we need to identify where in the plane $\sin(x) = \pm\sin(y)$. There will be infinitely many points and in fact infinitely many lines where these equations hold true. We have the obvious y=x and y=-x but we also have the perhaps less obvious $y = \pi - x$ and $y = \pi + x$. Due to the periodic nature of the sine function there are many more lines but there are obtained by adding on an integer multiple of 2π to one of the four already obtained.

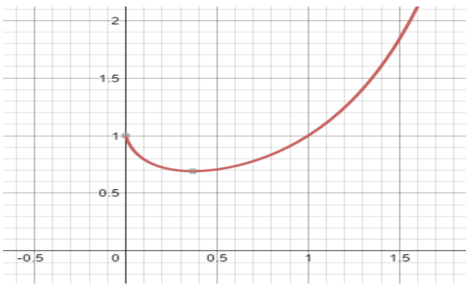
83. For $y = x ln(x)$ the minimum occurs at $(\frac{1}{e}, \frac{-1}{e})$ and the curve does go through the origin.



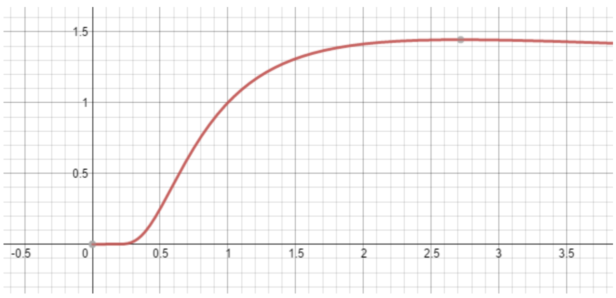
84. For $y = \frac{\ln(x)}{x}$ the maximum occurs at $(e, \frac{1}{e})$ and if we take logs and rearrange the given equation we see it is equivalent to $\frac{\ln(b)}{b} = \frac{\ln(a)}{a}$. From the graph we can deduce the only integer solution is when a=2 and by inspection we see b=4.



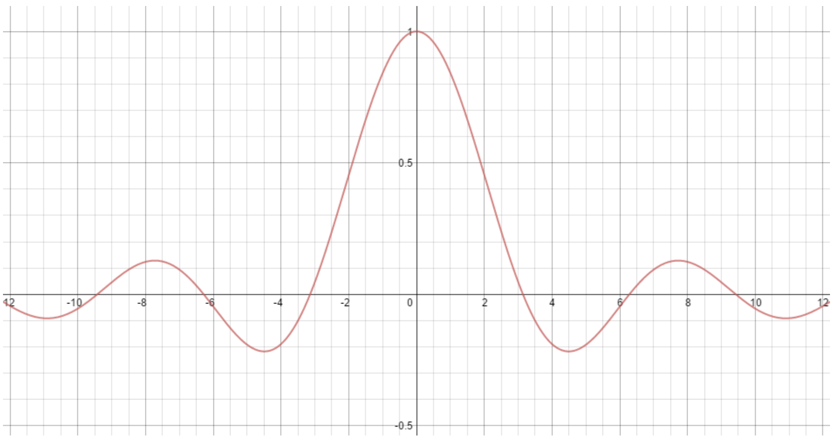
85. For y=x^x the minimum occurs at $(\frac{1}{e}, \frac{1}{e}^{\frac{1}{e}})$



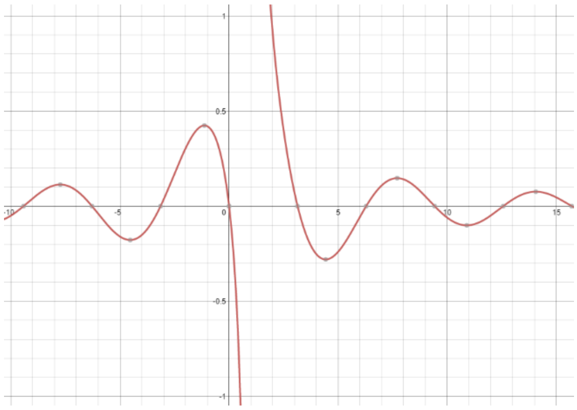
For $y = x^{\frac{1}{x}}$ the maximum occurs at $(e, e^{\frac{1}{e}})$



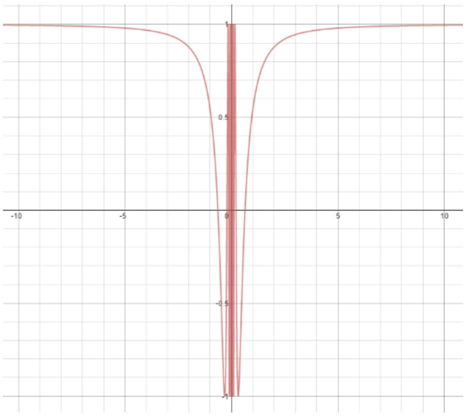
86. For $y = \frac{\sin(x)}{x}$ the curve goes through (0,1) and its maximums (except at x=0) lie on the curve $y = \frac{1}{x}$.

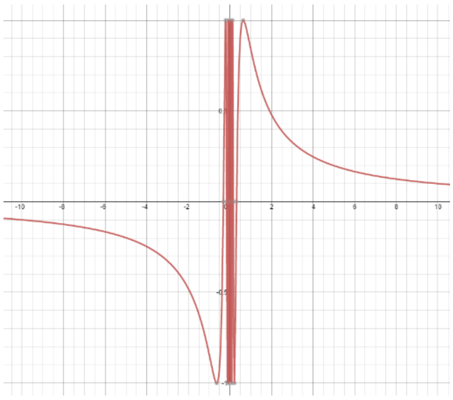


For $y = \frac{\sin(x)}{x-1}$ the graph has an asymptote at x=1.

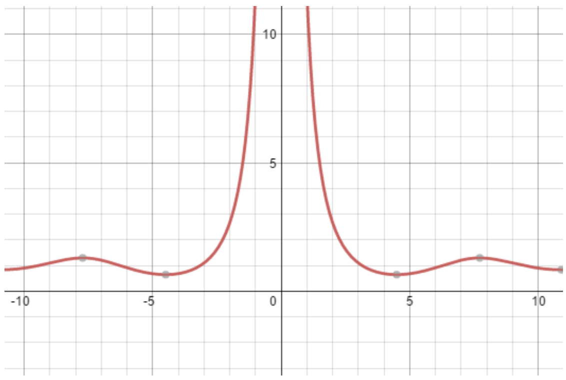


87. $y = \cos(\frac{1}{x})$ $y = \sin(\frac{1}{x})$

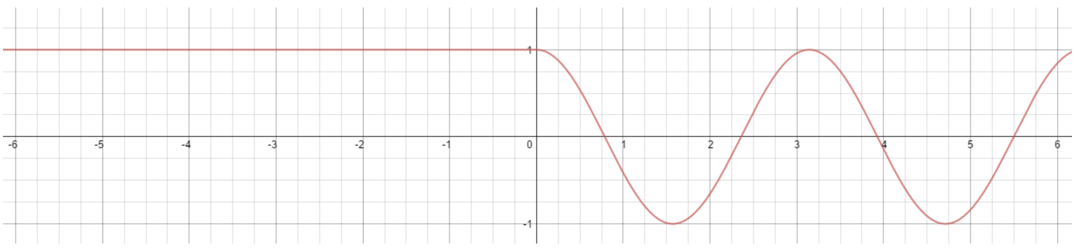




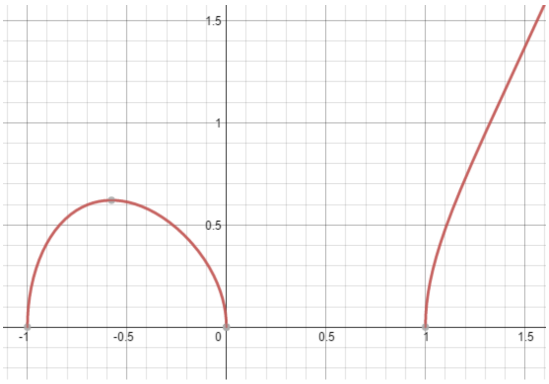
88. For $y = \frac{x+\sin(x)}{x-\sin(x)}$ the graph has an asymptote at x=0 and it oscillates above and below y=1.

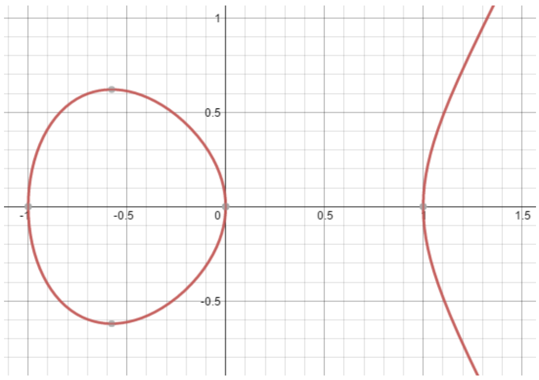


89. For x<0 the graph $y = \cos(x + |x|)$ is simply given by y=cos(0)=1 and for x>0 it is given by y=cos(2x).

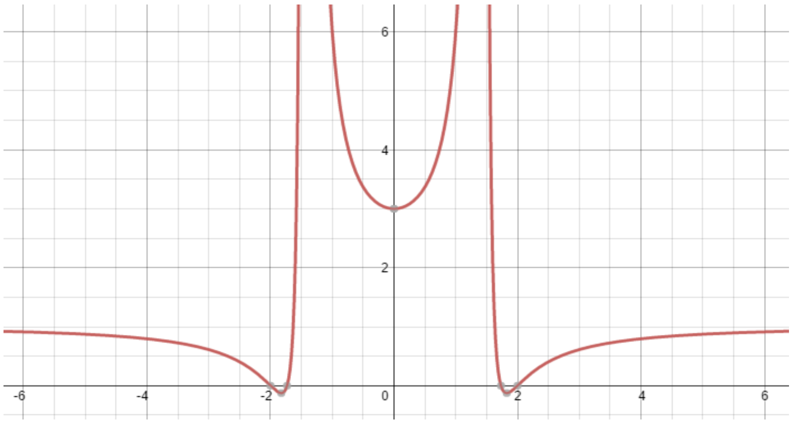


90. $y = \sqrt{x^3 - x}$ $y^2 = x^3 - x$

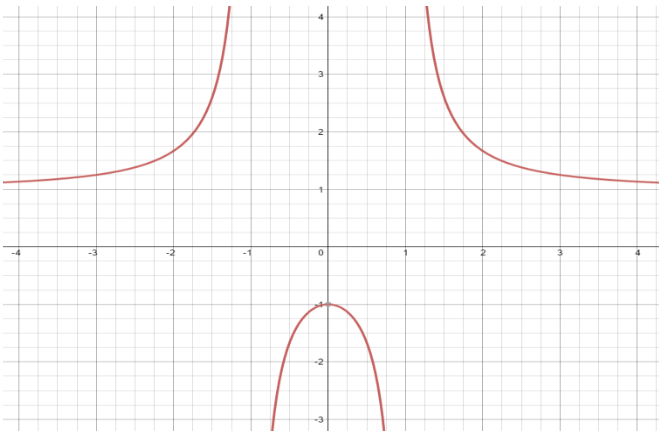




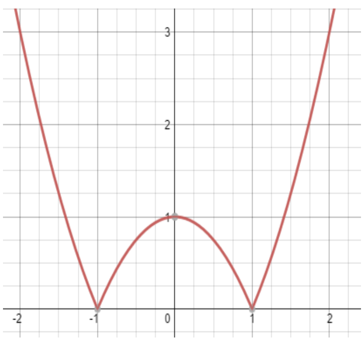
91. For $y = \frac{(x^2-4)(x^2-3)}{(x^2-2)^2}$ the graph has asymptotes at $x = \pm\sqrt{2}$ and y=1.

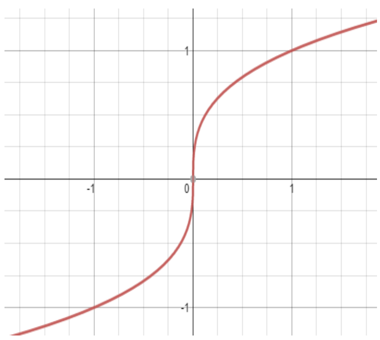


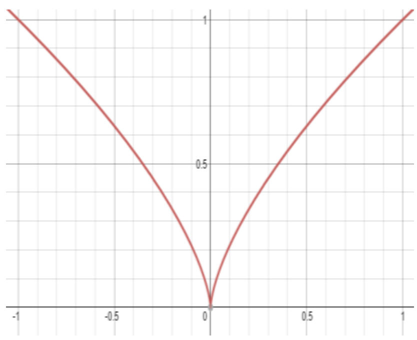
92. For $y = \frac{x^2+1}{x^2-1}$ the graph has asymptotes at $x = \pm 1$ and y=1.



93. y=|x²-1| $y = x^{\frac{1}{3}}$ $y = x^{\frac{2}{3}}$

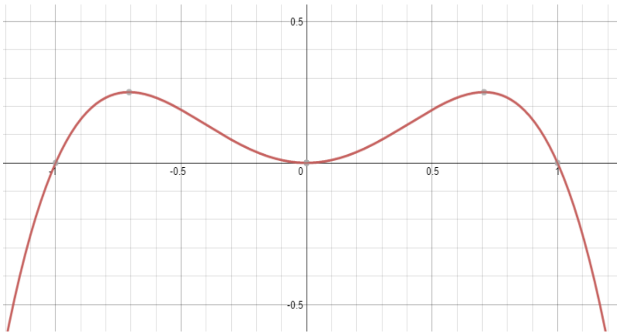




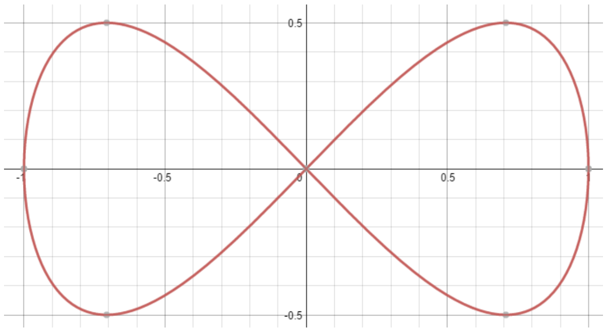


94.

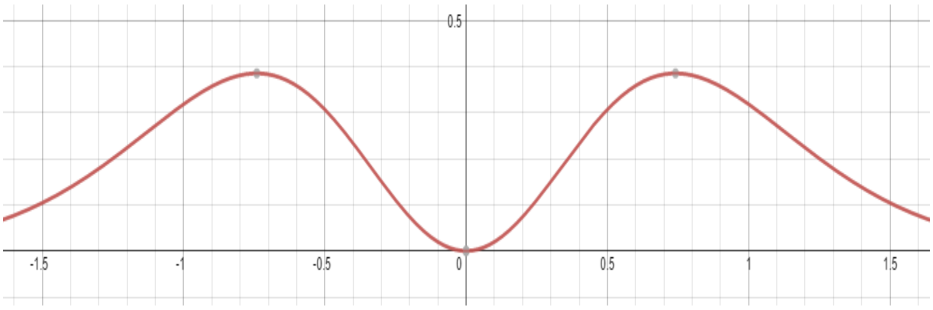
$y=x^2-x^4$



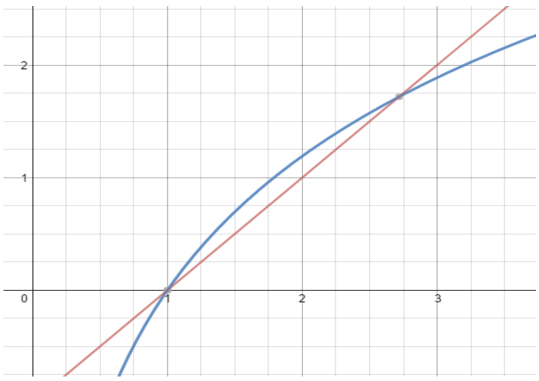
$y^2=x^2-x^4$



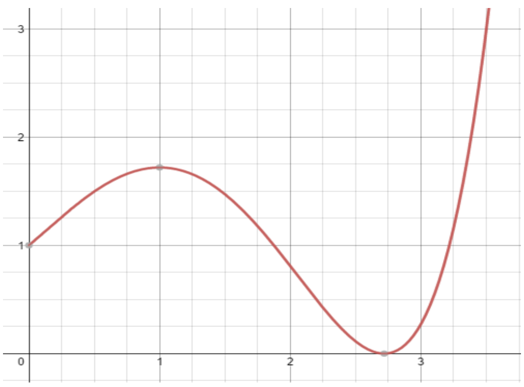
95. For $y = e^{-x^2} - e^{-3x^2}$ the maximums occur at $(\pm\sqrt{\ln(\sqrt{3})}, \frac{2\sqrt{3}}{9})$.



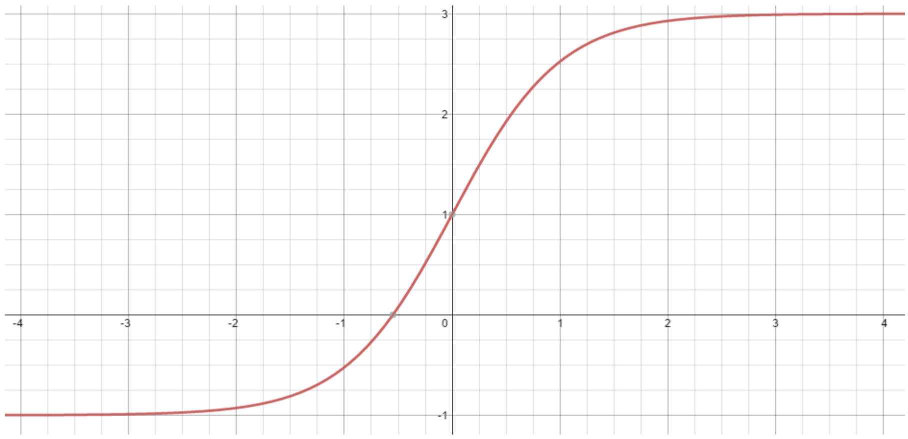
96. By sketching appropriate graphs it is clear there are two solutions to the following equation; $x - 1 = (e - 1)\ln(x)$ and by inspection we see that they are x=1 and x=e.



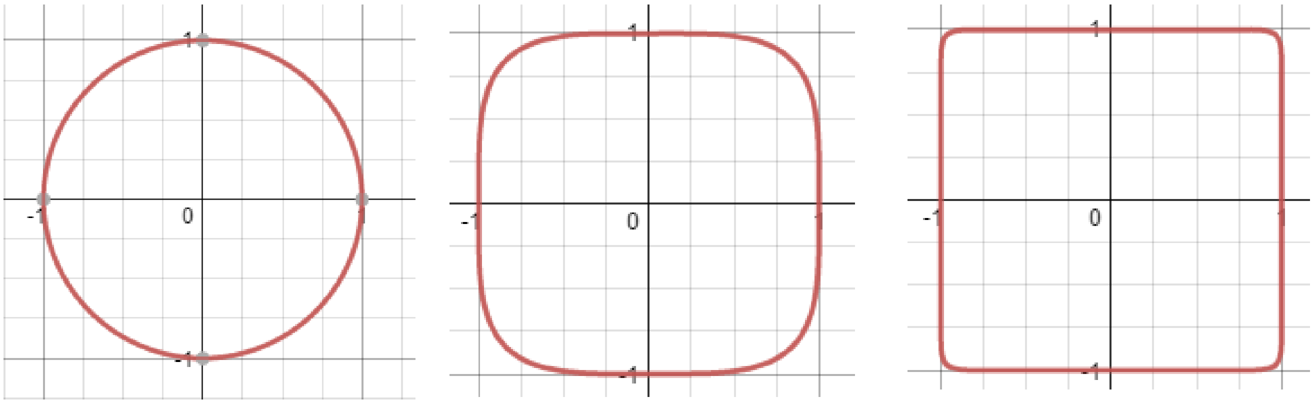
This helps us graph $y = e^x - x^e$ as its turning points occur precisely when $x - 1 = (e - 1)\ln(x)$.



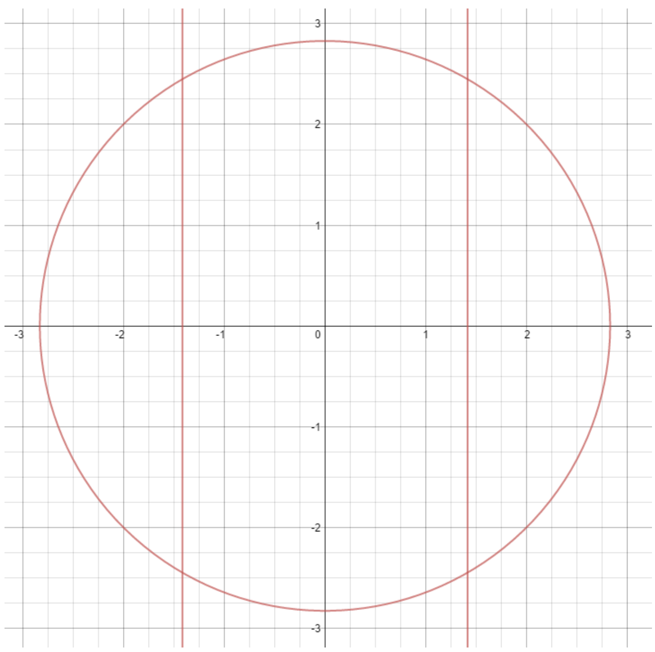
97. The equation for y can be written as $y = 3 - \frac{4}{e^{2x}+1}$ and thus the graph has asymptotes at y=-1 and y=3.



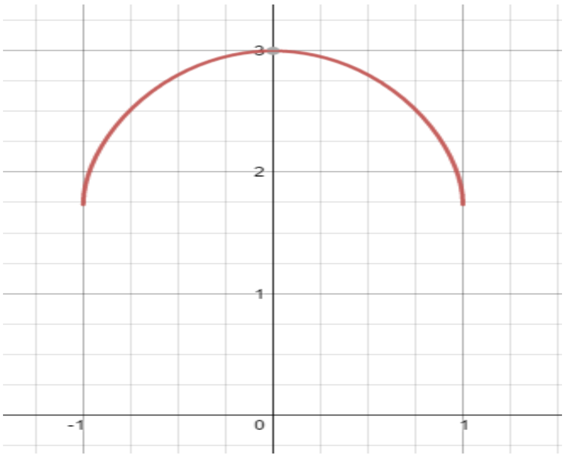
98. n=2 n=4 n=20



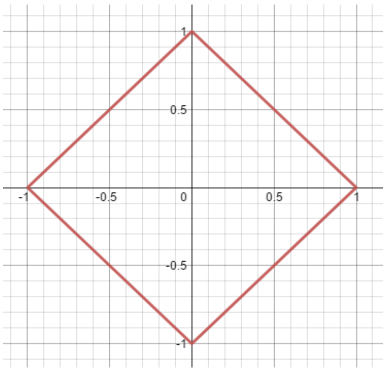
99. If both sides of the equation have the same sign, then we the curve we want it the circle $x^2 + y^2 = 8$ and if they have differing signs the curve we want is $x = \pm\sqrt{2}$.



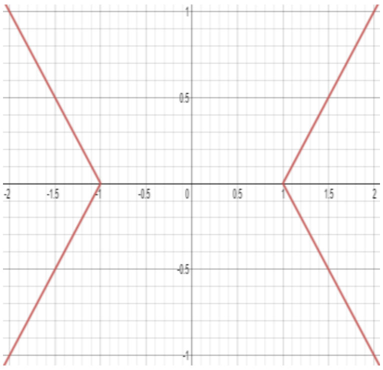
100. $y = \sqrt{1-x^2} + \sqrt{4-x^2}$



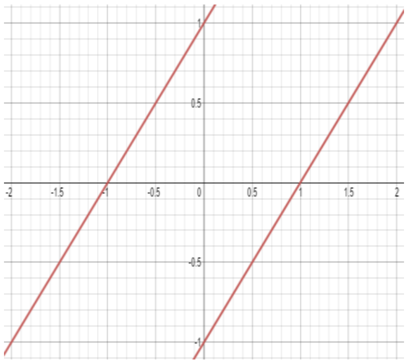
101. 1=|x|+|y|



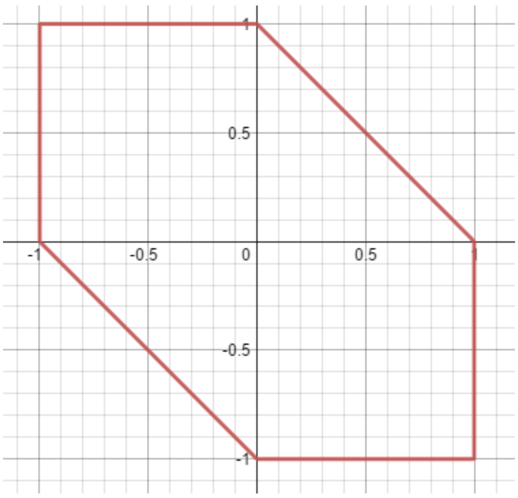
1=|x|-|y|



1=|y-x|



102. First notice if (x,y) satisfies |x|+|y|+|x+y|<2 then so does (-x,-y). In the first quadrant the inequality becomes x+y<1. In the second quadrant we have either y>-x which gives rise to the inequality y<1, or y<-x which gives rise to the inequality x<1. Hence the region we want is the interior of the following quadrilateral:



103. The quantity we wish to minimise is the distance of x from 1, 2, 4 and 6. It should be clear that we can begin at x=0 and move towards x=2 in order to reduce this distance. Once we reach x=2 our function attains the value of 7 and if we move towards 4 we find our function maintains the value of 7. If we move passed x=4 our function begins increasing again and thus the minimum value of the given function is 7.

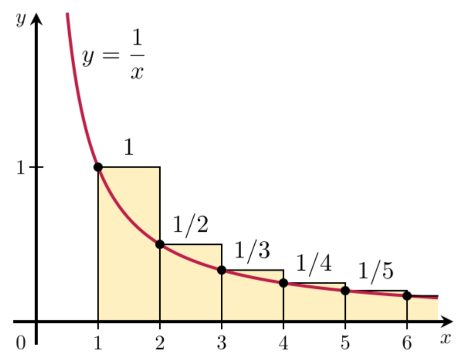
104. The first differential equation yields the solution $y=e^{kt}$ and for x>0 we can see y increases faster and faster as x increases. For the second part we know that when x=0 we have y=1, this tells us that the gradient of the solution curve will be positive. This means the value of y increases and as it does we can consider the new value of the gradient. Again the gradient will be positive so y will continue to increase and in fact whilst y is smaller than our large constant M the gradient will continue to be positive and y will continue to increase. As y gets closer to the value of M the gradient will continue to increase but at a slower and slower rate, from this we deduce that y=M is an asymptote to the solution curve we are looking for. Thus y increases continuously from 1 and tends to M as x gets large.

105. It is worth noting that if F(x) is an anti-derivative of f(x) then $\int_a^b f(x)dx = F(b) - F(a)$ and in particular that $\int_0^x f(t)dt = F(x) - F(0)$ (this is known as The Fundamental Theorem of Calculus Part 1). For this problem we want to find $\frac{d}{dx}(\int_0^x f(t)dt) = \frac{d}{dx}(F(x) - F(0))$ but this is simply $\frac{d}{dx}[F(x)] = f(x)$ as F(0) is a constant and F(x) is an anti-derivative of f(x). Thus by the Fundamental Theorem of Calculus (Part 1) the solution to the given problem is x^8e^x.

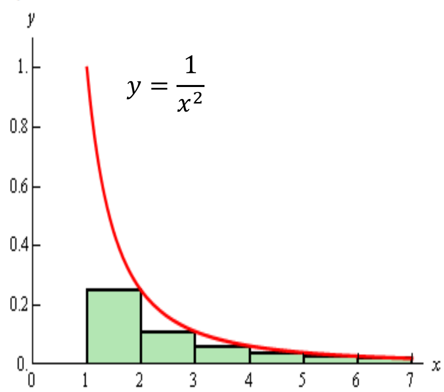
106. We can differentiate both sides of the given equation with respect to x to obtain $\frac{d}{dx}(\int_0^x f(t)dt) = 3f'(x)$ and then using the Fundamental Theorem of Calculus we arrive at $\frac{1}{3} = \frac{f'(x)}{f(x)}$. It is worth noting at this point that if we let x=0 in the original equation we deduce that $f(0) = -\frac{k}{3}$ and combining this with the previous equation we find that $f(x) = -\frac{1}{3}ke^{\frac{1}{3}x}$.

107. All three expressions are found in a similar manner which we will illustrate for i) only: let $y=\sinh^{-1}(x)$ which implies that sinh(y)=x. This can be rewritten as $\frac{e^y - e^{-y}}{2} = x$ which upon rearrangement becomes $e^{2y} - 2xe^y - 1 = 0$. This equation is easily solved for e^y by making use of the quadratic formula and finally by taking logarithms we deduce an expression for $\sinh^{-1}(x)$ (the positive root must be taken as $e^y>0$).
The three expressions we seek are; i) $\ln(x + \sqrt{x^2+1})$ ii) $\ln(x + \sqrt{x^2-1})$ iii) $\frac{1}{2}\ln(\frac{1+x}{1-x})$

108. The series $1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} \ldots$ is divergent, in other words it can be made to be as large as possible by taking more and more terms. This can be shown by considering the area under the curve $y = \frac{1}{x}$ from x=1 onwards. From the graph we see that the rectangles have area equal to our sum and the area under the curve from x=1 onwards is less than this. The curved area is given by $\int_1^t \frac{1}{x}dx = \ln(t)$ as t approaches infinity. From this we deduce that the curved area is infinite hence the required sum is also infinite.



The series $1 + \frac{1}{4} + \frac{1}{9} + \frac{1}{16} + \cdots$ is convergent, in other words the more terms you take the closer the sum gets to a fixed limit. This can be shown by considering the area under the curve $y = \frac{1}{x^2}$ from x=1 onwards. From the graph we see that the rectangles have area equal to one less than our sum and the area under the curve from x=1 onwards is more than this. The curved area is given by $\int_1^t \frac{1}{x^2}dx = 1 - \frac{1}{t}$ as t approaches infinity. From this we deduce that the area under the curve is finite hence the required sum is also finite (as an aside note that it is less than 2). This method of ascertaining convergence is known as the integral test.



109. The inequality is easily obtained by expanding the LHS, using the linearity property of integrals and noting that it must always be positive thus the quadratic (in μ) obtained must have a non-negative discriminant. Upon rearrangement the discriminant will yield the desired result.

The second inequality is derived by letting $f(x) \equiv (1+x^5)^{\frac{1}{2}}$ and $g(x) \equiv 1$ in the Cauchy-Schwarz inequality.

110. If we first find instead the integrals A+B and A-B and then add and subtract these we see that $A = \frac{x-\ln|\sin(x)+\cos(x)|}{2}$ and $B = \frac{x+\ln|\sin(x)+\cos(x)|}{2}$.

111. The first integral is (by inspection) $\frac{1}{5}sin^5(x)$ and the second is found by writing $\cos^3(x) \equiv \cos^2(x)\cos(x)$ and then replacing $\cos^2(x)$ by $1-\sin^2(x)$ which leads to $\frac{1}{7}sin^7(x) - \frac{1}{9}sin^9(x)$. This method will always work if we are integrating a positive integer power of sine multiplied by a positive odd integer power of cosine (note that both must have the same argument).

112. If we divide the top and bottom of the integrand by x^n we can integrate easily to obtain $I = \frac{1}{1-n}\ln|1+x^{1-n}|$ which can manipulated to give $I = \frac{1}{1-n}\ln|\frac{x+x^n}{x^n}|$.

113. Replacing x with a+b-x has the effect of reflecting the function in the line $x = \frac{a+b}{2}$ and hence does not change the value of the integral in question as the reflection preserves the area under the curve (we could also prove the stated result by using a substitution). If we apply the result to I it becomes $I = \int_4^8 \frac{\ln(x-3)}{\ln(9-x)+\ln(x-3)}dx$ which implies that $2I = \int_4^8 1dx$ and hence $I = 2$. If we apply the result to J it becomes $J = \int_0^{\frac{\pi}{2}} \frac{cos^{2000}(\theta)}{sin^{2000}(\theta)+cos^{2000}(\theta)}d\theta$ (as $\sin(90-\theta) \equiv \cos(\theta)$) which implies that $2J = \int_0^{\frac{\pi}{2}} 1d\theta$ and hence $J = \frac{\pi}{4}$.

114. If possible we would like to remove the surds from this integral which guides us towards the substitution $u^6 = x$ and upon this substitution we have that $I = \int_0^1 \frac{6u^5}{u^3+u^2}du$. This integral is easily solved if we first perform polynomial long division and then integrate term by term yielding the result $I = 5 - 6\ln(2)$.

115. This question is similar to question 108 and we will compare the given sum with an integral involving the function $f(x) = x^{-s}$. From a diagram similar to the first in the solution to question 108 it should be clear that the given sum is greater than $\int_1^\infty x^{-s}dx = \frac{1}{s-1}$. From a diagram similar to the second in the solution to question 108 it can be deduced that the given sum is less than $1 + \int_1^\infty x^{-s}dx = 1 + \frac{1}{s-1} = \frac{s}{s-1}$ as required.

116. The first integral I can be found by first multiplying the top and bottom of the fraction in the integrand by 1+sin(x) which leads to $I = \int \frac{1+\sin(x)}{1-sin^2(x)} = \int \frac{1+\sin(x)}{cos^2(x)}dx$. If we then split the integrand into two separate functions we deduce that $I = \tan(x) + \sec(x) + c$.

J can be found by using integration by parts, if parts is used twice the original integral will appear again on the RHS and from this it can be deduced that $J = \frac{e^x\sin(x) - e^x\cos(x)}{2}$.

K can be found by first substituting $u = e^x$, followed by u=tan(z).

This leads to $K = \int \frac{\sec(z)}{\tan(z)}sec^2(z)dz = \int \frac{\sec^2(z)}{\sin(z)}dz$ which after further manipulation gives

$K = \int cosec(z) + \tan(z)\sec(z)\,dz = -\ln|cosec(z) + \cot(z)| + \sec(z)$.

Replacing z yields $K = -\ln|cosec(\arctan(e^x)) + \cot(\arctan(e^x))| + \sec(\arctan(e^x)) + c$.

117. If we consider the functions being integrated in both cases they are clearly positive in the interval (0,1) and for I we have $y = \sqrt[4]{1-x^7}$ and for J we have $y = \sqrt[7]{1-x^4}$. Upon rearrangement we see that I gives us the area under the curve $y^4 + x^7 = 1$ and J gives us the area under the curve $y^7 + x^4 = 1$ but these two functions are inverses of each other (one is derived from the other by interchanging the roles of x and y). The graphs of two inverse functions are related by a reflection in the line y=x and hence the two integrals must be identical as both functions are decreasing and remain non negative on the interval (0,1).

118. We begin by making the substitution x=π-u in to the LHS which gives the integral $\int_{\frac{\pi}{2}}^{0} \frac{-(\pi-u)\sin(\pi-u)}{1+cos^2(\pi-u)} du.$
We can now use some trigonometrical identities to tidy the expression; sin(π-u)≡sin(u) and cos(π-u)=-cos(u) and also note that reversing the limits requires us to multiple the integrand by -1. We have shown that $\int_{\frac{\pi}{2}}^{\pi} \frac{xsin(x)}{1+cos^2(x)}\, dx = \int_0^{\frac{\pi}{2}} \frac{(\pi-x)sin(x)}{1+cos^2(x)}\, dx$ and this allows to write that $I = \int_0^{\frac{\pi}{2}} \frac{xsin(x)}{1+cos^2(x)}\, dx + \int_{\frac{\pi}{2}}^{\pi} \frac{xsin(x)}{1+cos^2(x)}\, dx$ is equivalent to $I = \int_0^{\frac{\pi}{2}} \frac{xsin(x)}{1+cos^2(x)}\, dx + \int_0^{\frac{\pi}{2}} \frac{(\pi-x)sin(x)}{1+cos^2(x)}\, dx = \int_0^{\frac{\pi}{2}} \frac{\pi \sin(x)}{1+cos^2(x)} dx$. This last integral is found by using the substitution v=cos(x) which leads to $I = \int_0^1 \frac{\pi}{1+v^2} dv = \pi(\arctan(v))\,_0^1$ and eventually yields that $I = \frac{\pi^2}{4}$.

119. If we write $\int_0^1 \frac{x^2}{\sqrt{1-x^2}}\, dx = \int_0^1 x \frac{x}{\sqrt{1-x^2}}\, dx$ and apply integration by parts with u=x we yield the desired result. This allows us to write $I = \int_0^1 \frac{x^2}{\sqrt{1-x^2}}\, dx = \int_0^1 \sqrt{1-x^2}\, dx$. The second form can be integrated by making the substitution x=cos(u) and making use of double angle formulae for cosine, eventually leading to $I = \frac{\pi}{4}$.

120. There are two methods that can be used to find the shortest distance from A to the curve $y=x^2+1$ and we will outline both here. Each method requires us first to introduce point $B(t,t^2+1)$ which will always lie on the given curve. The first method utilises the distance between two points formula and allows us to derive an expression for the square of the distance AB. The distance formula is quartic in t and can be minimised by differentiating the expression and setting the derivative to zero. The only real value of t obtained is 1 and hence we find that the point we desire is B(1,2). The second method requires us to note that the shortest distance will a perpendicular distance i.e. the line AB will be perpendicular to the tangent to the curve at B. To justify this imagine a circle centred at A starting with radius zero and slowly increasing, at some point it will touch the curve for the first time and this point of contact will be the point B. If we consider AB as a radius it should be clear it is perpendicular to the tangent to the curve at B due to the tangent-radius circle theorem. This means we can use the gradients of AB and the tangent to the curve at B to find the value of t we desire as perpendicular gradients have product -1. Of course both methods yield t=1 and thus B(1,2) and the distance required is $\sqrt{5}$.

The second problem can be solved by adapting the second method outlined above as the shortest distance from y=x to the curve must be perpendicular to the tangent to the curve at the point of contact and perpendicular to the line y=x. Let us introduce $C(t,t^2+1)$, at C the gradient of the curve must be 1 (the same as the gradient of the line y=x) which after minimal calculus gives us that t=0.5 and hence C(0.5,1.25). Now in order to find the required distance we must find where the normal to the curve at C meets the line y=x. After minimal calculus and coordinate geometry we find the normal meets the line y=x at D(0.875,0.875) which allows us to calculate the shortest distance as $\frac{3\sqrt{2}}{8}$.

The third problem may seem rather complicated as the shortest distance between the curves will lie on a common perpendicular and usually finding such a line is not easy. We are fortunate here because the two curves are inverses to each other and hence the graph of one is just a reflection of the other in the line y=x. This means we have already found our common perpendicular in the previous part and the shortest distance we want is simply twice the distance found in the previous part. Thus the shortest distance between the curves is $\frac{3\sqrt{2}}{4}$.

121. Consider a variable point $C(t,t^2)$ which lies on the curve $y=x^2$, as we have three coordinates we can now calculate the vectors $\overrightarrow{BA}$ and $\overrightarrow{BC}$. We have that $\overrightarrow{BA} = \binom{6}{4}$ and $\overrightarrow{BC} = \binom{t}{t^2+4}$ and remembering that the area of a triangle can be found as half the positive value of the determinant of the matrix formed by two vectors emanating from the same vertex of the triangle and representing its corresponding edges. The matrix thus formed is $M = \begin{pmatrix} 6 & t \\ 4 & t^2+4 \end{pmatrix}$ and we find that its determinant is minimised when t=3 hence the point we are looking for is C(3,9).

122. For the first part we consider one side of the triangle fixed along the x-axis and vary the third vertex. The third vertex will trace out the locus of an ellipse as the perimeter is fixed (total distance of a point on an ellipse from its two foci remains constant). As the area of any triangle is one half of the product of its base and its perpendicular height we deduce that the triangle of maximum area given a fixed perimeter must be isosceles (as this would maximise the height of our triangle within the ellipse). It is easily shown with the use of calculus that among isosceles triangles of fixed perimeter the equilateral triangle has the greatest area.
For the second part consider what happens if we let one side of the right-angle triangle tend to zero, the hypotenuse and other side length will both approach $\frac{P}{2}$. This means if the smallest side has length ε then the area of the triangle will be smaller than $\frac{\varepsilon P}{4}$ which can be made as small as we wish. Thus there is no right angled triangle, with fixed perimeter, of smallest area.

123. We can write $11^{10}-1 \equiv (10+1)^{10}-1$ which after expanding leaves only terms divisible by 100 hence $11^{10}-1$ is divisible by 100 as required. An alternative would be to make use of the following factorisation $x^{n-1}=(x-1)(x^{n-1}+x^{n-2}+...+x+1)$ and note that $11^9+11^8+...+11+1$ has last digit zero and hence is divisible by 10.

124. For any polynomial in x we can find the sum of its coefficients by evaluating its value at x=1 (think about this claim for a moment) hence the sum we want is 1.

125. We wish to find a polynomial with integer coefficients that has $x = \sqrt{2} + \sqrt{3}$ as a root, to help us we first calculate powers of x starting with x^2. We have that $x^2 = 5 + 2\sqrt{6}$ and from this $x^4 = 49 + 20\sqrt{6}$.
From these two expressions we can eliminate the irrational part as $x^4-x^2=-1$ which means that one such polynomial we seek is $x^4-x^2+1=0$.

126. If there is no term in x of odd degree then our polynomial is an even polynomial and $f(-x) \equiv f(x)$. If we replace x with –x in the given expression we find that the first bracket becomes the second and vice versa hence the expression is that of an even function and we deduce the desired result.

127. As the given polynomial is quadratic in x^2 its roots must be of the form $x = \{-b, -a, a, b\}$ and as we seek roots that are in arithmetic progression they must be of the form $x = \{-3a, -a, a, 3a\}$. This means our polynomial can be factored as $(x^2 - a)(x^2 - 9a)$ and upon expanding and comparing coefficients of each power of x with the given expression we find that $10a^2 = 3m + 2$ and $9a^4 = m^2$.
Upon elimination of a we find that $19m^2 - 108m - 36 = 0$ and hence $m = -\frac{6}{19}$ or $m = 6$.

128. If $\frac{p}{q}$ is a root of f(x) then $f\left(\frac{p}{q}\right) = 0$ which tells us that $0 = a_n(\frac{p}{q})^n + a_{n-1}(\frac{p}{q})^{n-1} + \cdots + a_1(\frac{p}{q}) + a_0$. If we multiply the previous equation by q^n we find that $0 = a_n p^n + a_{n-1}p^{n-1}q + \cdots + a_1 pq^{n-1} + a_0 q^n$. As $\frac{p}{q}$ is in lowest terms we know that p and q share no factors and thus as q divides the LHS of the previous equation it must also divide the RHS which implies $q|a_n$. A similar argument allows us to deduce that $p|a_0$. This result can aid us in finding rational roots of polynomials where factorisation may be difficult.

129. The four distinct collinear points can be viewed as the intersection points of the given curve and the straight line y=mx+c. At the points of intersection we have that $mx+c=2x^4+7x^3+3x-5$ which after rearrangement becomes $0=x^4+3.5x^3+(1.5-0.5m)x-2.5$. If we let the four distinct intersection points have x coordinates a, b, c and d then we can write the previous equation as $0=(x-a)(x-b)(x-c)(x-d)$ and by comparing x^3 coefficients in the two equations we find that a+b+c+d=-3.5. This implies the average we seek is -0.875.

130. Consider $f(x, y) = x^n + y^n$, when n is odd we have that $f(x, -x) = 0$ which tells us that $(x + y)$ is a factor of $x^n + y^n$. We can use this result to aid us here as $1^{99}+2^{99}+3^{99}+4^{99}+5^{99}$ can be rewritten as $(1^{99}+4^{99})+(2^{99}+3^{99})+(5^{99})$ where it is now clear each bracketed term is divisible by 5 hence the sum is also divisible by 5 as required.

131. If we express each term of the sum as a fraction with denominator equal to the lowest common multiple of 2, 3, ..., n we find all numerators will be even except for the single term whose original denominator was the highest power of 2 less than or equal to n. Thus the sum of the numerators is odd and the lowest common denominator is even hence the sum cannot be integer valued.

132. Denote the sum of the first n terms S(n). If n is odd we want the sum $S(n) = 0 + 1 + 1 + 2 + 2 + \ldots + \frac{n-1}{2} + \frac{n-1}{2}$ which can be simplified using the formula for the sum of the first k natural numbers: $\frac{1}{2}k(k+1)$.
Thus for n odd we have that $S(n) = \frac{n^2-4}{4}$.
For n even we want $S(n) = 0 + 1 + 1 + 2 + 2 + \ldots + \left(\frac{n}{2} - 1\right) + \left(\frac{n}{2} - 1\right) + \frac{n}{2}$ which simplifies to $S(n) = \frac{n^2}{4}$.
For the second part first note that (s+t)-(s-t)=2t and as the difference is always even s+t and s-t must have the same parity (both be even or bot be odd). This means that for n even we have
$S(s+t) - S(s-t) = \frac{(s+t)^2}{4} - \frac{(s-t)^2}{4} = st$. For n odd we have $S(s+t) - S(s-t) = \frac{(s+t)^2-1}{4} - \frac{(s-t)^2-1}{4} = st$ also.

133. First note that for k>2 we have k!=2x3x4x5x...xk>2^{k-1} which implies $\frac{1}{k!} < \frac{1}{2^{k-1}}$.
Thus $1 + \frac{1}{1!} + \frac{1}{2!} + \cdots + \frac{1}{n!} < 1 + 1 + \frac{1}{2} + \frac{1}{2^2} + \ldots + \frac{1}{2^{n-1}}$ but the RHS of this inequality can be simplified (using the formula for the sum of a geometric series) to yield $1 + \frac{1}{1!} + \frac{1}{2!} + \cdots + \frac{1}{n!} < 1 + \frac{1-(\frac{1}{2})^n}{1-\frac{1}{2}} = 3 - \frac{1}{2^{n-1}} < 3$.

134. The first property tells us that P(x,y)-P(y,x)≡0 and the second property tells us that P(x,y)≡(x-y)Q(x,y). Combining both identities we see (x-y)Q(x,y)-(y-x)Q(y,x)≡0 which can be rewritten as (x-y)(Q(x,y)+Q(y,x))≡0. From this we deduce that Q(x,y)+Q(y,x)≡0 and if we let y=x we find that Q(x,x)=0 and hence (x-y) is a factor of Q(x,y). This implies that $(x-y)^2$ is a factor of P(x,y) as required.

135. The given expression can be manipulated to give $(x-2y)^2 = xy$ and upon expansion and division by y^2 this yields $(\frac{x}{y})^2 - 5\left(\frac{x}{y}\right) + 4 = 0$. The second equation can be factored and gives solutions $\frac{x}{y} = 1$ or $\frac{x}{y} = 4$ but only one of these is a solution to the original problem because if x=y then x-2y<0 and the original equation would be undefined.

136. Let us write a and b in a more convenient form; $a = \frac{10^m-1}{9}$ and $b = 10^m + 5$. Next we calculate an expression for ab+1; $ab + 1 = \frac{(10^m-1)(10^m+5)}{9} + 1$ which can be manipulated to give $ab + 1 = \frac{10^{2m}+4(10^m)+4}{9} = (\frac{10^m+2}{3})^2$. Thus the square root we require is $\frac{10^m+2}{3} = \frac{10^m+2-3+3}{3} = \frac{10^m-1}{3} + 1 = 333 \ldots 3334$ where there are m-1 digit 3's.

137. If we let x denote our choice of base we have 10201≡x^4+2x^2+1≡$(x^2+1)^2$ thus 10201 is composite in all bases.

138. The number of integers less than or equal to 10^{30} that are a power of k is $10^{\frac{30}{k}}$. You may think then that the total we are looking for is $10^{15}+10^{10}+10^6$ but this number is too large because some powers of 2 are also powers of 3 for example. The total we want is $10^{15}+10^{10}+10^6-10^5-10^3-10^2+10^1$ and it is found by applying the inclusion exclusion principle. The 4th, 5th and 6th terms in the sum remove the 6th powers (which are both 2nd powers and 3rd powers and so have been counted twice), 10th powers and 15th powers respectively and the last term counts the number of 30th powers (which have been counted in the first three terms and removed in the next three).

139. Let $x = [x] + e$ where $0 \le e < 1$, so we have $nx = n[x] + ne$ which implies $[nx] = n[x] + (ne)$.
This tells us $\left[\frac{[nx]}{n}\right] = [x] + \left[\frac{[ne]}{n}\right] = [x]$, since $[ne] \le ne < n$. As an aside; the function (z) is often called the floor function.

140. Let us calculate the probability of picking three cards and not seeing a face card; there are 40 cards in the pack which are not face cards so the number of three card hands without a face card is $\binom{40}{3}$ and the total number of three card hands is $\binom{52}{3}$ hence the probability we want is $\frac{\binom{40}{3}}{\binom{52}{3}} \approx 0.44$. This means you are more likely to see face cards in your hand of three cards than not.

141. The given expression can be simplified to $(2\log_3 13)(5\log_{13} 3) = 10\,(\log_3 13)(\log_{13} 3) = 10$ (if the last equality is not clear consider the change of base formula for logarithms in the special case where the argument switches with the base).

142. The following ingenious solution is rather famous: Consider adding an outer shell to the 4x4x4 cube so as to make it a 6x6x6 cube. Imagine splitting it into 216 1x1x1 cubes. Each winning line on the 4x4x4 cube can be extended at each end by a single 1x1x1 cube in to the outer shell. Every winning line will have two unique end points within the shell hence the total number of winning lines must be $\frac{1}{2}(6^3 - 4^3) = 76$.

143. Without loss of generality assume that a<b. First note that the first digit of the product ab will not change if we divide a or b by a power of ten hence we can keep dividing a and b by ten until the quotient of both divisions is less than ten. We shall work with these quotients instead; denote these two numbers c+r and d+s where c and d are single digit integers (the first digits of a and b respectively) and $0 \leq r, s < 1$. Now we have two cases to examine: The first is that (c+r)(d+s)<10 in which case the first digit of the product is at least cd and as $c \geq 1$ then this first digit is greater than or equal to d. In the second case $(c+r)(d+s) \geq 10$ and since (c+r)(d+s)<(c+1)(9+1)=10(c+1) the product is a number between 10 and 90 inclusive whose first digit does not exceed c. Thus it is not possible for the first digit of the product to fall strictly between the first digits of the two numbers.

144. Every time the gambler lost his total was multiplied by 0.5 and every time he won his total was multiplied by 1.5 so if he played 2n games and won half of them his total would be multiplied by $(\frac{1}{2})^n(\frac{3}{2})^n = (\frac{3}{4})^n$.
As the overall multiplier is less than one we deduce that the gambler lost money.

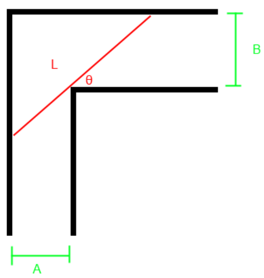
145. Let w_k denote the chance of an individual winning all of their opponent's money given that they currently have £k, we see that $w_k = \frac{1}{2}w_{k-1} + \frac{1}{2}w_{k+1}$ as the next game is either won or lost. This recurrence relation can be rearranged to give $w_{k+1} - w_k = w_k - w_{k-1}$ and as $w_0 = 0$ (if a player has no money they have lost the game hence have a zero chance of winning) we have that $w_2 - w_1 = w_1$. From this we know $w_2 = 2w_1$ and repeated application of the recurrence relation yields $w_k = kw_1$. We also know that $w_{a+b} = 1$ (when one player has £(a+b) they have won the game and hence their chance of winning is certain) from which we deduce $w_1 = \frac{1}{a+b}$ and hence the probability we desire is $w_a = \frac{a}{a+b}$.

146. Replacing x with 1-x in the given functional equation gives $(1-x)^2F(1-x)+F(x)=2(1-x)-(1-x)^4$. Solving this and the given equation simultaneously yields $F(x)=1-x^2$.

147. If we start at the end there are 3 choices for s_n it can be n-2, n-1 or n. If we move on to s_{n-1} we have four choices n-3, n-2, n-1, n but one of these choices has already been taken by s_n thus there are again 3 choices. We can continue in the same way until we reach s_2 for which there are only two choices remaining and thus there will only be one for s_1. Thus the total number of permutations satisfying the given condition is $2(3^{n-2})$.

148. Notice that $\binom{2n}{n} - \binom{2n}{n+1} \equiv \binom{2n}{n} - \frac{n}{n+1}\binom{2n}{n} \equiv \frac{1}{n+1}\binom{2n}{n}$ and as the difference between two binomial coefficients (always integer valued themselves) is always an integer the give expression is also always integer valued. The expression given is the value of the nth Catalan number and the sequence of Catalan numbers arises frequently in the study of Combinatorics (the theory of counting mathematical structures).

149. Using the diagram and some trigonometry we can show that $L(\theta) = asec(\theta) + bcosec(\theta)$. For the ladder to be carried around the corner it must have a length less than the smallest value $L(\theta)$ can take. In a sense we are minimising in order to maximise. If we differentiate $L(\theta)$ with respect to theta, set the derivative to zero and rearrange we find that $\tan(\theta) = (\frac{b}{a})^{\frac{1}{3}}$. Using the Pythagorean identities (or by drawing a suitable triangle) we can find the corresponding values of $\sec(\theta)$ and $cosec(\theta)$.



Upon substituting these values in to $L(\theta)$ we find its minimum value is $(a^{\frac{2}{3}} + b^{\frac{2}{3}})^{\frac{3}{2}}$ and hence this is also the maximum length of ladder that can be carried through the corridors.

150. For each position on the bracelet we have three choices giving 3^{11} arrangements. Unfortunately many of these arrangements lead to the same bracelet due to rotation. The three bracelets made up of a single colour are counted once. For each arrangement that contains at least two different coloured beads we can imagine rotating it by one position which would yield the same bracelet. This means every bracelet containing multiple colours is counted eleven times so there are in fact $\frac{3^{11}-3}{11}$ such bracelets.

In total this gives $3 + \frac{3^{11}-3}{11} = 16107$ bracelets. It is interesting to note two things; the first is that if number of beads on the bracelet is not prime the above method is invalid (can you see why?), the second is that we have provided a combinatorial proof of Fermat's Little Theorem which tells us that $a^p - a$ is always divisible by p where a is a positive whole number and p is prime.

151. The total number of outcomes when rolling a fair dice twelve times is 6^{12}. To calculate the required probability we need to know how many of those outcomes give us two of each number. Any outcome that is a rearrangement of $1, 1, 2, 2, 3, 3, 4, 4, 5, 5, 6, 6$ is what we need and there are $\frac{12!}{(2!)^6}$ of these. This is because there are 12! ways to arrange twelve items but some of the items are repeated so this is too large. If we just consider digit 1s: we can see that in any given arrangement they can be swapped around to yield the same arrangement, this is why we must divide by 2! and of course the same can be said of the other numbers. This leads to a probability of $\frac{12!}{(2!)^6 6^{12}}$.

For the second problem if the sequence is increasing then the outcome of each of the four rolls must be different, so we will first count in how many ways this can occur. There 10 choices for the outcome of the first roll, 9 choices for the second, 8 for the third and 7 for the fourth giving a total of $10\times9\times8\times7 = 5040$. If we have any set of four numbers there are 4! ways to arrange them but only one of these will put them in increasing order. This means the number of desirable outcomes for us is $\frac{5040}{4!} = 210$ and the required probability is $\frac{210}{10000} = 0.021$.