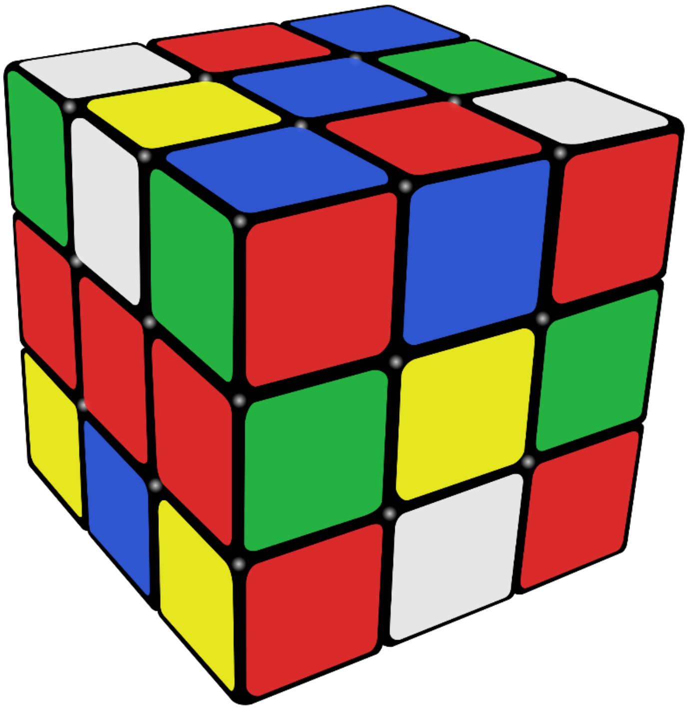
TBO's Problem Solving Booklet



151 questions to help prepare for undergraduate admissions interviews in mathematics.

1. How many ways can you arrange each of the digits 1 to 6 to create distinct 6 digit numbers?
How many of these contain the digits 1, 2 and 3 next to each other and in that order?
In how many arrangements does 5 occur before 1?
How many distinct 6 digit numbers are there in which all of the digits 1 to 5 appear?

2. The Power Set is formed from all subsets of a given set.
If a set contains n elements what is the cardinality of its Power Set?
How many subsets contain a given element x_1?

3. If a round table has n people sitting around it, what is the probability of person A sitting exactly k seats away from person B?

4. How can you maximise the number of regions n straight lines will divide the plane in to?
What is this maximum in terms of n?
What if we replace lines with circles?
What does this tell you about Venn Diagrams?

5. How many vertices and edges does a line segment have? A square? A cube? A tesseract?
Can you conjecture formulae for the number of edges and vertices of an n dimensional hypercube?
Can you give the coordinates of the vertices of a tesseract (where 4 edges coincide with the coordinate axes)?
What would the longest length between two vertices be?

6. In a football competition where every round played is a knockout match (i.e. a draw leading to a replay is not an option), how many matches will be played in the competition in total if there are n teams?

7. In how many ways can 2n opponents be paired in the first round of a tennis competition?
Can you come up with a more succinct expression for your previous solution (i.e. if I gave you a large value for n you could then use your calculator to calculate the answer quickly)?

8. How many distinct tessellations of the plane use only one regular polygon?
Why are there only five platonic solids?
Using Euler's Polyhedron Formula V-E+F=2 show that a platonic solid made of triangles must have 4, 8 or 20 faces.

9. The numbers 1 to 1000 are written on a blackboard. You randomly choose two numbers a and b from among them and replace them with their difference. You continue this process until you are left with a single number on the board, is it possible for you to be left with the number 1?

10. If I colour three faces of a cube red and the other faces blue, how many distinguishable colourings are there?

11. Every subset of the set (1, 2, 3, ..., n) either contains the element 1 or it does not.
By considering these two possibilities, show that

$\binom{n-1}{r-1} + \binom{n-1}{r} = \binom{n}{r}$. Explain why $\binom{n-2}{r-2} + 2\binom{n-2}{r-1} + \binom{n-2}{r} = \binom{n}{r}$.

12. 10 distinct points lie within a unit square, prove at least two of the points lie within $\frac{\sqrt{2}}{3}$ units of each other.

13. Consider an infinite chessboard, the squares of which have been filled with positive integers. Each of these integers is the arithmetic mean of four of its neighbours (above, below, left, right). Show that all the integers are equal to each other.

14. Two full decks of cards are shuffled and placed side by side. I take the top card from each pile and pair them up. What is the probability I have:
   i) 52 matching pairs?
   ii) 51 matching pairs?
   iii) 50 matching pairs?
   iv) 49 matching pairs?
   v) k matching pairs?

15. If n points are distributed around the circumference of a circle and each point is joined to every other point by a chord of the circle (assuming that no three chords intersect at a point inside the circle) in to how many regions is the circle divided?

16. 2n points are chosen in the plane such that no 3 are collinear, n are coloured blue and n are coloured red. Prove that it is always possible to join the n red points to the n blue points by line segments, such that no two line segments cross.

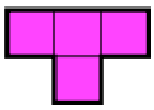
17. For $n>1$, the integers from 1 to n^2 are placed in the cells of an nxn chessboard. Show that there is a pair of horizontally, vertically, or diagonally adjacent cells whose value differs by at least $n+1$.

18. Say you have finitely many red and blue points on a plane with the interesting property: every line segment that joins two points of the same colour contains a point of the other colour. Prove that all the points lie on a single straight line.

19. n students are standing in a field such that the distance between each pair is distinct. Each student is holding a ball, and when the teacher blows a whistle, each student throws their ball to the nearest student. Prove that there is a pair of students that throw their balls to each other.

20. A longevity chain is a sequence of consecutive integers, whose digit sums are never a multiple of 9. What is the longest possible length of a longevity chain?

21. The T-tetromino is the shape made by joining four 1x1 squares edge to edge, as shown. A rectangle R has dimensions (2a)x(2b) where a and b are integers. The expression 'R can be tiled by T' means that R can be covered exactly by copies of T without gaps or overlaps.
   i) Can R be tiled by T when both a and b are even?
   ii) Can R be tiled by T when both a and b are odd?



22. I have a large supply of counters which I place in each of the 1x1 squares of an 8x8 chessboard (1 counter on each square). Each counter is red, white or blue. A particular pattern of coloured counters is called an arrangement. Determine whether there are more arrangements which contain an even number of red counters or more arrangements which contain an odd number of red counters.

23. Prove that it is impossible to have a cuboid for which the volume, the surface area and the perimeter are numerically equal (the perimeter of a cuboid is the sum of the lengths of all its twelve edges).

24. A two player game is played on a 5x5 grid. A token starts in the bottom left corner of the grid. On each turn, a player can move the token one or two units to the right, or to the leftmost square of the above row. The last player who is able to move wins. Determine which positions of the token are winning positions and which are losing. Generalize this problem to larger grids. How many winning positions are there on an mxn grid?

25. Two people play a game: There are n sweets in a pile and they each take it in turns to remove at least one sweet from the pile whilst ensuring they take no more than half of what remains. The person who removes the last sweet is the loser. Are there values of n for which the second player has a winning strategy?

26. Show that the family of concentric circles which have centre $(\frac{1}{3}, \sqrt{2})$ are such that each circle has exactly 1 lattice point on its boundary, and each lattice point is on a circle.

27. There are 6 ropes in a bag. In each step, two rope ends are picked at random, tied together and put back into a bag. The process is repeated until there are no free ends. What is the expected number of loops e_n at the end of the process? (Hint: Find a formula linking e_n and e_{n-1})

28. For each non-empty subset of integers (1, 2, 3, ..., n) consider the reciprocal of the product of the elements. Let S_n denote the sum of these products. Conjecture and prove a formula for S_n.

29. A thin rod is broken into three pieces. What is the probability that a triangle can be formed from the three pieces?

30. Given n consecutive positive integers, show that n! is a factor of their product.

31. The lengths of the sides of a triangle are in geometric progression with common ratio r.
Prove that $\frac{2}{1+\sqrt{5}} < r < \frac{1+\sqrt{5}}{2}$

32. Find the number of integer solutions the equation $|x|+|y| \leq 100$.

33. Are there any integer solutions to the equation $x^2+y^2=3z^2$ where x, y, z are co-prime?
Are there any integer solutions at all?

34. Sketch $x^2-ny^2=0$ where n is a natural number.
Find all natural solution pairs (x,y) in the case n=9.
Find all natural solution pairs (x,y) in the case n=10.

35. How many natural number solutions are there to the equation $\frac{1}{a}+\frac{1}{b}+\frac{1}{c}=1$ where a<b<c?

36. Find all positive integer solutions (x,y) to $x^2+y^2=2015$.
Will the following equation have any positive integer solutions $x^2+33y^2=555555555$?

37. If n, x, y, z are all positive integers, find all solutions of the equation $n^x+n^y=n^z$.

38. Use algebraic techniques to determine whether the following equation has any real solutions: $x^4+2x^3+3x^2+2x+1=0$.

39. How many solutions are there to the equation $|x|+|x-1|=0$?
How many solutions are there to the equation $|27x^2-48|+|6x^2-5x-4|=0$?
What are the solutions of the equation $|\sin(2x)|+|\cos(0.5x)|=0$?

40. If three positive real numbers a, b, c satisfy the following equations show that at least one of them must be 1 and hence deduce all solutions:

$$abc = 1 \qquad \text{and} \qquad a+b+c = 3 = \frac{1}{a}+\frac{1}{b}+\frac{1}{c}$$

41. Prove that the only solution to the equation $x^2+y^2+z^2=2xyz$ for integers x, y, z is x=y=z=0.

42. Show that no three real numbers a, b, c satisfy the equations $a+b+c=0=\frac{1}{a}+\frac{1}{b}+\frac{1}{c}$

43. If x and y are positive integers, find all solutions of the equation $2xy-4x^2+12x-5y=11$.

44. A right angled triangle has all of its sides an integer length.
If the length of the perimeter equals the area, find all such triangles.

45. Prove that $n^2(n^2-1)(n^2-4)$ is divisible by 360 whenever n is a natural number.

46. Find the last two digits of 99^n.

47. Which will be larger as $n\to\infty$; $2^{2^{2^n}}$ or 100^{100^n}?

48. Do you know a solution to the equation $5^x=4^x+3^x$? Are there any more? Prove it.

49. Prove that n^2-1 is divisible by 8 when n is odd.
Prove that n^5-n is divisible by 6 whenever n is a natural number.
Prove that n^5-n is divisible by 30 whenever n is a natural number.

50. Simplify $1^2-2^2+3^2-4^2+\cdots+(2n-1)^2-(2n)^2$.
Find $21^2-22^2+23^2-24^2+\cdots+39^2-40^2$.

51. Construct a counter example to the statement: When written in decimal notation, every square number has at most 1000 digits that are not 0 or 1.

52. Prove that 4^n-1 is divisible by 3 whenever n is a natural number.

53. A natural number from 1 to 1000000 is selected at random, what is the probability its cube ends in 11?

54. Given that $8<\pi^2<10$, show that $\frac{1}{\log_2\pi}+\frac{1}{\log_5\pi}>2$ and $\frac{1}{\log_2\pi}+\frac{1}{\log_\pi 2}>2$

55. Prove that there are infinitely many primes.

56. Prove that there are infinitely many primes of the form 4n+3.

57. Is $\log_2 3$ rational? Prove it.

58. Prove that 14^n+11 is never prime.

59. Let n be a natural number. Suppose a^n-1 is prime. Show that a=2 and that n must be prime (Mersenne Primes).
Comment on primes of the form 2^n+1 (Fermat Numbers).

60. Find all prime numbers p such that 2p-1 and 2p+1 are also prime.

61. Show that $3<\pi<4$.

62. If the ratio of consecutive Fibonacci numbers approaches a limit what must this limit be?

63. Find the exact value of $cos^2(1°)+cos^2(2°)+cos^2(3°)+\cdots+cos^2(89°)$

64. If a natural number n has N digits how many digits can n^2 have? What about n^n?
How would you write a formula for the number of digits of n?

65. Given that $\sum_1^\infty \frac{1}{r^2}=\frac{\pi^2}{6}$ find the exact value of $\sum_1^\infty \frac{1}{(2r-1)^2}$

66. Prove that if a, b, c are all odd then the quadratic equation $ax^2+bx+c=0$ cannot have rational roots.

67. Is $\tan(1°)$ irrational?
What about $\cos(1°)$?

68. Find $\lim_{n\to\infty} \left(\frac{1}{n} + \frac{(n-1)^2}{n^3} + \frac{(n-2)^2}{n^3} + \cdots + \frac{1}{n^3}\right)$

69. Evaluate $\prod_2^\infty \left(1 - \frac{1}{n^2}\right)$

70. Conjecture and prove a formula for 1x1!+2x2!+3x3!+...+nxn!

71. Is 1234567891011 a square number? Is 24681012141618202224?

72. Let f(x,y) be a function of two real variables which is not identically zero. If f(x,y)=k(f(y,x)) for all values of x and y, what are the possible values of k?

73. Let h(x)=x^3+ax, where a is a constant. When will an inverse to h(x) exist for all x?

74. Suppose that $f(0) = 0$ and that, for $x \neq 0$, $0 < \frac{f(x)}{x} < 1$

i) Show that $-\frac{1}{2} < \int_{-1}^{1} f(x)dx < \frac{1}{2}$

ii) How does the above inequality change if $0 < \frac{f(x)}{x^2} < 1$ instead?

75. Show that $\cos(n\theta) = f_n(\cos(\theta))$ for polynomials $f_n(x)$ satisfying $f_{n+1}(x) = 2xf_n(x) - f_{n-1}(x)$.
Find all the roots of $f_2(x) + f_3(x) = 0$, and write them in the form $\cos(\varphi)$ for suitable φ.

76. Consider the cubic curve given by the equation y=ax^3+bx^2+cx+d, find conditions on a, b, c, d which ensure the curve has a local maximum and a local minimum. Under these conditions, show that the curve has a point of inflection midway between the turning points.

77. Find all real valued functions which satisfy $(f(x+y))^2 \equiv (f(x))^2 + (f(y))^2$.

78. What is the domain and range of the functions; f(x)=ln(x), ff(x) and fff(x)?
What about $f^n(x)$?

79. Find the value of $1 + \cfrac{1}{1+\cfrac{1}{1+\cfrac{1}{\cdots}}}$

80. Find the smallest $a > 1$ such that $\frac{a+\sin(x)}{a+\sin(y)} \leq e^{y-x}$ for all $x \leq y$.

81. Let f(x) be a non-constant function satisfying the functional equation f(x+y)=f(x)f(y).
Show that f(n)=k^n for all integers n and for k=f(0).
Show also that the same holds for all rational numbers and that k>0.

82. Where in the plane is $\sin^2 x + \cos^2 y = 1$?

83. Sketch $y = xln(x)$

84. Sketch $y = \frac{\ln(x)}{x}$ and hence find all natural solutions of the equation $a^b = b^a$.

85. Sketch $y = (x)^x$ and $y = (x)^{\frac{1}{x}}$

86. Sketch $y = \frac{\sin(x)}{x}$ and $y = \frac{\sin(x)}{x-1}$

87. Sketch $y = \cos\left(\frac{1}{x}\right)$ and $y = \sin\left(\frac{1}{x}\right)$

88. Sketch $y = \frac{x+\sin(x)}{x-\sin(x)}$

89. Sketch $y = \cos(x + |x|)$ for $-2\pi < x < 2\pi$

90. Sketch $y = \sqrt{x^3 - x}$ and $y^2 = x^3 - x$

91. Sketch $y = \frac{x^4-7x^2+12}{x^4-4x^2+4}$

92. Sketch $y = \frac{x^2+1}{x^2-1}$

93. Sketch $y = |x^2 - 1|$ and comment on the derivative.
Sketch $y = x^{\frac{1}{3}}$ and comment on the derivative.
Sketch $y = x^{\frac{2}{3}}$ and comment on the derivative.

94. Sketch $y = x^2 - x^4$ and $y^2 = x^2 - x^4$ (consider the derivative at the origin carefully)

95. Sketch $y = e^{-x^2} - e^{-3x^2}$

96. By sketching appropriate graphs, find all solutions to the equation $x - 1 = (e - 1)\ln(x)$.
Hence sketch the graph with equation $y = e^x - x^e$.

97. Write $\frac{3e^x-e^{-x}}{e^x+e^{-x}}$ in the form $a + \frac{b}{e^{2x}+1}$ and hence sketch $y = \frac{3e^x-e^{-x}}{e^x+e^{-x}}$.

98. Sketch $x^{2n} + y^{2n} = 1$ for n=2 and 4.
Explain what happens to the graph as $n \to \infty$.

99. Sketch the curve $|3x^2+y^2-12|=|x^2-y^2+4|$.

100. Sketch $y = \sqrt{1 - x^2} + \sqrt{4 - x^2}$

101. Sketch i) $1 = |x| + |y|$ ii) $1 = |x| - |y|$ iii) $1 = |y - x|$

102. What is the area of the region in the Cartesian plane whose points (x,y) satisfy $|x|+|y|+|x+y|<2$?

103. Find the minimum value of the expression $|x - 1| + |x - 2| + |x - 4| + |x - 6|$.

104. Solve the differential equation $\frac{dy}{dx} = ky$ for $x > 0$ subject to the initial condition $x = 0, y = 1$ and k>0.
Sketch the solution to the differential equation $\frac{dy}{dx} = ky(1 - \frac{y}{M})$ where M is a large constant and the same initial conditions apply (without directly finding y).

105. Find $\frac{dy}{dx}$ when $y = \int_0^x t^8 e^t \, dt$.

106. Find $f(x)$ if $\int_0^x f(t) \, dt = 3f(x) + k$, where k is a constant.

107. Find explicit expressions for i) $\sinh^{-1}(x)$ ii) $\cosh^{-1}(x)$ iii) $\tanh^{-1}(x)$

108. Is the series $1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4}$ ... divergent? How about the series $1 + \frac{1}{4} + \frac{1}{9} + \frac{1}{16} + \cdots$?
Sketch on separate axes $y = \frac{1}{x}$ and $y = \frac{1}{x^2}$, considering your sketches and by using integration justify your claims.

109. By considering the inequality $\int_0^t (f(x) + \mu g(x))^2 \, dx \geq 0$, where μ is a constant, prove that, for all functions $f(x)$ and $g(x)$:

$$\left(\int_0^t f(x)g(x)dx\right)^2 \leq \left(\int_0^t (f(x))^2 \, dx\right)\left(\int_0^t (g(x))^2 \, dx\right) \quad \text{(Cauchy –Schwarz Inequality)}$$

Hence show that $\int_0^1 (1 + x^5)^{\frac{1}{2}} dx \leq \sqrt{\frac{7}{6}}$.

110. Let $A = \int \frac{\sin(x)}{\sin(x)+\cos(x)} dx$ and $B = \int \frac{\cos(x)}{\sin(x)+\cos(x)} dx$, find A and B.

111. Integrate $sin^4(x)cos(x)$ and $sin^6(x)cos^3(x)$, in general when will this method work?

112. Evaluate $I = \int \frac{1}{x^n + x} dx$.

113. Prove that for a continuous function $\int_a^b f(x)\, dx = \int_a^b f(a + b - x)\, dx$.
Hence evaluate i) $I = \int_4^8 \frac{\ln(9-x)}{\ln(9-x)+\ln(x-3)} dx$ ii) $J = \int_0^{\frac{\pi}{2}} \frac{sin^{2000}(\theta)}{sin^{2000}(\theta)+cos^{2000}(\theta)} \, d\theta$

114. Evaluate $I = \int_0^1 \frac{1}{\sqrt{x}+\sqrt[3]{x}} \, dx$.

115. By considering the graph of the function $f(x) = x^{-s}$ show that $\frac{1}{s-1} < 1 + 2^{-s} + 3^{-s} + \cdots < \frac{s}{s-1}$ whenever $s > 1$.

116. Evaluate i) $I = \int \frac{1}{1-\sin(x)} \, dx$ ii) $J = \int e^x \sin(x) \, dx$ iii) $K = \int \sqrt{e^{2x} + 1} \, dx$

117. Which of the following numbers is bigger and why; $I = \int_0^1 \sqrt[4]{1-x^7} \, dx$ or $= \int_0^1 \sqrt[7]{1-x^4} \, dx$?

118. Show that $\int_{\frac{\pi}{2}}^{\pi} \frac{xsin(x)}{1+cos^2(x)} \, dx = \int_0^{\frac{\pi}{2}} \frac{(\pi-x)sin(x)}{1+cos^2(x)} \, dx$.
Hence find $I = \int_0^{\pi} \frac{xsin(x)}{1+cos^2(x)} \, dx$.

119. Show that $\int_0^1 \frac{x^2}{\sqrt{1-x^2}} \, dx = \int_0^1 \sqrt{1-x^2} \, dx$.
Hence find $I = \int_0^1 \frac{x^2}{\sqrt{1-x^2}} \, dx$.

120. What is the shortest distance from the point A(3,1) to the curve with equation $y=x^2+1$?
What is the shortest distance from the line $y=x$ and the curve $y=x^2+1$?
What is the shortest distance between the two curves $y=x^2+1$ and $x=y^2+1$?

121. The points A(6,0) and B(0,-4) are points on a triangle, the third point lies on the graph of $y=x^2$, find the co-ordinates of the third point which minimises the area of the triangle.

122. If I have a triangle of fixed perimeter P what will the maximum area be?
Does there exist a right-angled triangle of fixed perimeter P of smallest area?

123. Prove, without directly calculating its value, that $11^{10}-1$ is divisible by 100.

124. Find the sum of the coefficients of the polynomial obtained after expanding and collecting terms of the product $(1-3x+3x^2-5x^3+5x^4)(1+3x-3x^2+5x^3-5x^4)$.

125. Find a polynomial with integer coefficients whose roots include $\sqrt{2} + \sqrt{3}$.

126. Prove that in the product $(1 - x + x^2 - x^3 + \cdots - x^{99} + x^{100})(1 + x + x^2 + \cdots + x^{99} + x^{100})$ after multiplying out and collecting terms, there does not appear a term in x of odd degree.

127. Determine m, an integer, so that the equation $x^4-(3m+2)x^2+m^2=0$ has four real solutions for x that form an arithmetic progression.

128. If the rational quantity $\frac{p}{q}$ (in lowest terms) is a root of $f(x) = a_n x^n + a_{n-1}x^{n-1} + \cdots + a_1 x + a_0$, prove that $p|a_0$ and $q|a_n$. Hence show that the nth root of an integer is either an integer itself or irrational.

129. Show that if four distinct points of the curve $y=2x^4+7x^3+3x-5$ are collinear then their average x-coordinate is some constant k. Find k.

130. Prove that $1^{99}+2^{99}+3^{99}+4^{99}+5^{99}$ is divisible by 5.

131. Show that if n is a positive integer greater than one then $\frac{1}{1}+\frac{1}{2}+\ldots+\frac{1}{n}$ is not an integer.

132. Consider the sequence 0, 1, 1, 2, 2, ..., r, r, r+1, r+1, ..., deduce the sum of the first n terms S(n).
Prove that S(s+t)-S(s-t)=st where s and t are positive integers and s>t.

133. Prove that for n a positive integer $1 + \frac{1}{1!} + \frac{1}{2!} + \cdots + \frac{1}{n!} < 3$.

134. Let P(x,y) be a polynomial in x and y such that: i) P(x,y)≡P(y,x) ii) (x-y) is a factor of P(x,y).
Deduce that $(x-y)^2$ is a factor of P(x,y).

135. If 2log(x-2y)=log(x)+log(y) find $\frac{x}{y}$.

136. Let a be the integer consisting of m digit 1's and b be the integer consisting of a digit 1 at the start, a digit 5 at the end and with m-1 digit 0's in between.
Show that ab+1 is a perfect square and find its square root.

137. Prove that 10201 is composite in any base.

138. How many integers from 1 to 10^{30} inclusive are perfect squares, cubes or fifth powers?

139. Prove that for any positive integer n and any real number x, $\left[\frac{[nx]}{n}\right] = [x]$ where [z] denotes the largest integer value less than or equal to z.

140. If you pick 3 cards from a randomly shuffled pack of cards are you more likely to see a face card than not?

141. Evaluate, without the use of a calculator, $(\log_3 169)(\log_{13} 243)$.

142. A three-dimensional version of noughts and crosses can be played with a 4x4 cube, the winner is the first player to get four noughts (or crosses) in a straight line.
How many winning lines are there?

143. Two positive numbers, a and b, with distinct first digits are multiplied together. Is it possible for the first digit of the product to fall strictly between the first digits of the two numbers?

144. A gambler played a game with his friend, he bet half of his money on the toss of coin; he won on heads and lost on tails. The game was repeated over and over and at the end the gambler had lost as many times as he had won. Did he make money, lose money or break even?

145. Alice and Bob play a fair game repeatedly for £1 a game. If originally Alice has £a and Bob has £b, what is Alice's chance of winning all of Bob's money, assuming that play continues until one person has lost all of his or her money?

146. Determine the function F(x) which satisfies the functional equation $x^2F(x)+F(1-x)=2x-x^4$.

147. There are n! permutations $(s_1, s_2, s_3, ..., s_n)$ of (1, 2, 3, ..., n).
How many of them satisfy $s_k>k-3$ for k=1, 2, 3, ..., n?

148. Is $\frac{1}{n+1}\binom{2n}{n}$ always integer valued when n is a positive integer?

149. If you are faced with a corridor of width m and another corridor of width n, which is perpendicular to the first, what is the maximum length of ladder you can carry through the corridors?
[You may model the ladder as a one-dimensional rod.]

150. A bracelet is made up of a combination of 11 red, yellow or blue beads. How many distinct bracelets can be made if you have at least 11 beads of each colour and if rotations are considered the same but reflections are not?

151. i) A regular fair dice is rolled twelve times, what is the probability of getting two of each number?
ii) A fair ten-sided dice is rolled four times, what is the probability that your sequence of rolls is increasing?