2025 May

数学月刊

Mathematics Monthly

Author :
Guanzhong Yang(student)
Yan Wang(teacher)

Mathematics changes our lives

PREFACE

This month, we are going to talk about one famous conclusion relating to chromatic number. If you haven't heard this terminology yet, don't worry! Just read through my thesis, known as *Proof of the tightness of a lower bound on the chromatic number of a graph*, you'll then have a better understanding towards the application of linear algebra!

If you have other brilliant questions or knowledge willing to learn, email to anmiciuangray@163.com **for surprising rewards!**

1. abstract

I confirm the tightness of a conjecture in Wocjan and Elphick (2013) [3] about a lower bound of the chromatic number of a graph with the help of matrix analysis.

2. Introduction

In this note, G is a simple graph with $N \geq 1$ vertices, A_G refers to its adjacency matrix, and $\chi(G)$ represents its chromatic number, which is the smallest number of colors needed to color the vertices so that no two adjacency vertices share the same color.

Since A_G is a real symmetric matrix, according to Spectral Theorem, it has real eigenvalues and its eigenvectors can be chosen to be orthonormal.

Definition 1.1 *For a matrix with all real eigenvalues, we denote all its eigenvalues by μ_1, ..., μ_N, in a non-increasing order, with the first π eigenvalues being positive and last δ eigenvalues being negative(counting multiplicities). In such way, s^+ and s^- are defined as*

$$s^+ := \mu_1^2 + \mu_2^2 + \ldots + \mu_\pi^2,$$
$$s^- := \mu_{N-\delta+1}^2 + \mu_{N-\delta+2}^2 + \ldots + \mu_N^2.$$

Although Ando and Lin in [1] has proven that

$$\chi(G) \geq 1 + \max\{\frac{s^+}{s^-}, \frac{s^-}{s^+}\}, \tag{1.1}$$

whether this inequality is tight seems not to be known. The purpose of this note is to provide a brief answer for this question by showing that equality of (1.1) holds for certain kinds of graphs.

Definition 1.2 *For a n-partite graph G, its vertex set V can be partitioned into n pairwise disjoint subsets V_1, V_2, ..., V_n(where $V_i \cap V_j = \emptyset$ for $i \neq j$) such that every edge $(u, v) \in E$ connects vertices from different subsets.*
For a n-partite graph G, it is defined as a symmetric n-partite graph if and only if
- $|V_1| = |V_2| = \ldots = |V_n| = t$, $V_i = \{v_{ij}: 1 \leq j \leq t\}$;
- *the vertices of G can be numbered in a way that, if for some $1 \leq j_1, j_2 \leq t$, $1 \leq i_1, i_2 \leq n$, $i_1 \neq i_2$, $v_{i_1 j_1}$ is adjacent to $v_{i_2 j_2}$, then for all $1 \leq l_1, l_2 \leq n$, $l_1 \neq l_2$, $v_{l_1 j_1}$ is adjacent to $v_{l_2 j_2}$.*

Definition 1.3 *For a symmetric n-partite graph G with cardinality $|V_i| = t$ for each subset, its detail matrix M = M(G) is defined as, for $1 \leq j_1, j_2 \leq t$, $b_{j_1 j_2} = 1$ if and only if $v_{1 j_1}$ is adjacent to $v_{2 j_2}$.*

Theorem 1.1 *The equality of (1.1) holds for G if G is a bipartite graph or a symmetric n-partite graph with its detail matrix M = M(G) being positive semidefinte.*

Definition 1.4 *For a matrix A with all real eigenvalues, we denote all its eigenvalues by μ_1, ..., μ_N, in a non-increasing order, with the first π eigenvalues being positive and last δ eigenvalues being negative(counting multiplicities). In such way, we define its corresponding positive-eigen matrix B = B(A)* and negative-eigen matrix C = C(A) as

$$B := \sum_{i=1}^{\pi} \lambda_i \boldsymbol{v}_i \boldsymbol{v}_i^T \text{ and } C := \sum_{i=N-\delta+1}^{N} \lambda_i \boldsymbol{v}_i \boldsymbol{v}_i^T,$$

where $\boldsymbol{v}_i$ refers to the corresponding eigenvector for eigenvalue λ_i.

Theorem 1.2 *The equality of (1.1) holds for G if and only if there exists a permutation matrix P such that PA_GP^T satisfies the following conditions:*

- *PA_GP^T is partitioned into $\chi(G) \times \chi(G)$ blocks $PA_GP^T = [(PA_GP^T)_{ij}]_{i,j=1}^{\chi(G)}$ with $(PA_GP^T)_{11}$, $(PA_GP^T)_{22}$, ..., $(PA_GP^T)_{\chi(G)\chi(G)}$ being zero matrices;*
- *if the corresponding positive-eigen matrix B = B(PA_GP^T) and negative-eigen matrix C = C(PA_GP^T) are expressed in the forms of*

$$\begin{pmatrix} F \\ G \\ \vdots \end{pmatrix} \left(F^T \; G^T \cdots\right),$$

where F, G, ... are suitably partitioned matrices, then the modules of the l^{th} column of F, G, ... are all the same for any possible l;
- *if B and C are partitioned into $\chi(G) \times \chi(G)$ blocks as the way PA_GP^T does, then all the elements of the non-diagonal blocks of B is the fixed negative multiple of the corresponding elements of the non-diagonal blocks of C.*

3. Main result

Lemma 2.1 *For a bipartite graph G, if the vertices of G is numbered in a way such that $A_G = \begin{pmatrix} 0 & Z \\ Z^T & 0 \end{pmatrix}$, where Z is a matrix, then if A_G has an eigenvalue k, it will also have an eigenvalue $-k$.*

Proof.
By singular value decomposition, $Z = U\Sigma V^T$. In such way, $ZZ^T = U\Sigma\Sigma^T U^T$, $Z^TZ = V\Sigma\Sigma^T V^T$.
Since $A_G^2 = \begin{pmatrix} ZZ^T & 0 \\ 0 & Z^TZ \end{pmatrix}$, if $\boldsymbol{u}_i$ represents the i^{th} column of U, $\boldsymbol{v}_i$ represents the i^{th} column of V, σ_i represents the corresponding i^{th} diagonal element of $\Sigma\Sigma^T$, then

$$A_G^2 \begin{pmatrix} \boldsymbol{u}_i \\ \boldsymbol{v}_i \end{pmatrix} = \sigma_i \begin{pmatrix} \boldsymbol{u}_i \\ \boldsymbol{v}_i \end{pmatrix}.$$

This implies,

$$A_G \begin{pmatrix} \boldsymbol{u}_i \\ \boldsymbol{v}_i \end{pmatrix} = \sqrt{\sigma_i} \begin{pmatrix} \boldsymbol{u}_i \\ \boldsymbol{v}_i \end{pmatrix}, \; A_G \begin{pmatrix} \boldsymbol{u}_i \\ -\boldsymbol{v}_i \end{pmatrix} = -\sqrt{\sigma_i} \begin{pmatrix} \boldsymbol{u}_i \\ -\boldsymbol{v}_i \end{pmatrix},$$

so the eigenvalues of adjacency matrix A_G appear in pairs (k, $-k$). If A_G has an eigenvalue k, it will also have an eigenvalue $-k$.

Lemma 2.2 *For a symmetric n-partite graph G, if the vertices of G is numbered in a way such that*

$$A_G = \begin{pmatrix} 0 & M & M & \cdots & M \\ M & 0 & M & \cdots & M \\ M & M & 0 & \cdots & M \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ M & M & M & \cdots & 0 \end{pmatrix},$$

where M = M(A_G) is the detail matrix of A_G, then the eigenvalues of A_G satisfies $\frac{s^+}{s^-} = n - 1$ if M is positive semidefinte.

Proof. Assume $\boldsymbol{u}_i$ is the i^{th} eigenvector of M, and λ_i is the corresponding i^{th} eigenvalue of M, then

$$\begin{pmatrix} 0 & M & M & \cdots & M \\ M & 0 & M & \cdots & M \\ M & M & 0 & \cdots & M \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ M & M & M & \cdots & 0 \end{pmatrix}\begin{pmatrix} \boldsymbol{u}_i \\ \boldsymbol{u}_i \\ \boldsymbol{u}_i \\ \vdots \\ \boldsymbol{u}_i \end{pmatrix} = (n-1)\lambda_i \begin{pmatrix} \boldsymbol{u}_i \\ \boldsymbol{u}_i \\ \boldsymbol{u}_i \\ \vdots \\ \boldsymbol{u}_i \end{pmatrix}, \begin{pmatrix} 0 & M & M & \cdots & M \\ M & 0 & M & \cdots & M \\ M & M & 0 & \cdots & M \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ M & M & M & \cdots & 0 \end{pmatrix}\begin{pmatrix} \boldsymbol{u}_i \\ -\boldsymbol{u}_i \\ 0 \\ \vdots \\ 0 \end{pmatrix} = -\lambda_i \begin{pmatrix} \boldsymbol{u}_i \\ -\boldsymbol{u}_i \\ 0 \\ \vdots \\ 0 \end{pmatrix},$$

$$\begin{pmatrix} 0 & M & M & \cdots & M \\ M & 0 & M & \cdots & M \\ M & M & 0 & \cdots & M \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ M & M & M & \cdots & 0 \end{pmatrix}\begin{pmatrix} \boldsymbol{u}_i \\ 0 \\ -\boldsymbol{u}_i \\ \vdots \\ 0 \end{pmatrix} = -\lambda_i \begin{pmatrix} \boldsymbol{u}_i \\ 0 \\ -\boldsymbol{u}_i \\ \vdots \\ 0 \end{pmatrix}, \ldots, \begin{pmatrix} 0 & M & M & \cdots & M \\ M & 0 & M & \cdots & M \\ M & M & 0 & \cdots & M \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ M & M & M & \cdots & 0 \end{pmatrix}\begin{pmatrix} \boldsymbol{u}_i \\ 0 \\ 0 \\ \vdots \\ -\boldsymbol{u}_i \end{pmatrix} = -\lambda_i \begin{pmatrix} \boldsymbol{u}_i \\ 0 \\ 0 \\ \vdots \\ -\boldsymbol{u}_i \end{pmatrix}$$

If M is positive semidefinite, then λ_i is non-negative for all i, which implies,

$$\frac{s^+}{s^-} = \sum_{\lambda_i > 0} \frac{((n-1)\lambda_i)^2}{(-\lambda_i)^2 + (-\lambda_i)^2 + \ldots + (-\lambda_i)^2} = n - 1.$$

Proof of Theorem 1.1 For any graph G, when we re-number its vertices, the corresponding adjacency matrix will shift from A_G to PA_GP^T, where P is a permutation matrix.
Since A_G and PA_GP^T are similar, their eigenvalues are the same. Thus, if certain relationship for the eigenvalues of PA_GP^T holds, then the same relationship will also hold for the eigenvalues of A_G.

If G is a bipartite graph, then according to Lemma 2.1, the eigenvalues of PA_GP^T appear in pairs $(k, -k)$. This implies, for A_G,

$$\frac{s^+}{s^-} = \sum_{\lambda_i > 0} \frac{(\lambda_i)^2}{(-\lambda_i)^2} = 1,\ \chi(G) = 1 + \max\{\frac{s^+}{s^-}, \frac{s^-}{s^+}\}.$$

If G is a symmetric n-partite graph with its detail matrix M = *M(G)* being positive semidefinte, then according to Lemma 2.2, $\frac{s^+}{s^-} = n - 1$ for PA_GP^T. This implies, for A_G,

$$\chi(G) = 1 + \max\{\frac{s^+}{s^-}, \frac{s^-}{s^+}\}.$$

To conclude, the inequality of (1.1) is tight if G is a bipartite graph or a symmetric n-partite graph with its detail matrix M = M*(G)* being positive semidefinte.

Lemma 2.3 *For a real positive semidefinite matrix* $X = [X_{ij}]_{i,j=1}^r = \begin{pmatrix} F \\ G \\ \vdots \end{pmatrix} \begin{pmatrix} F^T & G^T & \cdots \end{pmatrix}$, *where* F, G, ... *are suitably partitioned matrices, it satisfies*

$$\|X\|^2 \le r \sum_{i=1}^r \|X_{ii}\|^2 \tag{2.1}$$

and equality holds if and only if the modules of the l^{th} *column of* F, G, ... *are all the same for any possible* l.

Proof. The inequality in (2.1) can be proven by [1, Lemma 2.1] and [2, p. 209].
The only part that consists of inequality in [1, Lemma 2.1] is that, since $X = [X_{ij}]_{i,j=1}^r$ is real positive semidefinite, so is $\begin{pmatrix} X_{ii} & X_{ij} \\ X_{ij}^T & X_{jj} \end{pmatrix}$, which satisfies

$$\|X_{ij}\|^2 \le \|X_{ii}\| \|X_{jj}\| \le \frac{\|X_{ii}\|^2 + \|X_{jj}\|^2}{2}. \tag{2.2}$$

Thus, in order to prove our desired argument, we need to find the condition that holds the equality of (2.2).

Let $\begin{pmatrix} X_{ii} & X_{ij} \\ X_{ij}^T & X_{jj} \end{pmatrix} = \begin{pmatrix} D \\ E \end{pmatrix} \begin{pmatrix} D^T & E^T \end{pmatrix}$, where D and E are suitably partitioned matrices, so

$$X_{ii} = DD^T,\ X_{jj} = EE^T,\ X_{ij}=DE^T.$$

Let $D = [d_{ij}]_{i,j=1,1}^{m,k}$, $E = [e_{ij}]_{i,j=1,1}^{n,k}$, then

$$[X_{ii}]_{gh} = \sum_{f=1}^{k} d_{gf}d_{hf},\ [X_{jj}]_{gh} = \sum_{f=1}^{k} e_{gf}e_{hf},\ [X_{ij}]_{gh} = \sum_{f=1}^{k} d_{gf}e_{hf}.$$

We first focus on the first half of (2.2).

$$\|X_{ij}\|^2 \le \|X_{ii}\|\|X_{jj}\|,$$

$$\sum_{f=1}^{k}\sum_{g=1}^{m}\sum_{h=1}^{n}(d_{gf}e_{hf})^2 \le \sqrt{\sum_{f=1}^{k}\sum_{g=1}^{m}\sum_{h=1}^{m}(d_{gf}d_{hf})^2}\sqrt{\sum_{f=1}^{k}\sum_{g=1}^{n}\sum_{h=1}^{n}(e_{gf}e_{hf})^2},$$

$$\sum_{f=1}^{k}\left(\sum_{g=1}^{m}(d_{gf})^2\sum_{h=1}^{n}(e_{hf})^2\right) \le \sqrt{\sum_{f=1}^{k}\left(\sum_{g=1}^{m}(d_{gf})^2\right)^2}\sqrt{\sum_{f=1}^{k}\left(\sum_{h=1}^{n}(e_{hf})^2\right)^2}.$$

This implies, with $u_f := \sum_{g=1}^{m}(d_{gf})^2$, $w_f := \sum_{h=1}^{n}(d_{hf})^2$,

$$\sum_{f=1}^{k} u_f w_f \le \sqrt{\sum_{f=1}^{k} u_f^2}\sqrt{\sum_{f=1}^{k} w_f^2}. \tag{2.3}$$

Since the equality of (2.3) holds if and only if $u_f = \lambda w_f$, then the first half equality of (2.2) holds if and only if $u_f = \lambda w_f$.

We then focus on the second half of (2.1).

$$\|X_{ii}\|\|X_{jj}\| \le \frac{\|X_{ii}\|^2+\|X_{jj}\|^2}{2}. \tag{2.4}$$

Since the equality of (2.4) holds if and only if $\|X_{ii}\| = \|X_{jj}\|$, then the second half equality of (2.2) holds if and only if $\|X_{ii}\| = \|X_{jj}\|$.

That implies,

$$\sqrt{\sum_{f=1}^{k}\sum_{g=1}^{m}\sum_{h=1}^{m}(d_{gf}d_{hf})^2} = \sqrt{\sum_{f=1}^{k}\sum_{g=1}^{n}\sum_{h=1}^{n}(e_{gf}e_{hf})^2},$$

$$\sum_{f=1}^{k} u_f^2 = \sum_{f=1}^{k} w_f^2.$$

Then we combine these two auxiliary results, which implies,

$$\sum_{f=1}^{k} u_f^2 = \sum_{f=1}^{k} \lambda^2 u_f^2,$$

$$\lambda = 1.$$

Therefore, the modules of the l^{th} column of D and E are all the same for any possible l.

Similar arguments hold for any i and j, so for a real positive semidefinite matrix $X = [X_{ij}]_{i,j=1}^{r} = \begin{pmatrix} F \\ G \\ \vdots \end{pmatrix} \begin{pmatrix} F^T & G^T & \cdots \end{pmatrix}$, the equality of (2.1) holds if and only if the modules of the l^{th} column of F, G, ... are all the same for any possible l.

Lemma 2.4 *Let $X = [X_{ij}]_{i,j=1}^{r}$ and $Y = [Y_{ij}]_{i,j=1}^{r}$ be two real positive semidefinite matrices conformally partitioned. If the diagonal blocks of X and Y coincide and $XY = 0$, then*

$$\|X\|^2 \le (r-1)\|Y\|^2 \tag{2.5}$$

and the equality holds if and only if all the elements of the non-diagonal blocks of X is the fixed negative multiple of the corresponding elements of the non-diagonal blocks of Y.

Proof. The inequality in (2.5) can be proven by [1, Theorem 2.2].
The only part that consists of inequality in [1, Theorem 2.2] is that, for real positive semidefinite matrices *X* and *Y,*

$$\sum_{i=1}^{r}\|X_{ii}\|^2 = -\sum_{i\neq j}\operatorname{tr}(X_{ij}^T Y_{ij}) \le \sum_{i\neq j}\|X_{ij}\|\|Y_{ij}\| \le \sqrt{\sum_{i\neq j}\|X_{ij}\|^2}\sqrt{\sum_{i\neq j}\|Y_{ij}\|^2}. \tag{2.6}$$

Thus, in order to prove our desired argument, we need to find the condition that holds the equality of (2.6).

Let $X_{ij} = [[X_{ij}]_{ab}]_{a,b=1,1}^{m_{ij},n_{ij}}$, $Y_{ij} = [[Y_{ij}]_{ab}]_{a,b=1,1}^{m_{ij},n_{ij}}$, then

$$-\sum_{i\neq j}\sum_{a=1}^{m_{ij}}\sum_{b=1}^{n_{ij}}[X_{ij}]_{ab}[Y_{ij}]_{ab} \leq \sqrt{\sum_{i\neq j}\sum_{a=1}^{m_{ij}}\sum_{b=1}^{n_{ij}}[X_{ij}]_{ab}^2}\sqrt{\sum_{i\neq j}\sum_{a=1}^{m_{ij}}\sum_{b=1}^{n_{ij}}[Y_{ij}]_{ab}^2}. \tag{2.7}$$

Since the equality of (2.7) holds if and only if $[X_{ij}]_{ab} = \sigma[Y_{ij}]_{ab}$ where $i \neq j$, $1 \leq a \leq m_{ij}$, $1 \leq b \leq n_{ij}$, $\sigma \leq 0$, the equality of (2.5) holds if and only if all the elements of the non-diagonal blocks of X is the fixed negative multiple of the corresponding elements of the non-diagonal blocks of Y.

Proof of Theorem 1.2 we first prove the necessity of Theorem 1.2.
According to Lemma 2.3, 2.4 and [1, Proof of Conjecture 1.1], for any graph G, after its vertices being re-numbered and corresponding positive-eigen matrix B = B(PA_GP^T) and negative-eigen matrix $C = C(PA_GP^T)$ being suitably partitioned, if the aforementioned three conditions hold, then the equalities of (2.1) and (2.5) will hold, resulting in that the equality of (1.1) will hold.

We then prove the sufficiency of Theorem 1.2.
According to Lemma 2.3, 2.4 and [1, Proof of Conjecture 1.1], for any graph G, if the equality of (1.1) holds, then the equalities of (2.1) and (2.5) must hold. In such way, after the vertices of G being re-numbered and corresponding positive-eigen matrix B = B(PA_GP^T) and negative-eigen matrix $C = C(PA_GP^T)$ being suitably partitioned, the aforementioned three conditions must hold.

To conclude, the equality of (1.1) holds for G if and only if there exists a permutation matrix P such that PA_GP^T satisfies the following conditions:

• PA_GP^T is partitioned into $\chi(G) \times \chi(G)$ blocks $PA_GP^T = [(PA_GP^T)_{ij}]_{i,j=1}^{\chi(G)}$ with $(PA_GP^T)_{11}$, $(PA_GP^T)_{22}$, ..., $(PA_GP^T)_{\chi(G)\chi(G)}$ being zero matrices;

• if the corresponding positive-eigen matrix B = B(PA_GP^T) and negative-eigen matrix C = C(PA_GP^T) are expressed in the forms of

$$\begin{pmatrix} F \\ G \\ \vdots \end{pmatrix}\left(F^T\ G^T\ \cdots\right),$$

where F, G, ... are suitably partitioned matrices, then the modules of the l^{th} column of F, G, ... are all the same for any possible l;

• if B and C are partitioned into $\chi(G) \times \chi(G)$ blocks as the way PA_GP^T does, then all the elements of the non-diagonal blocks of B is the fixed negative multiple of the corresponding elements of the non-diagonal blocks of C.

4. Reference

[1] Ando, Tsuyoshi, and Minghua Lin. "Proof of a conjectured lower bound on the chromatic number of a graph." Linear Algebra and its Applications, vol. 485, 15 Nov. 2015, pp. 480-84.
[2] Horn, Roger A., and Charles R. Johnson. Topics in Matrix Analysis. Cambridge University Press, 1991, p. 209.
[3] Wocjan, Pawel, and Clive Elphick. "New spectral bounds on the chromatic number encompassing all eigenvalues of the adjacency matrix." Electronic Journal of Combinatorics, vol. 20, no. 3, 14 Sept. 2012, p. 39.