OXBRIDGE INTERVIEW NOTEBOOK

by ECFDPB(CN, JNFLS)

for 2024.12

TOP TIPS

1. you should be confident for your interviews!

2. slow down! be silent and think for 10 seconds once you see a question!

3. will only a fking university deny you!**

WISH THE BEST LUCK BE WITH YOU!

UNIVERSITY OF CAMBRIDGE

UNIVERSITY OF OXFORD

IMPERIAL

知识点总整理

CALCULUS

1. 积分时分母的 polynomial 项高于分子的 polynomial:
① partial?　　② by part/substitution.
③ 通过上下同除x^n后让 LHS=ln|分母|?
④ 如果是三角函数 inverse 相关, 别忘记可以分母配方.
2. 积分时分子的 polynomial 项高于分母的 polynomial: 化出非分数项.
3. 积分时有很多 sin/cos:
① 和差化积?　　② 换成 $\tan\frac{1}{2}x = t$?
③ 上下同乘个 cos/sin/sec/csc 后将式子化成 tan/cot 相关? 可能跟 by part 相关, recall cosx•secx = 1, sinx•cscx = 1. 这个方法尤其当分子上缺少可以 substitution 的东西时有用.
④ substitution? 考虑 $(\sin^2x)' = 2\sin x\cdot\cos x$, $(\cos^2x)' = -2\sin x\cdot\cos x$, $(\sec x)' = \tan x\cdot\sec x$, $(\csc x)' = -\cot x\cdot\csc x$; 如果分子分母都是 sin/cos 考虑 ln|分母|?
⑤ 拆分? $1 = \sin^2x + \cos^2x$, $\sin 2x = 2\sin x\cdot\cos x$, $\sec^2x = 1 + \tan^2x$, $\csc^2x = 1 + \cot^2x$.
4. 困难函数积分时考虑考虑奇偶函数
5. differential equation:
① $\frac{dy}{dx} + f(x)y = g(x)$: $\frac{d}{dx}[e^{\int f(x)dx}y] = e^{\int f(x)dx}g(x)$.　　② $\frac{d^2y}{dx^2} + a\frac{dy}{dx} + by = f(x)$: GS=CF+PI.
③ $\frac{dy}{dx} = f(\frac{y}{x})$: let $u = \frac{y}{x}$, $y = ux$, $\frac{dy}{dx} = u+x\frac{du}{dx}$, so we have $x\frac{du}{dx} = f(u)-u$.
$\frac{dy}{dx} = \frac{ax+by+c}{Ax+By+C}$: (1) let X = x+h, Y = y+k, so we may get $\frac{dY}{dX} = \frac{mX+nY}{MX+NY} = f(\frac{Y}{X})$.
(2) If (1) doesn't work, $\frac{dy}{dx} = \frac{ax+by+c}{\lambda(ax+by)+c}$, let w = ax+by.
④ $\frac{dy}{dx} + f(x)y = g(x)y^n$, n≠0,1: $y^{-n}\frac{dy}{dx} + f(x)y^{1-n} = g(x)$:
let $z = y^{1-n}$, $\frac{dz}{dx} = (1-n)y^n\frac{dy}{dx}$, so we have $\frac{dz}{dx} + (1-n)f(x)z = (1-n)g(x)$.
⑤ $\frac{d^2y}{dx^2} = f(x,\frac{dy}{dx})$: let $p = \frac{dy}{dx}$, so we have $\frac{dp}{dx} = f(x,p)$.
⑥ $\frac{d^2y}{dx^2} = f(y,\frac{dy}{dx})$: let $p = \frac{dy}{dx}$, $\frac{d^2y}{dx^2} = \frac{dp}{dx} = \frac{dp}{dy}\frac{dy}{dx} = p\frac{dp}{dy}$, so we have $p\frac{dp}{dy} = f(y,p)$.
⑦ 勇于凑格式, 求倒数/substitution 简单的 term.
6. 某些求导题要用 definition 去做: $\lim_{h\to0}\frac{f(x+h)-f(x)}{h}$.
7. you may express "x approaches to a" as $x = \lim_{h\to0} a+h$.
8. 求极值问题一般求导即可.

POLYNOMIAL

1. 当看到根进行计算, 考虑根与系数关系. 当看到很规律的$a^i+b^i+c^i+$...时, 可以转化成 polynomial 用根与系数关系求解. 别忘了如果$\sum a_i x^i = 0$, $\sum a_i S_{i+k} = 0$.
2. 有些时候对 polynomial 求 mod n, 然后可能会发现留有一项 m, 那么得到 m mod n 的关系.
3. polynomial 只有有限个零点/stationary point, 如果有无限个零点/stationary point 的话那 polynomial 就是一个常数.
4. 对于复杂的表达式, 先考虑考虑 substitution. 考虑考虑$a^2x^2+a^2y^2=a^2$三角换元? 考虑考虑是奇函数还是偶函数?

5. Remainder Theorem 问题凑格式: assume remainder is $\sum a_i x^i$.
6. 求极限: ① 洛必达.　　　② Taylor expansion.
7. 有些时候将$\sqrt{a^2+b^2}$/$\sqrt{a^2-b^2}$等相关式子看成与直角三角形/圆/向量相关.
8. $m^4 + n^4 = (m^2 + n^2)^2 - 2m^2n^2$.
9. $[x^m - 1] | [x^n - 1]$ when m|n.
10. $a^n - b^n = (a - b)(a^{n-1} + a^{n-2}b + a^{n-3}b^2 + \cdots)$.
11. $a^n + b^n = (a + b)(a^{n-1} - a^{n-2}b + a^{n-3}b^2 - \cdots)$[但是这里 n 必须为奇数].
12. $|f(x)| = \sqrt{f(x)^2}$. 看到 absolute 想到平方解决. 不要人为地换算成 absolute.

GRAPH SKETCHING

1. 画图题第一件事: 观察!
2. domain, asymptotes, approximation, derivatives, stationary points, zero points.
3. domain! 考虑是否存在负半轴.
4. asymptotes! 考虑分子分母是否同时为 0, if yes, then maybe it is not an asymptote.
5. smooth! 一般都是连续的. 尤其考虑$\sqrt{f(x)}$, $f(\sqrt{x})$, $f(x^2)$, $[f(x)]^2$等情况下 y=0 时的斜率. 遇到相关问题一定要先求导: I wonder what will be the tangent of zero points.
6. 对称轴! f(x,y)=f(y,x) 说明 y=x 对称, f(x,y)=f(-x,-y)说明原点对称, f(x,y)=f(-y,-x)说明 y=-x 对称. 有些时候可以靠对称轴和 asymptotes 去大概的画.
7. 当 sinx/cosx 存在且求导困难时, 将其看成摆动, 求完 asymptotes/lower boundary/upper boundary 后大概的去画. 但是注意: tangent point may not be turning point.
8. 考虑考虑 polar form? 考虑考虑将两项画到同一个图中然后找差? 考虑考虑两边增加 constant 或者是恒等式然后简化形式或 change the form?
9. 比大小问题:
① Talyor 展开?　　② 特殊点大小+导数大小.　　③ 相减/相除。
④ 当看到两项比大小且左右两边有相同元素, 尝试把所有相同元素归类到一边后考虑同一个 function 在不同 x 值下的 y 的大小.

NUMBER THEORY

1. $a^2 \equiv 0$ or 1 (mod 4); $a^3 \equiv 0$ or 1 or -1 (mod 9).
2. cannot be integer = numerator is not divisible by denominator.
3. "consecutive integers""prime"之类的词都是在引诱你去用数论.
4. n 进制的小数可以用来表示$\frac{m}{n^k-1}$: 先找$\frac{1}{n^k-1} = n^{-k} + n^{-2k} + n^{-3k} + \ldots$, 随后乘 m 即可
5. parity: 奇偶性

SERIES

1. 记得说: assume convergence.
2. 巧用积分/求导进行简便计算.
3. 对于有规律的数列来说, 大部分都可以用错位相减计算.
4. 对于很多项相加/乘, 考虑先将几项组合起来.
5. $F_n = \frac{1}{\sqrt{5}}[(\frac{1+\sqrt{5}}{2})^n - (\frac{1-\sqrt{5}}{2})^n]$.

TRIGONOMETRIC

1. $\cos x = \frac{e^x + e^{-x}}{2}$, $\sin x = \frac{e^x - e^{-x}}{2i}$.

2. 1 - tanx•tan(45-x) = tanx + tan(45-x); tanx•tan(90-x) = 1.
3. arctan/arcsin/arccos 求角度和:
① trigonometric 角度和公式.
② De Moivre’s Theorem: arg(xy)=arg(x)+arg(y), 将角度化成 complex 之后直接计算.
4. 判断是否周期性: f(x) = f(x+T)时, T 是否为常数.
5. 看到π相关与π不相关比大小: ① function? ② geometry meaning: arc > chord?

MECHANICS

1. change in total energy = work done. 考虑有没有外力做工.
2. without resultant force and resultant torque(moment), a system is in equilibrium.
3. 受力分析: 重力, 外力, 弹力(支持力), 摩擦力.
4. upthrust numerically equals to the gravity of water displaced by object.
5. collide: sum of momentum unchanged: $m_1u_1 + m_2u_2 = m_1v_1 + m_2v_2$.

   sum of kinetic energy may change: $\frac{1}{2}m_1u_1^2 + \frac{1}{2}m_2u_2^2 \geq \frac{1}{2}m_1v_1^2 + \frac{1}{2}m_2v_2^2$.

   coefficient of restitution: $\frac{v_2 - v_1}{u_1 - u_2}$.
6. 整体隔离列式子: 绳子之间力不变.
7. moment: 力矩(Fd). momentum: 动量(mv). coefficient of restitution: 碰撞系数.
   viscous force: 粘滞力. coefficient of friction: 摩擦系数.
   gravity: 重力. normal force: 支持力. upthrust: 浮力(ρgV). density: 密度.

PROBABILITY

1. 概率计算方法.

① 逐分法:

分成 n 组, 每组N_i人 → 一步一步来, 先分什么再分什么.

[注意考虑第一步操作要不要算重复, 最左最右固定则无需考虑;
A 或 B 只能去一个问题, 想清楚在当前情况下, 有多少人可能去, 再要多少人]

② 隔板法:

一共 n 人分成 m 组 → n-1 个空需要 m-1 个板子.

不相邻问题, n 组不能相邻的元素与 m 组可以相邻的其余元素
→ 可以相邻的其余元素先排,剩下 n 组有序插 m+1 的空.

左右都有空位问题: 空位中间插人.

[考虑一共有多少个空? 左右固定类空有多少个? 有没有必要插空? 为什么不能插空?]

注意: n 个数据分成 k 组时:

板子之间不为空时看空和板子$\binom{n-1}{k-1}$, 板子之间可以为空时看整体$\binom{n+k-1}{k-1}$.

路径问题: 向上 m 步,向右 n 步, 看成整体用 $\binom{m+n}{m}$.

③ 捆绑法:

要相邻的元素看成一个元素.

[注意要不要考虑内部分类, (AB)C&(BA)C; 还要考虑重复, (EE)E&E(EE)]

④ 先分堆后排序:

对于题目提出‘至多’‘至少’类问题, 先分出保底元素后再依次加码看不同组合.

找到不同种组合后再考虑放要依次把组合放到盒子里的问题.

[考虑有没有重复? 题目有没有说盒子不一样?]

⑤ 排除法:
快速地通过大除小出结果, 一般分类讨论太复杂的时候就用它.
⑥ 染色问题:
跳格分类法, 即每次涂颜色都跳一格; 做这种题目时画树状图会好看一些.
⑦ 图像法:
将大小于关系化成解析式画在坐标轴上求面积占多少百分比.
2. 对于很多次重复的或雷同的操作, 大部分都可以用 **induction/recurrence relation** P_n=f(P_{n-1},P_{n-2})解决, 建议写出文字含义之后去找对应概率. **general function** 可用 **diagonalised matrix** 求出.
3. 容斥原理[**inclusion and exclusion principle]:** 不重复地求全部情况.
4. use Binomial Expansion to calculate probability.
5. symmetry will be useful.

PROOF

1. 证明 **statement:** 正常推导/**contradiction/induction[**尤其当存在 **n** 或者数很大的时候有用**]**
2. "find out something" can be "never exist"

经典积分题

1. $\int \frac{1}{x+x^n} dx$

基本逻辑: 想 substitution, 那就要让分子上有东西, 或者是消去分母的一些项

LHS = $\int \frac{x^{-n}}{x^{-n+1}+1} dx$

2. $\int \frac{1}{1+\sin x} dx$

基本逻辑: 想 substitution, 那就要让分子上有东西, 或者是消去分母的一些项

LHS = $\int \frac{1-\sin x}{(1+\sin x)(1-\sin x)} dx$

3. $\int \frac{1}{\sin^2 x+2\cos^2 x} dx$

基本逻辑: 想 substitution, 那就要让分子上有东西, 或者是消去分母的一些项

LHS = $\int \frac{\sec^2 x}{(\sin^2 x+2\cos^2 x)\sec^2 x} dx = \int \frac{\sec^2 x}{\tan^2 x+2} dx$

4. $\int \sin 3x \sin 5x \, dx$

基本逻辑: 和差化积

5. $\int \frac{\sin x\cos x}{\sin^4 x+\cos^4 x} dx$

基本逻辑: $(\sin^2 x)' = 2\sin x\cdot\cos x$, if $u = \sin^2 x$:

LHS = $\frac{1}{2}\int \frac{1}{u^2+(1-u)^2} du = \frac{1}{2}\int \frac{1}{2u^2-2u+1} du$

基本逻辑: 注意如果分母是 quadratic 尝试配方

LHS = $\frac{1}{2}\int \frac{1}{(2u^2-2u+\frac{1}{2})+\frac{1}{2}} du = \frac{1}{2}\int \frac{1}{2(u-\frac{1}{2})^2+\frac{1}{2}} du$

6. $\int \sqrt{e^{2x}+1}\, dx$

基本逻辑: $\sqrt{a^2+1}$考虑用 a=sinhx 或者是 a=tanx

if $e^x = \sinh u$, $e^x \frac{dx}{du} = \cosh u$, $dx = \frac{\cosh u}{\sinh u} du$

LHS = $\int \frac{\cosh^2 u}{\sinh u} du = \int \frac{1+\sinh^2 u}{\sinh u} du = \int \operatorname{csch} u + \sinh u \, du$

7. $\int \frac{1+\sin x}{\sin x(1+\cos x)} dx$

基本逻辑: 如果一堆 sinx/cosx 尝试万能公式

$t = \tan\frac{x}{2}$, $\frac{dt}{dx} = \frac{1}{2}\sec^2\frac{x}{2} = \frac{1+t^2}{2}$, $dx = \frac{2}{1+t^2}dt$

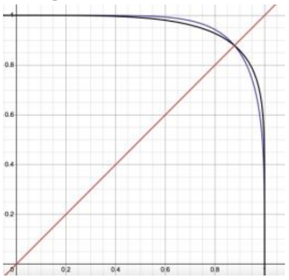
LHS = $\int \frac{1+2t+t^2}{2t} dt$

8. which one is larger, $\int_0^1 \sqrt[4]{1-x^7}\,dx$ or $\int_0^1 \sqrt[7]{1-x^4}\,dx$?

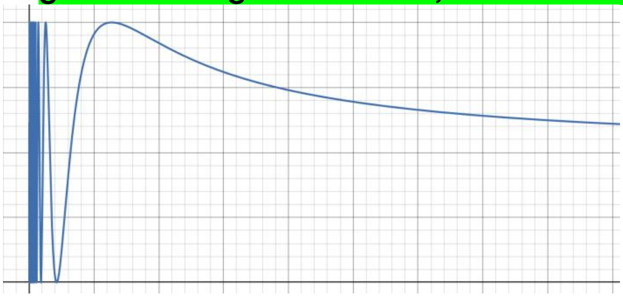
基本逻辑: 画图对比积分

左边的为: $y = \sqrt[4]{1-x^7}$, $x^7 + y^4 = 1$. 同理得到右边的为: $x^4 + y^7 = 1$

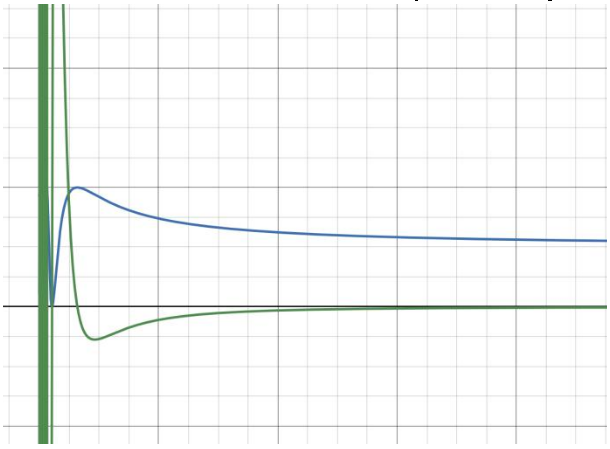
沿 y=x 对称, 面积相等



9. **given the original function, sketch its derivative on the same graph**



基本逻辑: 找 turning point, 选取几个点比较斜率, smooth
注意: 当Δy 相同时, Δx 越小, |gradient|越大



经典画图题

1. $\frac{\sin x}{x}$

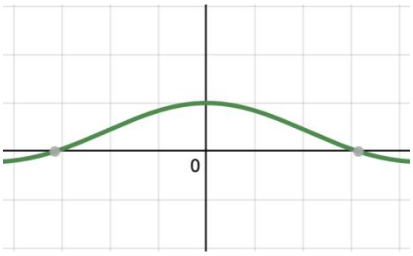
① domain/asymptotes/approximation:

当 $x \to 0$, 分子分母同时趋近于 0, 因此不一定有 asymptotes. 当 $x \to 0$ 时, $\frac{\sin x}{x} \to 1$

② consider sinx as oscillation

③ derivatives/stationary point/zero point:

求导之后得到, when stationary point, x = tanx, 由此可知从 x=0 到第一个零点没有拐点

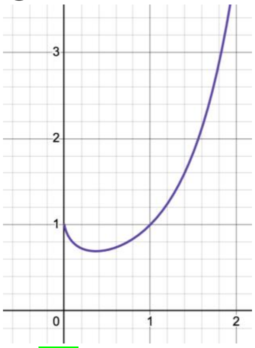


2. x^x

① domain: 指数有 x, 没有负半轴

② approximation: $x \to 0$, $x\ln x = \frac{\ln x}{\frac{1}{x}} \to \frac{\frac{1}{x}}{-\frac{1}{x^2}} \to 0$, $y \to 1$

③ derivatives: 分类讨论 $x<e^{-1}$ and $x>e^{-1}$即可

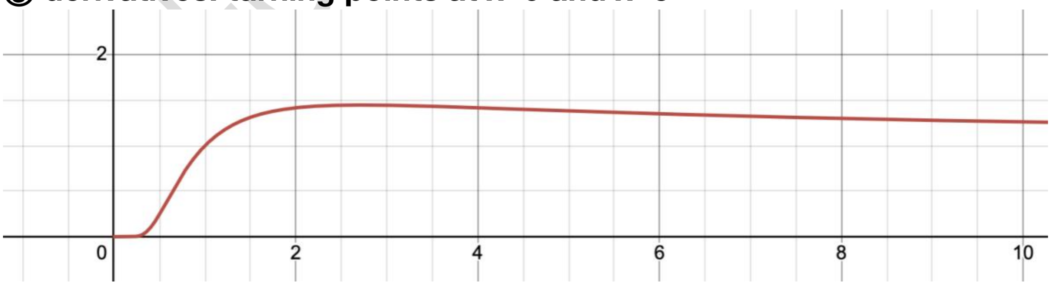


3. $x^{\frac{1}{x}}$

① domain: 指数有 x, 没有负半轴

② asymptotes/approximation: $x \to 0$, $\frac{1}{x}\ln x \to \infty \cdot (-\infty) \to -\infty$, $y \to 0$; $x \to \infty$, $\frac{\ln x}{x} \to \frac{\frac{1}{x}}{1} \to 0$, $y \to 1$

③ derivatives: turning points at x=0 and x=e



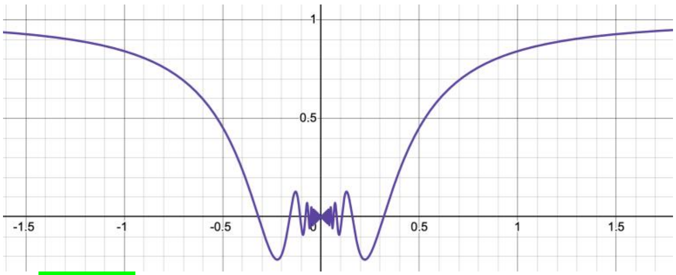
4. $x\sin\frac{1}{x}$

① asymptotes/approximation: 当 $x \to \infty$, $x\sin\frac{1}{x} = \frac{\sin\frac{1}{x}}{\frac{1}{x}} \to 1$

② consider $\sin\frac{1}{x}$ as oscillation, 先画出来 sinx 后沿|x|=1 翻转成 $\sin\frac{1}{x}$

③ **derivatives/stationary point/zero point:**

求导之后得到, **when stationary point,** $\frac{1}{x} = \tan\frac{1}{x}$, 由此可知从 **x=1** 后没有拐点



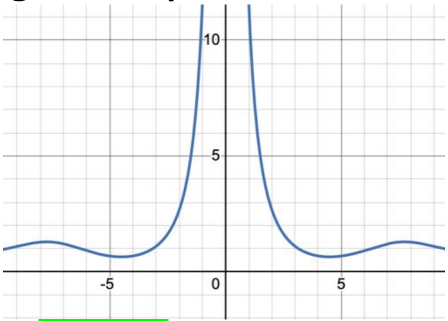
5. $\frac{x+\sin x}{x-\sin x}$

① **asymptotes/approximation:**

when $x \to 0$, $\frac{x+\sin x}{x-\sin x} \to \frac{1+\cos x}{1-\cos x} \to \infty$; **when** $x \to \infty$, $\frac{x+\sin x}{x-\sin x} \to 1$

② **consider sinx as oscillation**

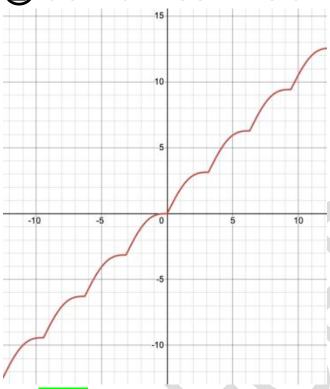
③ **no zero point**



6. $x + \sin|x|$

① **consider sinx as oscillation**

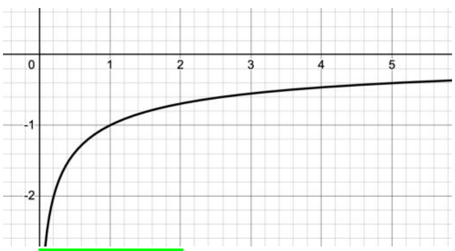
② **derivatives: it is always increasing**



7. $\frac{\ln x}{1-x}$

① **domain/asymptotes/approximation:**

当 $x \to 1$, 分子分母同时趋近于 **0**, 因此不一定有 **asymptotes.** 当 $x \to 1$ 时, $\frac{\ln x}{1-x} \to \frac{\frac{1}{x}}{-1} \to -1$



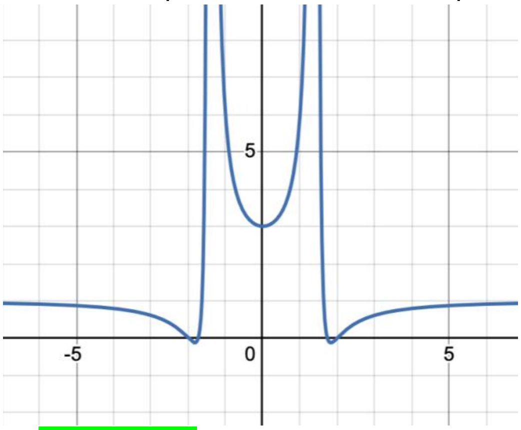
8. $\frac{x^4-7x^2+12}{x^4-4x^2+4}$

① **asymptotes:** 注意, **asymptotes** 两边并不代表一定一正一负

② 记住这种函数一定是 smooth 的!

③ $\sqrt{f(x)}$, f($\sqrt{x}$), f(x^2), $[f(x)]^2$相关问题, 一定要先求导: I wonder what will be the tangent of zero points

先画$\frac{u^2-7u+12}{u^2-4u+4}$, 然后转成$\frac{x^4-7x^2+12}{x^4-4x^2+4}$



9. $y^2=x^2-x^4$

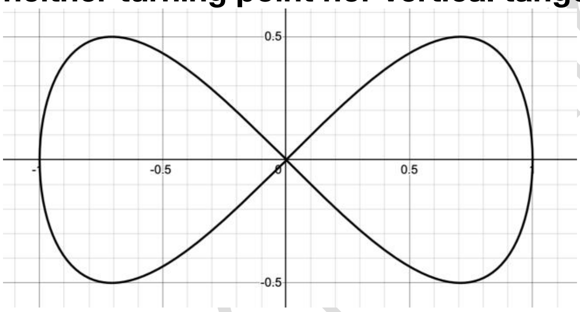
① 记住这种函数一定是 smooth 的!

② $\sqrt{f(x)}$, f($\sqrt{x}$), f(x^2), $[f(x)]^2$相关问题, 一定要先求导: I wonder what will be the tangent of zero points

先画 y=x^2-x^4, 然后转成y^2=x^2-x^4

for x=±1, $\frac{1}{\sqrt{f(x)}}$ has term $\frac{1}{\sqrt{(x\pm1)}}$, f'(x) does not have related term, so totally $[\sqrt{f(x)}]'$ has term $\frac{1}{\sqrt{(x\pm1)}}$, vertical tangent

for x=0, $\frac{1}{\sqrt{f(x)}}$ has term $\frac{1}{x}$, f'(x) has term x, so totally $[\sqrt{f(x)}]'$ does not have related term, neither turning point nor vertical tangent



10. y^2=x(x+1)$(x-2)^4$

① 记住这种函数一定是 smooth 的!

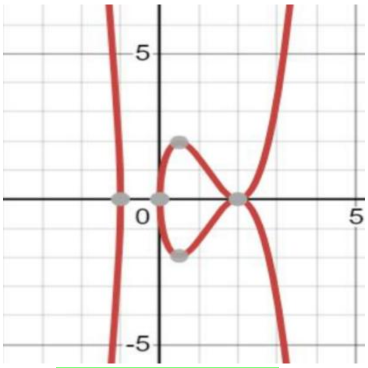
② $\sqrt{f(x)}$, f($\sqrt{x}$), f(x^2), $[f(x)]^2$相关问题, 一定要先求导: I wonder what will be the tangent of zero points

先画 y=x(x+1)$(x-2)^4$. 注意到$[\sqrt{f(x)}]' = \frac{f'(x)}{2\sqrt{f(x)}}$.

for x=-1, $\frac{1}{\sqrt{f(x)}}$ has term $\frac{1}{\sqrt{(x+1)}}$, f'(x) does not have related term, so totally $[\sqrt{f(x)}]'$ has term $\frac{1}{\sqrt{(x+1)}}$, vertical tangent

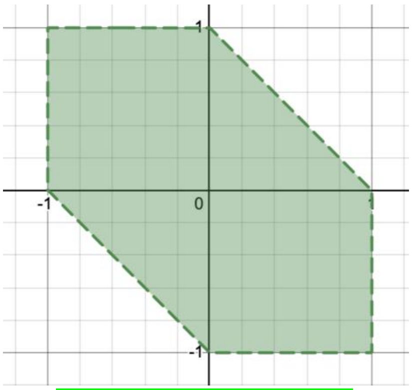
for x=0, $\frac{1}{\sqrt{f(x)}}$ has term $\frac{1}{\sqrt{x}}$, f'(x) does not have related term, so totally $[\sqrt{f(x)}]'$ has term $\frac{1}{\sqrt{x}}$, vertical tangent

for x=2, $\frac{1}{\sqrt{f(x)}}$ has term $\frac{1}{(x-2)^2}$, f'(x) has term $(x-2)^3$, so totally $[\sqrt{f(x)}]'$ has term (x-2), turning point



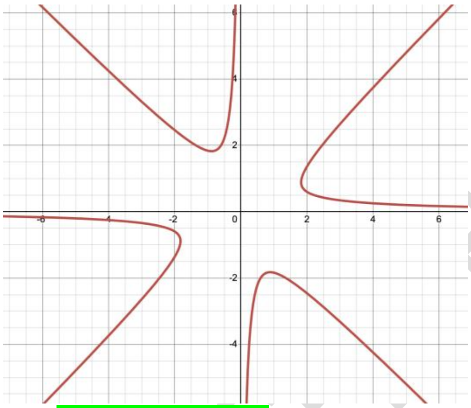
11. $|x|+|y|+|x+y| < 2$

① if (x,y) is a solution, so do (-x,-y) and (-y,-x), 所以说 y=x, y=-x 对称
根据这一点分类讨论即可画出



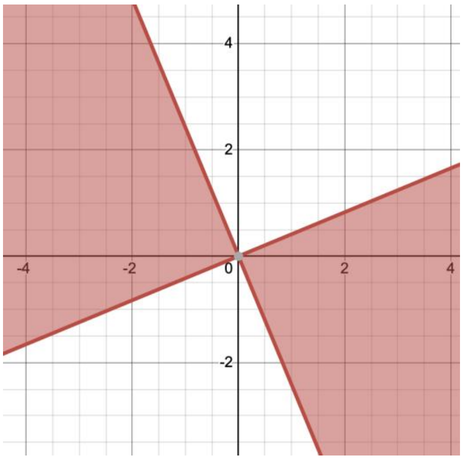
12. $xy(x^2-y^2) = x^2+y^2$

① 注意到一堆 square, 考虑 polar form. 此处$r^2 = \frac{4}{\sin 4\theta}$



13. $x^2-2xy-y^2 \geq 0$

① 注意到一堆 square, 考虑 polar form
② 注意到一堆 square, 考虑凑平方差等格式
方法一: 平方差
$x^2-2xy+y^2 \geq 2y^2$, $(x-y)^2 \geq 2y^2$, $(x-y)^2-2y^2 \geq 0$, $(x-y-\sqrt{2}y)(x-y+\sqrt{2}y) \geq 0$
$[x-(1+\sqrt{2})y][x-(1-\sqrt{2})y] \geq 0$
so both terms are positive or negative
(1) $x \geq (1+\sqrt{2})y$, $x \geq (1-\sqrt{2})y$. thus $\frac{1}{1+\sqrt{2}}x \geq y$, $\frac{1}{1-\sqrt{2}}x \leq y$
(2) $x \leq (1+\sqrt{2})y$, $x \leq (1-\sqrt{2})y$. thus $\frac{1}{1+\sqrt{2}}x \leq y$, $\frac{1}{1-\sqrt{2}}x \geq y$
方法二: polar form
$r^2\cos^2\theta - 2r^2\cos\theta\sin\theta - r^2\sin^2\theta \geq 0$, $\cos^2\theta - 2\cos\theta\sin\theta - \sin^2\theta \geq 0$, $\cos 2\theta \geq \sin 2\theta$


4
2
-4
-2
0
2
4
-2

经典数论题

1. e 是无理数

assume e = $\frac{p}{q}$, p and q are co-prime:

e = 1 + $\frac{1}{1!}$ + $\frac{1}{2!}$ + $\frac{1}{3!}$ + ...

(q!)•e = (q!)•(1 + $\frac{1}{1!}$ + $\frac{1}{2!}$ + $\frac{1}{3!}$ + ...) = integer term + $\frac{1}{q+1}$ + $\frac{1}{(q+1)(q+2)}$ + ...

< integer term + $\frac{1}{q+1}$ + $\frac{1}{(q+1)^2}$ + ...

LHS is integer but RHS is not

2. $\sum_{k=1}^{n}\frac{1}{k}$, for n > 1, 永远不是整数

if LHS = $\frac{p}{q}$, q = $\prod_{k=1}^{n} k$, LHS not an integer means q∤p:

设[1,n]中最大的能被多个 2 整除的数为2^m, 则化简之后2^m|q, 2^m∤p [考虑$\frac{1}{2^m}$那一项]

3. given that x^3 + lx^2 + mx + n = 0, l, m, n are all integers, prove that: if x∉Z then x∉Q

if x∈Q but x∉Z, so x = $\frac{p}{q}$, p and q are co-prime and q≠1

$(\frac{p}{q})^3$ + l$(\frac{p}{q})^2$ + m$\frac{p}{q}$ + n = 0, p^3 + lp^2q + mpq^2 + nq^3 = 0, p^3 = -(lp^2q + mpq^2 + nq^3)

whereas, q|RHS but q∤LHS, so impossible

4. a polynomial f(x) with integer coefficients such that f(19)=f(94)=1994. given that the absolute value of constant term is less than 1000, find the value of constant term

recall that f(x) = $\sum a_i x^i$, so f(19) ≡ a_0 (mod 19), f(94) ≡ a_0 (mod 94), 因为含 x 项都消去了

a_0 = 208

5. let P(x,y) be a polynomial in x and y such that: i) P(x,y)=P(y,x); ii) (x-y) is a factor of P(x,y). deduce that $(x-y)^2$ is a factor of P(x,y)

according to ii), P(x,y)=(x-y)Q(x,y), we want to prove that (x-y)|Q(x,y)
P(x,y)=P(y,x), (x-y)Q(x,y)=(y-x)Q(y,x), Q(x,y)=-Q(y,x) for all x and y
when x=y, Q(x,x)=-Q(x,x), Q(x,x)=0
so x=y is a root of Q(x,y), thus (x-y)|Q(x,y)

6. 猜测 f(x)表达式相关

① 一般来说考虑特殊值: x=0, x=1, x=y, x=-y etc.
② 某些时候考虑通过求导的定义进行求解
③ 某些时候通过转化成已经知道的关系式进行求解
④ 某些时候要猜测一个答案[根据 degree etc.], 并且证明答案正确[number of roots is our friend]

if f(x+y) = f(x) + f(y) and f(x) is differentiable, find f(x)

when y=0: f(x) = f(x) + f(0), f(0) = 0

f'(0) = $\frac{f(0+\Delta x)-f(0)}{\Delta x}$ = $\frac{f(\Delta x)}{\Delta x}$

f'(x) = $\frac{f(x+\Delta x)-f(x)}{\Delta x}$ = $\frac{f(x)+f(\Delta x)-f(x)}{\Delta x}$ = $\frac{f(\Delta x)}{\Delta x}$ = f'(0)

so f(x) has constant derivatives, f(x) = Ax + B
since f(0) = 0, f(x) = Ax

if f(xy) = f(x) + f(y) and f(x) is differentiable, x is positive, find f(x)

consider f($e^u e^v$) = f(e^{u+v}) = f(e^u) + f(e^v). if g(x) = f(e^x), then g(x+y) = g(x) + g(y)
thus g(x) = Ax, f(x) = A•lnx

if f(x+y) = f(x)f(y) and f(x) is differentiable, find f(x)

consider ln[f(x+y)] = ln[f(x)f(y)] = ln[f(x)] + ln[f(y)]. if g(x) = ln[f(x)], then g(x+y) = g(x) + g(y)
thus g(x) = Ax, f(x) = e^{Ax}

if $[f(x+y)]^2 = [f(x)]^2 + [f(y)]^2$, find f(x)
when y=0: $[f(x)]^2 = [f(x)]^2 + [f(0)]^2$, $f(0)=0$
when x=-y: $[f(0)]^2 = [f(x)]^2 + [f(-x)]^2 = 0$, f(x) = 0
if f(x)f(y) = f(x+y) + xy, find f(x)
when y = 0: f(x)f(0) = f(x), f(0) = 1
when x = -y: f(x)f(-x) = f(0) - x^2 = 1 - x^2
注意 RHS is degree of 2, assume f(x) = ax + 1. by calculate, f(x) = ±x + 1
if f(x) is a polynomial and f(sinx)=f(sin2x), prove that f(x) is a constant for all x
P(sin1) = P(sin$\frac{1}{2}$) = P(sin$\frac{1}{4}$) = ...
consider P(x) - P(sin1), it has infinite zero points
but a polynomial only has finite zero points, unless it is a constant
if f(x^2)=$(f(x))^2$, find polynomial f(x)
assume f(x) = $\sum_{i=0}^{n} a_i x^i$,
since f(x^2)=$(f(x))^2$, a_n = 1
assume g(x) = $\sum_{i=0}^{k} a_i x^i$, f(x) = x^n + g(x), the degree of g(x) is k
f(x^2) = x^{2n} + g(x^2) = x^{2n} + $a_k x^{2k}$ + ...
$(f(x))^2$ = x^{2n} + 2x^ng(x) + $(g(x))^2$ = x^{2n} + 2$a_k x^{n+k}$ + ...
so a_k=0, impossible since the degree of g(x) is k
final answer: f(x)=0, or f(x)=1, or f(x)=x^n
if f(x)f(-x)=f(x^2), find polynomial f(x)

if a is a root, so does a^2. since we only has finite roots, a=0, or a=1, or a=$e^{\pm i\frac{\pi}{3}}$
f(x) = $x^p(x-1)^q(1+x+x^2)^q$, 经过演算, p+q 为偶数
7. prove that the only solution to the equation x^2+y^2+z^2=2xyz for integers x, y, z is x=y=z=0
基本思想: contradiction+转转不已
假设 x, y, z 最大共享2^k的 factor, 那么当(x,y,z) = (a2^k,b2^k,c2^k)时, a^2+b^2+c^2=2^{k+1}abc
因为 a, b, c 不全为偶数, 因此不成立, 唯一的可能就只有(a,b,c)=(0,0,0)了
8. if n, x, y, z are all positive integers, find all solutions of the equation n^x+n^y=n^z
① 两边都有指数的时候, 尝试把指数归到一边
n^{x-z} + n^{y-z} = 1, 因为 common factor, only possibility: n^{x-z} = n^{y-z} = $\frac{1}{2}$
9. prove that for any positive integers n>1, n^4+4^n is not a prime
① 如果两项分别为m^4 + n^4 for some m, n, LHS = $(m^2+n^2)^2$ - $2m^2n^2$
in such way, LHS = $(2^n + n^2)^2$ - $2^{n+1}n^2$, 随后按照 n 的奇偶分类讨论即可
10. prove that f(x) = Asinx + Bcos($\sqrt{2}$x) is not periodical
① if periodical: f(x+T) - f(x-T) = 0
2A•cosx•sinT + 2B•sin($\sqrt{2}$x)•sin($\sqrt{2}$T) = 0
when x=$\frac{\pi}{2}$, then $\sqrt{2}$T=nπ, T=$\frac{n\pi}{\sqrt{2}}$; when x=0, T=mπ; n and m are integers
since $\frac{n\pi}{\sqrt{2}}$ ≠mπ, impossible
11. calculate (1+tan1)×(1+tan2)×......×(1+tan45)
对于很多项相加/乘, 考虑先将几项组合起来
(1+tan1)(1+tan44) = 1 + tan1 + tan44 + tan1•tan44

Since tan45 = $\frac{\tan 1 + \tan 44}{1 - \tan 1 \cdot \tan 44}$ = 1, tan1 + tan44 = 1 - $\tan 1 \cdot \tan 44$

(1+tan1)(1+tan44) = 2, 同理求别的项, 易得答案

12. calculate $\sqrt{3-x}$ - $\sqrt{x+1}$ > $\frac{1}{2}$

let A=$\sqrt{3-x}$, B=$\sqrt{x+1}$. since A^2+B^2=4, assume A=2siny, B=2cosy, 0<y<$\frac{\pi}{2}$

2siny - 2cosy > $\frac{1}{2}$, 2siny > 2cosy + $\frac{1}{2}$ > 0, $(2\sin y)^2$ > $(2\cos y + \frac{1}{2})^2$

$32\cos^2$y + 8cosy - 15 < 0

13. a clock has 12 portions, and each portion is either blue or red. totally there are 6 blue portions and 6 red portions. prove that there exists a line which can divide the clock into two half parts, with each part has 3 blue portions and 3 red portions

只用关注 blue portion 的个数就行, 因为非 blue 即 red

随便画一条线, 然后数其右边的 blue portion 的个数, 设为 n

然后 clockwisely rotate $\frac{\pi}{6}$, 每次 rotate 完之后都数其右边的 blue portion 的个数

上述操作重复 6 次后, 这时线回到原位置, 其右边 blue portion 的个数为(6-n)

注意, 在这个过程中, 线右边 blue portion 的个数应该是 continuous 变化的, 所以说一定存在一条线可以满足条件

14. 极限的定义相关

① 有些时候公式如果不能求解, 就要去用$\lim_{\Delta x \to 0} \frac{f(x+\Delta x) - f(x)}{\Delta x}$的定义求解

let f(x) be a differentiable equation, evaluate $\lim_{x \to a} \frac{af(x)-xf(a)}{x-a}$

since x→a, we may assume that x=a+Δx

LHS = $\lim_{\Delta x \to 0} \frac{af(a+\Delta x) - (a+\Delta x)f(a)}{a+\Delta x-a}$ = $\lim_{\Delta x \to 0} \frac{[af(a+\Delta x) - af(a)] - \Delta x f(a)}{\Delta x}$ = af'(a) - f(a)

let f(x)=x+2x^2sin$\frac{1}{x}$ and given that f(0)=0, find f'(0)

f'(0) = $\lim_{\Delta x \to 0} \frac{f(\Delta x) - f(0)}{\Delta x}$ = $\lim_{\Delta x \to 0} \frac{\Delta x + 2\Delta x^2 \cdot \sin\frac{1}{\Delta x}}{\Delta x}$ = $\lim_{\Delta x \to 0} 1 + 2\Delta x \cdot \sin\frac{1}{\Delta x}$ = 1

15. given that x, y, z are prime numbers, find all possible triples (x,y,z) such that x+y^2=4z^2

4z^2-y^2=(2z-y)(2z+y)=x

since x is prime, and 2z+y cannot be 1 as z, y are positive, so 2z-y=1, 2z+y=x

x	y	z	
1	0	2	mod 3
0	1	1	mod 3
2	2	0	mod 3

so one of them must be divisible by 3, meaning that one of them is 3

by trials, (x,y,z)=(7,3,2) or (11,5,3)

经典概率题

1. if I had a cube and six colours and painted each side a different colour, how many (different) ways could I paint the cube?

① 对于首尾相接/旋转重复题, 固定一个 **element.** 这里先固定一个 **top**

② 涂色题最有用的就是间隔涂色. 选择一个 **bottom,** 剩下的随便排完后除去重复.

total 30

2. if I colour three faces of a cube red and the other faces blue, how many distinguishable colourings are there?

①对于重复多次使用的颜色+首尾相接/旋转重复题, 并不一定能固定一个重复多次使用的颜色作为 **indicator,** 因为其本身不能起到一个定位的作用

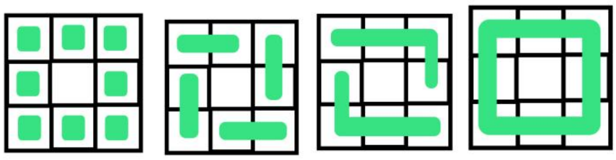
there are only 2 possible configurations, three blue faces meeting at a point and three blue faces wrapping around the cube joined edge to edge

3. there is a white board on a wall which has 3×3 small squares in it. if I have nine colors to paint it and each color can be used repeatedly, how many different combinations can I get? does the number change if I can turn it around? what if the white board becomes a glass?

① 有些时候每一种情况的重复次数不一样, 因此要分类讨论

② 有些时候用 **Venn Graph** 可以清楚看出每一部分的真正数量[容斥原理]

第二问:



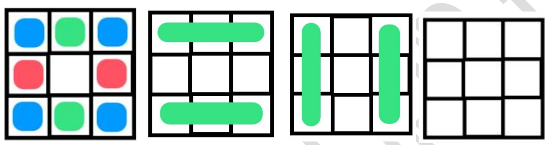
第一种情况: 单格子重复. 这种情况下有9^2种情况, 每种情况出现 **1** 次

第二种情况: 双格子重复. 这种情况下有9^3-9^2种情况, 每种情况出现 **1** 次

第三种情况: 四格子重复. 这种情况下有9^5-9^3种情况, 每种情况出现 **2** 次

第四种情况: 八格子重复. 这种情况下有9^9-9^5种情况, 每种情况出现 **4** 次

第三问:



第一种情况: 上下左右对称. 这种情况下有9^4种情况, 每种情况出现 **1** 次

第二种情况: 上下对称. 这种情况下有9^6-9^4种情况, 每种情况出现 **2** 次

第三种情况: 左右对称. 这种情况下有9^6-9^4种情况, 每种情况出现 **2** 次

第四种情况: 啥也不对称. 这种情况下有9^9-(9^6+9^6-9^4)种情况, 每种情况出现 **4** 次

4. how many configurations are possible when putting n indistinguishable balls into k distinguishable bins?

把将球放入箱子里的过程看作隔板:

在这种问题中, 隔板之间可以为 **0 element,** 因此看成 **n** 个 **elements** 和**(k-1)**个板子的排列组合.

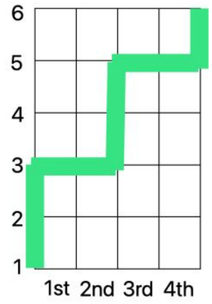
total $\binom{n+k-1}{n}$

5. a fair die is rolled four times. find the probability that each of the final three rolls is at least as large as the roll preceding it

① 对于一些概率问题需要对其进行建模

由此, 该问题便变成了从左下角有多少种方法移动到右上角

4 步 **R, 5** 步 **U, total** $\binom{9}{4}$



6. 3 girls and 4 boys were standing in a circle. what is the probability that two girls are together but one is not with them?

① 对于首位相接/旋转重复题, 固定一个 element
② 放在一起 → 看成一个 element; 不能放在一起 → 隔板
一共是 GGGBBBB, 两个 GG 呆在一起可以看作是一个 element A, 注意 group 内部也有顺序
A 和 G 不能在一起所以说插空法: B ( ) B ( ) B ( ) B ( )
先排 BBBB 再插 A 与 G, 考虑 A 与 G 的分法: $\frac{4\times3\times2}{4}$ ×4×3×6 = 432

7. There are 25 people in a classroom. find out the likelihood that there exists at least one person to born in every month

① 有些时候用 Venn Graph 可以清楚看出每一部分的真正数量[容斥原理]
我们考虑不满足以上条件的情况的情况:

如果我们先设定 1 个 empty month, 剩下的每个人都有11^{25}个选择: all $\binom{12}{1}\cdot 11^{25}$

如果我们先设定 2 个 empty month, 剩下的每个人都有10^{25}个选择: all $\binom{12}{2}\cdot 10^{25}$

...
注意到, 在一定数量的 empty month 时, 剩下的 month 中也有可能有 empty
所以说为了避免重复推导, 用容斥原理

$$1 - \sum_{n=1}^{11} \frac{\binom{12}{n}(12-n)^{25}(-1)^{n-1}}{12^{25}}$$

8. two identical decks of cards, each containing N cards, are shuffled randomly. we say that a k-matching occurs if the two decks agree in exactly k places. show that the probability that there's a k-matching is

$$\frac{1}{k!}\left(1-\frac{1}{1!}+\frac{1}{2!}-\dots+\frac{(-1)^{N-k}}{(N-k)!}\right).$$

① 有些时候用 Venn Graph 可以清楚看出每一部分的真正数量[容斥原理]. 尤其是出现 alternative signs 的时候就应该想到容斥原理
这种问题叫 disarrange, 我们一般这样考虑: 我们要 k 个配对, 那就 disarrange (N-k)个
(1) 先将所有的都排好, 然后选(N-k)个 disarrange:

$\frac{N!}{(N-k)!k!}$ (N-k)! = $\frac{N!}{k!}$ •1

(2) 容斥原理, 有些 element 可能没有被 disarrange:
先考虑 1 个 element 已知(应该被 disarrange)但是没有 disarrange, 那就是先选出来该 element 之后剩下的排列:

$\frac{N!}{(N-k)!k!}\frac{(N-k)!}{(N-k-1)!1!}(N-k-1)! = \frac{N!}{k!}\cdot\frac{1}{1!}$

(3) repeat this step and we will eventually get that number of all situation that k elements are disarranged is

$$\frac{N!}{k!}\left(1-\frac{1}{1!}+\frac{1}{2!}-\dots+\frac{(-1)^{N-k}}{(N-k)!}\right)$$

given that totally we have $N!$ situations, so the possibility is

$$\frac{1}{k!}(1-\frac{1}{1!}+\frac{1}{2!}-...+\frac{(-1)^{N-k}}{(N-k)!})$$

9. if we flip a coin and generate a sequence of length n, what is the probability that the number of heads is even?

① 有些时候如果以 even, odd 等形式加以区分, 则可能跟 binomial expansion 有关

② 有些时候 symmetry 可以节省大量时间

方法一: binomial expansion

P(even number of head) = $\sum_{i\ is\ even}(\frac{1}{2})^i(\frac{1}{2})^{n-i}\binom{n}{i}$

P(odd number of head) = $\sum_{i\ is\ odd}(\frac{1}{2})^i(\frac{1}{2})^{n-i}\binom{n}{i}$

P(even number of head) - P(odd number of head) = $(\frac{1}{2}-\frac{1}{2})^n$

方法二: symmetry

when n is odd, we have two situations: (1) odd H+even T or (2) even H+ odd T

for every specific case in (1), if we turn its H into T and turn its T into H, it will become a case in (2), with the probability unchanged.

just like a bijection: the possibility for us to get a specific case in (1) equals to the possibility for us to get a corresponding case in (2)

so, when n is odd, P(even number of head) = $\frac{1}{2}$ for n is odd

when n is even, consider it as n = k+1, so k is odd

first throw k toss, with the possibility P(even number of head for first k toss) = $\frac{1}{2}$

then throw the last toss, with the possibility P(last toss is head) = $\frac{1}{2}$

totally P(even number of head) = $\frac{1}{2}$ for n is even

10. there are 6 ropes in a bag. In each step, two rope ends are picked at random, tied together and put back into a bag. the process is repeated until there are no free ends. what is the expected number of loops en at the end of the process?

① 对于很多次重复的或雷同的操作, 大部分都可以用 induction/recurrence relation $P_n=f(P_{n-1},P_{n-2})$解决

设 E(n)代表 n ropes 时 expected number of final loops

E(2)时, 拿起一端, 有$\frac{1}{3}$的概率造成一个 additional loop, 随后情况变化成了 E(1)

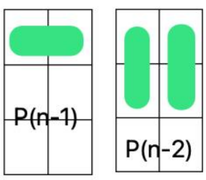
E(3)时, 拿起一端, 有$\frac{1}{5}$的概率造成一个 additional loop, 随后情况变化成了 E(2)

...

E(n) = $\frac{1}{2n-1}$ + E(n-1) = $\sum_{k=1}^{n}\frac{1}{2k-1}$

11. how many ways are there to cover a 2×n rectangular grid with 2×1 tiles?

① 对于很多次重复的或雷同的操作, 大部分都可以用 induction/recurrence relation $P_n=f(P_{n-1},P_{n-2})$解决



设 P(n)代表 2×n rectangular grid 中 2×1 tiles 的放法

那么如图, P(n) = P(n-1) + P(n-2) = F_n

12. if we flip a coin and generate a sequence of length n, what is the probability we do not see two heads in a row? set up a recursion equation and then solve it to find u_n.

① 对于很多次重复的或雷同的操作, 大部分都可以用 induction/recurrence relation P_n=f(P_{n-1},P_{n-2})解决

assume initial n toss satisfies our condition, for (n+1)th toss:
if nth toss is H, we want T for (n+1)th toss
if nth toss is T, we want H/T for (n+1)th toss

after several toss, notice H can only follow T, whereas H can follow both H and T
we find that for nth toss, the number of H is F_{n-1}, the number of T is F_n

so $u_{n+1} = u_n \cdot \left(\frac{F_n}{F_{n-1}+F_n} \cdot 1 + \frac{F_{n-1}}{F_{n-1}+F_n} \cdot \frac{1}{2}\right)$

13. let u_n be the probability that n tosses of a fair coin contain no run of 4 heads. find a recurrence relation for u_n and use it to show $u_8 = \frac{208}{256}$.

① 有些时候如果正着想不方便可以试试反着想
if P_n represents the probability that n tosses contain runs of 4 heads
then $P_n = P_{n-1} + (1-P_{n-4}) \cdot (\frac{1}{2})^4$
相当于 P[前 n-1tosses 就有问题] + P[前 n-1tosses 以 HHH 结尾, 然后 nth toss 是 H]

14. the probability of obtaining a head when a biased coin is tossed is p. the coin is toss repeatedly until n heads occur in a row. let X be the total number of tosses required for this to happen. find the expected value of X.

方法一: 通过考虑X_{n-1}得到X_n
$E(X_n) = p[E(X_{n-1})+1] + q[E(X_{n-1})+1+E(X_n)]$
相当于[在已有 n-1 个连续 H 时下一个是 H]+[在已有 n-1 个连续 H 时下一个是 T 所以说重来]
$E(X_n) = \frac{E(X_{n-1})+1}{p}$
如果要 find general form 可以用 diagnolised matrix: this recurrence relation seems like a repeated calculation, making me recall the diagnolised matrix

$$\begin{pmatrix} \frac{1}{p} & \frac{1}{p} \\ 0 & 1 \end{pmatrix} \begin{pmatrix} E(X_{n-1}) \\ 1 \end{pmatrix} = \begin{pmatrix} E(X_n) \\ 1 \end{pmatrix}$$

$$\begin{pmatrix} \frac{1}{p} & \frac{1}{p} \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 1 \\ 1-p & 0 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & \frac{1}{p} \end{pmatrix} \begin{pmatrix} 1 & 1 \\ 1-p & 0 \end{pmatrix}^{-1}$$

$$\begin{pmatrix} E(X_n) \\ 1 \end{pmatrix} = \begin{pmatrix} 1 & 1 \\ 1-p & 0 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & \frac{1}{p} \end{pmatrix}^n \begin{pmatrix} 1 & 1 \\ 1-p & 0 \end{pmatrix}^{-1} \begin{pmatrix} E(X_0) \\ 1 \end{pmatrix}$$

方法二: 考虑一旦 T 就 start anew
$E(X_n) = q(1+E(X_n)) + pq(2+E(X_n)) + p^2q(3+E(X_n)) + ... + p^{n-1}q(2+E(X_n)) + p^n n$

15. given any n integers $a_1, a_2, . . . , a_n$, is there a non-empty subset of these whose sum is divisible by n?

① 多种 element 相关+一定存在, 一般就用 pigeon hole

考虑a_1, a_1+a_2, $a_1+a_2+a_3$, ..., $a_1+a_2+a_3+...+a_n$

如果p^{th} term ≡ q^{th} term (mod n), then p^{th} term - q^{th} term is what we want

if all of them are not equal, then one of them k^{th} term ≡ 0 (mod n)

16. a thin rod is broken into three pieces. What is the probability that a triangle can be formed from the three pieces? What about if we want the triangle to be acute (e.g. all the internal angles are less than 90 degrees)?

① 对于 continuous 的概率情况, 将大小于关系化成解析式画在坐标轴上求面积占多少百分比

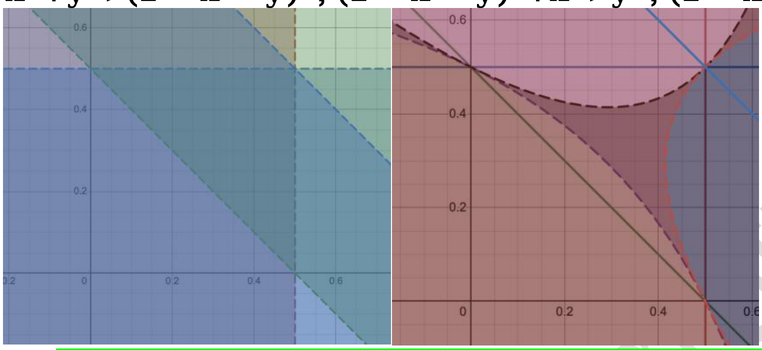
第一问:

因为两边之和大于第三边, if x+y+z=1: $x<\frac{1}{2}$, $y<\frac{1}{2}$, $x+y>\frac{1}{2}$, x+y<1

第二问:

对于锐角来说, $a^2+b^2>c^2$, so:

$x^2+y^2>(1-x-y)^2$, $(1-x-y)^2+x^2>y^2$, $(1-x-y)^2+y^2>x^2$



17. there are 6*3 seats on a plane. if there are 7 men and 10 women, how many situations are there such that there are both men and women for every column and row?

① convert this question into others method using bijection

label 所有的男生, ABCDEFG

然后在 row1, row2, row3 上分配他们, 再从 column1, column2, ..., column6 上分配他们

然后只需要计算 【row1, row2, row3 上分配他们】* 【column1, column2, ..., column6 上分配他们】就可以得到男生的 combination 的个数

最后一段用容斥原理即可

row: $\sum_{i=0}^{3}(-1)^i\binom{3}{i}(3-i)^7$

column: $\sum_{i=0}^{6}(-1)^i\binom{6}{i}(6-i)^7$

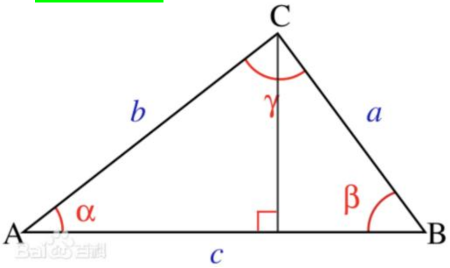
total(consider women): $[\sum_{i=0}^{3}(-1)^i\binom{3}{i}(3-i)^7]$ * $[\sum_{i=0}^{6}(-1)^i\binom{6}{i}(6-i)^7]$ * 11!

18. a biased coin has probability of landing heads equal to p. if the coin is tossed n times and let X be the number of times it lands heads in total. suppose now that the value of p is unknown. however, it's observed that k heads are obtained after tossing the coin n times. what is the value of p which makes this event most likely? that is, what value of p maximises P(X = k)?

consider Normal Distribution: when n is large, binomial expansion tends to be a normal distribution. for Normal Distribution, peak occurs at the middle

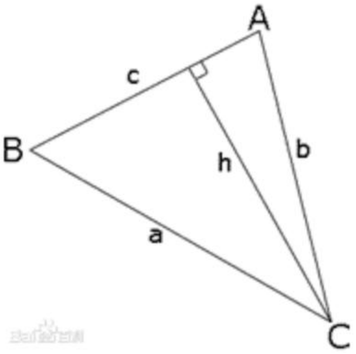
要证明的结论

1. 余弦定理



$a = b\cos\gamma + c\cos\beta$, $a^2 = a\cdot b\cdot\cos\gamma + a\cdot c\cdot\cos\beta$
$b = a\cos\gamma + c\cos\alpha$, $b^2 = a\cdot b\cdot\cos\gamma + b\cdot c\cdot\cos\alpha$
$c = a\cos\beta + b\cos\alpha$, $c^2 = a\cdot c\cdot\cos\beta + b\cdot c\cdot\cos\alpha$
$a^2 + b^2 = a\cdot b\cdot\cos\gamma + a\cdot c\cdot\cos\beta + a\cdot b\cdot\cos\gamma + b\cdot c\cdot\cos\alpha = c^2 + 2a\cdot b\cdot\cos\gamma$
$c^2 = a^2 + b^2 - 2a\cdot b\cdot\cos\gamma$

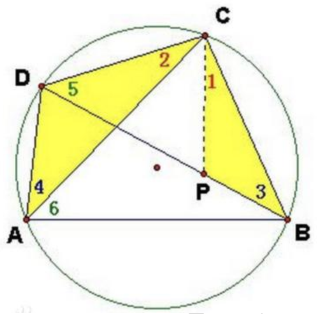
2. 正弦定理



$\sin A = \frac{h}{b}$, $\sin B = \frac{h}{a}$, $h = b\sin A = a\sin B$, $\frac{\sin A}{a} = \frac{\sin B}{b}$

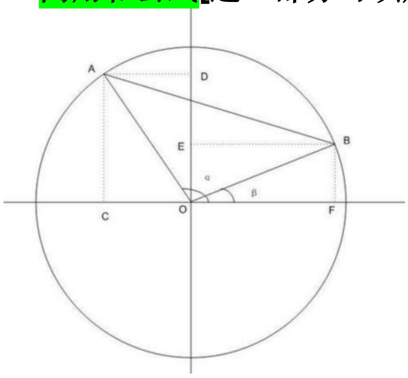
$S = \frac{1}{2}b\cdot c\cdot\sin A = \frac{1}{2}c\cdot h$

3. 托勒密定理



$AC\cdot BD = AB\cdot CD + AD\cdot BC$, 这一部分可以用$e^{i\theta}$和和差化积直接证明

4. 两角和公式[这一部分可以用$e^{i(\alpha+\beta)}$直接证明, 这里给出纯几何方法]



$AB^2 = OA^2 + OB^2 - 2OA\cdot OB\cdot\cos(\alpha - \beta) = 2 - 2\cos(\alpha - \beta)$ [余弦定理]

$AB^2 = DE^2 + CF^2$ [勾股定理]

$= (OD - OE)^2 + (OF + OC)^2$

$= (OA\sin\alpha - OB\sin\beta)^2 + (OB\cos\beta - OA\cos\alpha)^2$

$= (\sin\alpha - \sin\beta)^2 + (\cos\beta - \cos\alpha)^2$

$= 2 - 2\sin\alpha\sin\beta - 2\cos\alpha\cos\beta$

$\cos(\alpha - \beta) = \sin\alpha\sin\beta + \cos\alpha\cos\beta$

通过改α, β的大小$[\frac{\pi}{2}-x]$便可以得到全部的公式.

5. 求导&重要极限的证明

$(\ln x)' = \lim\limits_{\triangle x\to 0}\frac{\ln(x+\triangle x)-\ln x}{\triangle x} = \lim\limits_{\triangle x\to 0}\frac{1}{\triangle x}\ln(1+\frac{\triangle x}{x}) = \lim\limits_{\triangle x\to 0}\ln(1+\frac{\triangle x}{x})^{\frac{1}{\triangle x}}$

$= \lim\limits_{\triangle x\to 0}\ln(1+\frac{\triangle x}{x})^{\frac{x}{\triangle x}\cdot\frac{1}{x}} = \frac{1}{x}$

证明$(e^x)'$时让 $y = e^x$, $\ln y = x$ 即可.

$(\sin x)' = \lim\limits_{\triangle x\to 0}\frac{\sin(x+\triangle x)-\sin x}{\triangle x} = \lim\limits_{\triangle x\to 0}\frac{\sin x\cdot\cos\triangle x + \cos x\cdot\sin\triangle x - \sin x}{\triangle x}$

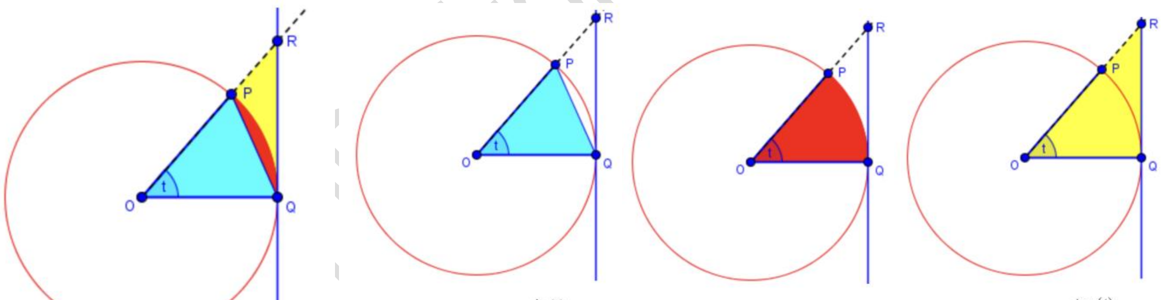
$= \lim\limits_{\triangle x\to 0}\frac{(\cos\triangle x-1)}{\triangle x}\cdot\sin x + \cos x = \lim\limits_{\triangle x\to 0}\frac{-2\sin^2\frac{\triangle x}{2}}{\triangle x}\cdot\sin x + \cos x$

$= \lim\limits_{\triangle x\to 0}[\frac{\sin\frac{\triangle x}{2}}{\frac{\triangle x}{2}}\cdot\frac{1}{2}\cdot(-2\sin\frac{\triangle x}{2})]\cdot\sin x + \cos x = \lim\limits_{\triangle x\to 0} -\sin\frac{\triangle x}{2}\cdot\sin x + \cos x$

$= \cos x$

证明$(\cos x)'$时证明$(\sin(\frac{\pi}{2}-x))'$即可. 证明其余三角函数/反三角函数的导数时化成 cos/sin 证明即可.

$\lim\limits_{x\to 0}\frac{\sin x}{x} = 1.$



Area(△OPQ) = $\frac{\sin(t)}{2}$ Area(sectorOPQ) = $\frac{t}{2}$ Area(△OPR) = $\frac{\tan(t)}{2}$

$\frac{\sin x}{2} \le \frac{x}{2} \le \frac{\tan x}{2}$, $1 \le \frac{x}{\sin x} \le \frac{1}{\cos x}$, 剩下的就是 sandwich rule 了.

$(f(x)\cdot g(x))' = \lim\limits_{\triangle x\to 0}\frac{f(x+\triangle x)\cdot g(x+\triangle x) - f(x)\cdot g(x)}{\triangle x}$

$= \lim\limits_{\triangle x\to 0}\frac{[f(x+\triangle x)\cdot g(x+\triangle x) - f(x)\cdot g(x+\triangle x)] + [f(x)\cdot g(x+\triangle x) - f(x)\cdot g(x)]}{\triangle x}$

$= \lim\limits_{\triangle x\to 0}[f'(x)\cdot g(x+\triangle x) + f(x)g'(x)] = f'(x)g(x) + f(x)g'(x)$

$[f(g(x))]' = \lim\limits_{\triangle x\to 0}\frac{f(g(x+\triangle x)) - f(g(x))}{\triangle x} = \lim\limits_{\triangle x\to 0}\frac{f(g(x+\triangle x)) - f(g(x))}{g(x+\triangle x) - g(x)}\cdot\frac{g(x+\triangle x) - g(x)}{\triangle x} = f'(g(x))\cdot g'(x)$

6. 三角万能代换$(\tan\frac{x}{2})$

if $t = \tan\frac{x}{2}$:

$\tan x = \frac{2t}{1-t^2}$, $\sin x = \frac{2\cos\frac{x}{2}\cdot\sin\frac{x}{2}}{\cos^2\frac{x}{2}+\sin^2\frac{x}{2}} = \frac{2\tan\frac{x}{2}}{1+\tan^2\frac{x}{2}} = \frac{2t}{1+t^2}$, $\cos x = \frac{\cos^2\frac{x}{2}-\sin^2\frac{x}{2}}{\cos^2\frac{x}{2}+\sin^2\frac{x}{2}} = \frac{1-\tan^2\frac{x}{2}}{1+\tan^2\frac{x}{2}} = \frac{1-t^2}{1+t^2}$

7. 反双曲函数

$y = \sinh x = \frac{e^x - e^{-x}}{2}$: $\sinh x + \cosh x = \sinh x + \sqrt{\sinh^2 x + 1} = e^x$, $x = \ln(y + \sqrt{y^2+1})$

$y = \cosh x = \frac{e^x + e^{-x}}{2}$: $\cosh x + \sinh x = \cosh x + \sqrt{\cosh^2 x - 1} = e^x$, $x = \ln(y + \sqrt{y^2-1})$

$y = \tanh x = \frac{e^x - e^{-x}}{e^x + e^{-x}}$:

$y = \frac{e^x - e^{-x}}{e^x + e^{-x}} = \frac{e^{2x}-1}{e^{2x}+1}$, $y\cdot(e^{2x}+1) = e^{2x} - 1$, $e^{2x} = \frac{1+y}{1-y}$, $x = \frac{1}{2}\ln(\frac{1+y}{1-y})$

8. $e^{ix} = \cos x + i\sin x$: 泰勒展开

9. 立体的表面积/体积/重心: 积分

10. 统计相关

why $Var(X) = \frac{\sum(x-E(X))^2}{n} = \frac{\sum x^2}{n} - [E(X)]^2$?

$\frac{\sum(x-E(X))^2}{n} = \frac{\sum x^2 - 2xE(X) + [E(X)]^2}{n} = \frac{\sum x^2}{n} - 2E(X)\frac{\sum x}{n} + [E(X)]^2 = \frac{\sum x^2}{n} - 2[E(X)]^2 + [E(X)]^2$

$= \frac{\sum x^2}{n} - [E(X)]^2$

for X~Geo(p), find E(X) and Var(X)

方法一: definition

$E(X) = \sum_{i=1}^{\infty} ipq^{i-1} = p\sum_{i=1}^{\infty} iq^{i-1} = p\frac{d}{dq}[\sum_{i=1}^{\infty} q^i] = p\frac{d}{dq}[\frac{1}{1-q}] = \frac{1}{p}$

$Var(X) = \sum_{i=1}^{\infty} i^2pq^{i-1} - [E(X)]^2 = \sum_{i=1}^{\infty} i(i-1)pq^{i-1} + \sum_{i=1}^{\infty} ipq^{i-1} - \frac{1}{p}^2$

$= pq\sum_{i=2}^{\infty} i(i-1)q^{i-2} + \frac{1}{p} - \frac{1}{p}^2 = pq\frac{d^2}{dq^2}[\sum_{i=1}^{\infty} q^i] + \frac{1}{p} - \frac{1}{p}^2$

$= pq\frac{d^2}{dq^2}[\frac{1}{1-q}] + \frac{1}{p} - \frac{1}{p}^2 = \frac{q}{p^2}$

方法二: generating function

$G_t(x) = \frac{pt}{1-qt}$, $E(X) = G_t'(1)$, $Var(X) = G_t''(1) + G_t'(1) - [G_t'(1)]^2$

for X~B(n,p), find E(X) and Var(X)

方法一: definition

$E(X) = \sum_{i=0}^{n} ip^iq^{n-i}\binom{n}{i} = p\sum_{i=1}^{n} ip^{i-1}q^{n-i}\binom{n}{i} = p\frac{d}{dq}[(p+q)^n] = np(p+q)^{n-1} = np$

$Var(X) = \sum_{i=0}^{n} i^2p^iq^{n-i}\binom{n}{i} - [E(X)]^2$

$= p^2\sum_{i=2}^{n} i(i-1)p^{i-2}q^{n-i}\binom{n}{i} + p\sum_{i=1}^{n} ip^{i-1}q^{n-i}\binom{n}{i} - n^2p^2$

$= p^2\frac{d^2}{dq^2}[(p+q)^n] + np - n^2p^2 = n(n-1)p^2 + np + n^2p^2 = np(np-p+1-np) = npq$

方法二: generating function

$G_t(x) = (q+pt)^n$, $E(X) = G_t'(1)$, $Var(X) = G_t''(1) + G_t'(1) - [G_t'(1)]^2$

explain why $E(aX+b) = aE(X)+b$, $Var(aX+b) = a^2Var(X)$

you may start by giving some verbal explanations: Var(x) represents the separations

$E(aX+b) = \frac{\sum ax+b}{n} = a\frac{\sum x}{n} + \frac{nb}{n} = aE(X)+b$

Var(aX+b) = $\frac{\sum(ax+b-aE(X)+b)^2}{n}$ = $\frac{\sum(ax-aE(X))^2}{n}$ = $a^2\frac{\sum(x-E(X))^2}{n}$ = a^2Var(X)

explain why $\bar{X}\sim(\mu,\frac{\sigma^2}{n})$[Central Limit Theorem]

Var($\bar{X}$(n)) = Var($\frac{1}{n}(X_1+X_2+...+X_n)$) = $\frac{1}{n^2}$Var($X_1+X_2+...+X_n$) = $\frac{1}{n^2}$[Var(X_1)+Var(X_2)+...+Var(X_n)]

= $\frac{n}{n^2}$Var(X) = $\frac{\sigma^2}{n}$