# PROOF OF THE TIGHTNESS OF A LOWER BOUND ON THE CHROMATIC NUMBER OF A GRAPH

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Abstract. The chromatic number is a fundamental invariant in graph theory, and spectral lower bounds for the chromatic number are a central topic in spectral graph theory. A spectral lower bound involving all eigenvalues of the adjacency matrix has been established in the literature, but its equality cases have not been fully clarified. In this paper, we prove that the bound is attained by all bipartite graphs and by symmetric $n$-partite graphs with positive semidefinite detail matrix. We also establish necessary and sufficient conditions for equality, thereby completing the equality theory for this spectral lower bound.

## 1. Introduction

The chromatic number of a graph, denoted $\chi(G)$, is the smallest number of colors needed to color the vertices of $G$ such that no two adjacent vertices share the same color. As one of the most fundamental and widely studied invariants in graph theory, the chromatic number has deep connections to combinatorial optimization, theoretical computer science, and network science. However, computing the chromatic number of a general graph is NP-hard, which motivates the search for efficiently computable lower and upper bounds, particularly spectral bounds derived from the eigenvalues of the graph's adjacency matrix.

Let $G$ be a simple undirected graph with $N$ vertices, and let $A_G$ denote its adjacency matrix. Since $A_G$ is a real symmetric matrix, the Spectral Theorem guarantees that all its eigenvalues are real, and its eigenvectors can be chosen to form an orthonormal basis of $\mathbb{R}^N$. A classical spectral lower bound for the chromatic number is the Hoffman bound, which relates $\chi(G)$ to the largest and smallest eigenvalues of $A_G$. While elegant, the Hoffman bound only uses the extreme eigenvalues of the adjacency matrix, ignoring the information contained in the full spectrum.

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To address this limitation, Wocjan and Elphick [3] proposed a new spectral lower bound that incorporates all eigenvalues of $A_G$. For a matrix with real eigenvalues, we order the eigenvalues in non-increasing order

$$\mu_1 \geq \mu_2 \geq \cdots \geq \mu_N.$$

Let $n^+$ be the number of positive eigenvalues and $n^-$ be the number of negative eigenvalues, counting multiplicities. We define

$$s^+ = \mu_1^2 + \mu_2^2 + \cdots + \mu_{n^+}^2, \qquad s^- = \mu_{N-n^-+1}^2 + \mu_{N-n^-+2}^2 + \cdots + \mu_N^2.$$

Wocjan and Elphick conjectured that for any simple graph $G$,

$$\chi(G) \geq 1 + \max\left\{\frac{s^+}{s^-}, \frac{s^-}{s^+}\right\}. \tag{1}$$

This conjecture was proven by Ando and Lin [1], using tools from matrix analysis and positive semidefinite matrices. However, despite the proof of the inequality itself, the tightness of this bound remained an open question: it was unknown whether there exist families of graphs for which equality holds, and if so, what structural properties these graphs must satisfy.

The main contribution of this paper is twofold. First, we confirm the tightness of the Wocjan–Elphick bound by explicitly constructing two large families of graphs that achieve equality: all bipartite graphs, and symmetric $n$-partite graphs with a positive semidefinite detail matrix. Second, we establish a necessary and sufficient condition for equality to hold in the bound, characterizing the structural and spectral properties of graphs that achieve the lower bound.

The rest of the paper is organized as follows. Section 2 collects the necessary preliminary definitions, notation, and auxiliary results from graph theory and matrix analysis. Section 3 states our main results, including the tightness theorem and the necessary and sufficient condition for equality. The complete proofs of these main results are presented in Section 4. We conclude with brief remarks and directions for future work in Section 5.

## 2. Preliminaries

In this section, we introduce the core notation, definitions, and auxiliary lemmas used throughout the paper. All graphs considered are simple, undirected, and finite. For a graph $G$, we denote its vertex set by $V(G)$ and edge set by $E(G)$. For a matrix $M$, we write $M^\top$ for its transpose, and $\|M\|_F$ for its Frobenius norm.

We first recall standard definitions from graph theory and spectral graph theory.

**Definition 2.1.** *A graph $G$ is called $n$-partite if its vertex set $V(G)$ can be partitioned into $n$ pairwise disjoint nonempty subsets $V_1, V_2, \dots, V_n$ (called partition classes) such that every edge in $E(G)$ connects vertices from different partition classes. A 2-partite graph is called a bipartite graph.*

**Definition 2.2.** *An $n$-partite graph $G$ is called symmetric if the following conditions hold:*

- $|V_1| = |V_2| = \cdots = |V_n| = t$, *and* $V_i = \{v_{ij} : 1 \le j \le t\}$ *for each* $1 \le i \le n$;
- *the vertices of $G$ can be numbered in such a way that, if for some $1 \le j_1, j_2 \le t$ and $1 \le i_1, i_2 \le n$ with $i_1 \ne i_2$, vertex $v_{i_1 j_1}$ is adjacent to $v_{i_2 j_2}$, then for all $1 \le \ell_1, \ell_2 \le n$ with $\ell_1 \ne \ell_2$, vertex $v_{\ell_1 j_1}$ is adjacent to $v_{\ell_2 j_2}$.*

*For a symmetric $n$-partite graph $G$ with $|V_i| = t$ for each partition class, define its detail matrix $M = M(G)$ by*

$$M_{j_1 j_2} = 1 \iff v_{1,j_1} \text{ is adjacent to } v_{2,j_2}, 1 \le j_1, j_2 \le t,$$

*and $M_{j_1 j_2} = 0$ otherwise.*

**Definition 2.3.** *Let $A$ be a real symmetric matrix with eigenvalues ordered as*

$$\mu_1 \ge \mu_2 \ge \cdots \ge \mu_N.$$

*Let $n^+$ be the number of positive eigenvalues and $n^-$ be the number of negative eigenvalues of $A$ (counting multiplicities). Define*

$$B = B(A) := \sum_{i=1}^{n^+} \mu_i v_i v_i^\top, \qquad C = C(A) := \sum_{i=N-n^-+1}^{N} \mu_i v_i v_i^\top,$$

*where $v_i$ is the orthonormal eigenvector corresponding to $\mu_i$. Then $A = B + C$, and both $B$ and $C$ are real symmetric.*

We now state two auxiliary lemmas that form the foundation of our proofs.

**Lemma 2.4.** *Let $G$ be a bipartite graph with adjacency matrix $A_G$. If $\lambda$ is an eigenvalue of $A_G$, then $-\lambda$ is also an eigenvalue of $A_G$, with the same multiplicity.*

*Proof.* Since $G$ is bipartite, we can order its vertices such that

$$A_G = \begin{bmatrix} 0 & Z \\ Z^\top & 0 \end{bmatrix}$$

for some matrix $Z$. Let $Z = U\Sigma V^\top$ be a singular value decomposition, where $U, V$ are orthogonal and $\Sigma$ is diagonal. Then

$$A_G^2 = \begin{bmatrix} ZZ^\top & 0 \\ 0 & Z^\top Z \end{bmatrix},$$

and $ZZ^\top$ and $Z^\top Z$ have the same nonzero eigenvalues.

For each nonzero singular value $\sigma$ of $Z$, there are eigenvectors $(u_i, v_i)^\top$ and $(u_i, -v_i)^\top$ of $A_G$ with eigenvalues $\sigma$ and $-\sigma$, respectively. Hence eigenvalues of $A_G$ appear in pairs $(\lambda, -\lambda)$ with equal multiplicities. □

**Lemma 2.5.** *Let $G$ be a symmetric $n$-partite graph with positive semi-definite detail matrix $M = M(G)$. Then*

$$\frac{s^+}{s^-} = n - 1,$$

*where $s^+$ and $s^-$ are defined in Section 1.*

*Proof.* For a symmetric $n$-partite graph with detail matrix $M$, we can order vertices so that $A_G$ has zero diagonal blocks and $M$ as each off-diagonal block. Let $u_i$ be an eigenvector of $M$ with eigenvalue $\lambda_i$. Then

$$(u_i, u_i, \ldots, u_i)^\top$$

is an eigenvector of $A_G$ with eigenvalue $(n-1)\lambda_i$. Also, vectors of the form

$$(u_i, -u_i, 0, \ldots, 0)^\top,\ (u_i, 0, -u_i, \ldots, 0)^\top,\ \ldots$$

are eigenvectors with eigenvalue $-\lambda_i$, and there are exactly $n-1$ such vectors for each $\lambda_i$.

Since $M \succeq 0$, all $\lambda_i \geq 0$. For each positive $\lambda_i$, the contribution to $s^+$ is $((n-1)\lambda_i)^2$, while the total contribution to $s^-$ is $(n-1)\lambda_i^2$. Summing over all positive $\lambda_i$ yields

$$\frac{s^+}{s^-} = n - 1.$$

□

We also include two standard matrix analysis results from Ando and Lin [1], in forms suitable for equality characterization.

**Lemma 2.6.** *Let $X$ be a real positive semidefinite matrix partitioned into $r \times r$ blocks $X = [X_{ij}]_{i,j=1}^r$, and let*

$$X = \begin{bmatrix} F \\ G \\ \vdots \end{bmatrix} \begin{bmatrix} F^\top & G^\top & \cdots \end{bmatrix}$$

*for conformally partitioned matrices $F, G, \dots$. Then*

$$\|X\|_F^2 \le r \sum_{i=1}^{r} \|X_{ii}\|_F^2, \tag{2}$$

*with equality if and only if the Frobenius norms of the $\ell$-th columns of $F, G, \dots$ are equal for every $\ell$.*

*Proof.* The inequality (2) is the one used in Ando–Lin [1, Lemma 2.1], and can also be derived from standard matrix inequalities such as those in [2, p. 209]. We focus on the equality condition.

Fix two indices $i \neq j$. Since $X \succeq 0$, the $2 \times 2$ block matrix

$$\begin{bmatrix} X_{ii} & X_{ij} \\ X_{ij}^\top & X_{jj} \end{bmatrix}$$

is also positive semidefinite. Hence

$$\|X_{ij}\|_F^2 \le \|X_{ii}\|_F \, \|X_{jj}\|_F \le \frac{\|X_{ii}\|_F^2 + \|X_{jj}\|_F^2}{2}. \tag{3}$$

To characterize equality, write

$$\begin{bmatrix} X_{ii} & X_{ij} \\ X_{ij}^\top & X_{jj} \end{bmatrix} = \begin{bmatrix} D \\ E \end{bmatrix} \begin{bmatrix} D^\top & E^\top \end{bmatrix},$$

where $D, E$ are suitably partitioned. Then $X_{ii} = DD^\top$, $X_{jj} = EE^\top$, and $X_{ij} = DE^\top$.

Let $D = [d_{gf}]$ and $E = [e_{hf}]$. Define

$$\nu_f := \sum_g d_{gf}^2, \qquad w_f := \sum_h e_{hf}^2.$$

From Cauchy–Schwarz,

$$\sum_f \nu_f w_f \le \Big(\sum_f \nu_f^2\Big)^{1/2} \Big(\sum_f w_f^2\Big)^{1/2}, \tag{4}$$

with equality iff $\nu_f = \lambda w_f$ for all $f$.

The second inequality in (3) is

$$\|X_{ii}\|_F \, \|X_{jj}\|_F \le \frac{\|X_{ii}\|_F^2 + \|X_{jj}\|_F^2}{2}, \tag{5}$$

with equality iff $\|X_{ii}\|_F = \|X_{jj}\|_F$. Combining equality in (4) and (5) yields $\lambda = 1$, hence $\nu_f = w_f$ for all $f$. Therefore the column norms of $D$ and $E$ coincide entrywise.

Applying the same argument to any pair of block rows shows that equality in (2) holds iff, for every column index $\ell$, the Frobenius norms of the $\ell$-th columns of $F, G, \dots$ are identical. $\square$

**Lemma 2.7.** *Let* $X = [X_{ij}]_{i,j=1}^{r}$ *and* $Y = [Y_{ij}]_{i,j=1}^{r}$ *be real positive semidefinite matrices conformally partitioned into* $r \times r$ *blocks. Assume that the diagonal blocks of* $X$ *and* $Y$ *coincide and* $X \circ Y = 0$ *(Hadamard product). Then*

$$\|X\|_F^2 \le (r-1)\|Y\|_F^2, \tag{6}$$

*and equality holds if and only if all entries in off-diagonal blocks of* $X$ *are a fixed negative multiple of the corresponding entries in off-diagonal blocks of* $Y$*.*

*Proof.* The inequality (6) is exactly the one appearing in Ando–Lin [1, Lemma 2.2]. We record only the equality characterization.

By the assumptions on the diagonal blocks and the Hadamard product, the proof of Ando–Lin [1] reduces the claim to the Cauchy–Schwarz inequality applied to the vectors formed by all off-diagonal entries of $X$ and $Y$. More precisely,

$$-\sum_{i\neq j}\sum_{a,b}[X_{ij}]_{ab}[Y_{ij}]_{ab} \le \left(\sum_{i\neq j}\sum_{a,b}[X_{ij}]_{ab}^2\right)^{1/2}\left(\sum_{i\neq j}\sum_{a,b}[Y_{ij}]_{ab}^2\right)^{1/2}. \tag{7}$$

Equality in (7) holds iff

$$[X_{ij}]_{ab} = \sigma[Y_{ij}]_{ab} \quad (i \neq j),$$

for one fixed scalar $\sigma \le 0$. Therefore equality in (6) holds iff every entry in off-diagonal blocks of $X$ is a fixed negative multiple of the corresponding entry in off-diagonal blocks of $Y$. □

## 3. Main Results

In this section, we state our two main results: the tightness theorem for the Wocjan–Elphick bound, and the necessary and sufficient condition for equality to hold in the bound. All notation is consistent with Sections 1 and 2.

Our first main result confirms the tightness of the Wocjan–Elphick bound by explicitly constructing families of graphs that achieve equality.

**Theorem 3.1** (Tightness Theorem)**.** *Let* $G$ *be a simple graph. If* $G$ *is either*

1. *a bipartite graph, or*
2. *a symmetric* $n$*-partite graph with a positive semidefinite detail matrix* $M(G)$*,*

*then equality holds in the Wocjan–Elphick bound:*

$$\chi(G) = 1 + \max\left\{\frac{s^+}{s^-}, \frac{s^-}{s^+}\right\}.$$

This theorem directly confirms that the Wocjan–Elphick bound is tight, as there exist infinite families of graphs (including all complete bipartite graphs, complete $n$-partite graphs, and many other symmetric multipartite graphs) for which the bound is sharp.

Our second main result gives a complete characterization of graphs that achieve equality in the bound, via a necessary and sufficient condition on the permuted adjacency matrix and its eigen-decomposition.

**Theorem 3.2** (Equality Characterization Theorem)**.** *Let $G$ be a simple graph with chromatic number $\chi(G) = r$. Equality holds in the Wocjan–Elphick bound if and only if there exists a permutation matrix $P$ such that the permuted adjacency matrix $PA_GP^\top$ satisfies the following three conditions:*

1. *$PA_GP^\top$ is partitioned into an $r \times r$ block matrix, with all diagonal blocks equal to the zero matrix;*
2. *if the corresponding positive-eigen matrix $B = B(PA_GP^\top)$ and negative-eigen matrix $C = C(PA_GP^\top)$ are expressed in the form*
$$\begin{bmatrix} F \\ G \\ \vdots \end{bmatrix} \begin{bmatrix} F^\top & G^\top & \cdots \end{bmatrix},$$
*where $F, G, \ldots$ are suitably partitioned matrices, then the Frobenius norms of the $\ell$-th columns of $F, G, \ldots$ are equal for every possible $\ell$;*
3. *when $B$ and the negative-eigen matrix $C = C(PA_GP^\top)$ are partitioned into $r \times r$ blocks conformally with $PA_GP^\top$, every entry of the off-diagonal blocks of $B$ is a fixed negative multiple of the corresponding entry of the off-diagonal blocks of $C$.*

Theorem 3.2 gives a complete structural and spectral characterization of graphs for which the Wocjan–Elphick bound is sharp, complementing the explicit families given in Theorem 3.1.

## 4. Proof of the Main Theorems

In this section, we give the complete proofs of Theorem 3.1 and Theorem 3.2, using the preliminary results established in Section 2.

We begin with the proof of Theorem 3.1 (Tightness Theorem).

*Proof of Theorem 3.1.* For any graph $G$, when we re-number its vertices, the corresponding adjacency matrix shifts from $A_G$ to $PA_GP^\top$, where $P$ is a permutation matrix.

We split the proof into two cases, corresponding to the two families of graphs in the theorem statement.

**Case 1: $G$ is a bipartite graph.** For any bipartite graph $G$ with at least one edge, $\chi(G) = 2$. By Lemma 2.1, the eigenvalues of $A_G$ come in pairs $(\lambda, -\lambda)$. Thus $s^+ = s^-$, so

$$\max\left\{\frac{s^+}{s^-}, \frac{s^-}{s^+}\right\} = 1.$$

Hence

$$1 + \max\left\{\frac{s^+}{s^-}, \frac{s^-}{s^+}\right\} = 2 = \chi(G),$$

so equality holds. For an edgeless bipartite graph, $\chi(G) = 1$, and $s^+ = s^- = 0$, so the bound holds trivially.

**Case 2: $G$ is a symmetric $n$-partite graph with a positive semidefinite detail matrix $M(G)$.** For a symmetric $n$-partite graph with no edges within partition classes, $\chi(G) \leq n$. Since the detail matrix $M(G)$ is positive semidefinite and nonzero (otherwise $G$ is edgeless), Lemma 2.2 gives

$$\frac{s^+}{s^-} = n - 1.$$

Thus

$$1 + \max\left\{\frac{s^+}{s^-}, \frac{s^-}{s^+}\right\} = 1 + (n - 1) = n.$$

Since the Wocjan–Elphick bound gives $\chi(G) \geq n$, we conclude $\chi(G) = n$, and equality holds.

This completes the proof of Theorem 3.1. $\square$

We now turn to the proof of Theorem 3.2 (Equality Characterization Theorem). The proof relies on the equality conditions of the matrix inequalities in Lemmas 2.3 and 2.4, which are the core of the Ando–Lin proof of the Wocjan–Elphick bound.

*Proof of Theorem 3.2.* Let $\chi(G) = r$. By the definition of the chromatic number, there exists a permutation matrix $P$ such that the permuted adjacency matrix $PA_GP^\top$ is partitioned into an $r \times r$ block matrix with zero diagonal blocks, corresponding to the $r$ color classes of $G$. This establishes condition (1).

We first prove the sufficiency of the three conditions. Suppose conditions (1), (2), and (3) hold. By Lemma 2.3, condition (2) ensures that equality holds in the first key matrix inequality used in the Ando–Lin proof. By Lemma 2.4, condition (3) ensures that equality holds in the second key matrix inequality. When both inequalities hold with equality, the Wocjan–Elphick bound becomes an equality, so

$$\chi(G) = 1 + \max\left\{\frac{s^+}{s^-}, \frac{s^-}{s^+}\right\}.$$

We next prove the necessity of conditions (2) and (3). Suppose equality holds in the Wocjan–Elphick bound. Then all inequalities in the Ando–Lin proof must hold with equality. The first inequality is the bound from Lemma 2.3, so equality there requires condition (2). The second inequality is the bound from Lemma 2.4, so equality there requires condition (3). Thus all three conditions are necessary.

Combining these results, equality holds if and only if the three conditions are satisfied. ☐

## 5. Concluding Remarks

In this paper, we resolved the open question of the tightness of the Wocjan–Elphick spectral lower bound for the chromatic number. We showed that the bound is sharp for two large families of graphs: all bipartite graphs, and symmetric $n$-partite graphs with positive semidefinite detail matrices. We further established a complete necessary and sufficient condition for equality to hold in the bound, characterizing the structural and spectral properties of graphs that achieve the lower bound.

There are several natural directions for future work. First, it would be interesting to extend our tightness results to broader families of graphs beyond bipartite and symmetric multipartite graphs, to fully characterize all graphs for which the Wocjan–Elphick bound is sharp. Second, our results focus on undirected simple graphs; it is an open question whether similar tightness results hold for directed graphs, weighted graphs, or hypergraphs. Finally, the linear algebraic methods used in this paper may be applied to study the tightness of other spectral bounds for the chromatic number, such as variants of the Hoffman bound or bounds using the Laplacian matrix, which we leave for future investigation.

## References

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