

MATH 40002

Analysis

GROUPS

• countability a set S is countably infinite if and only if there exists a bijection $f: \mathbb{N} \rightarrow S$

* intuition: we can make a list $S = \{s_1, s_2, \dots\} \forall i, j \in \mathbb{N}, i \neq j, s_i \neq s_j$, such that $\forall n \in \mathbb{N}, s_n = f(n)$

① assume $S \subseteq \mathbb{N}$ is infinite, then S is countably infinite.

proof:

• we first construct a function

define $f: \mathbb{N} \rightarrow S$ such that $f(0) = \min(S)$, and if $f(0), f(1), \dots, f(n-1)$ are defined already, $f(n) = \min(S - \{f(0), f(1), \dots, f(n-1)\})$. (achievable due to well-ordering rule)

• we then prove f is injective

$f(0) < f(1) < f(2) < \dots$, so f is injective

• we then prove f is surjective

assume f is not surjective $\Rightarrow S - \text{Im}(f) \neq \emptyset, S - \text{Im}(f) \subseteq \mathbb{N}$

$\Rightarrow S - \text{Im}(f) \neq \emptyset, S - \text{Im}(f) \subseteq \mathbb{N}$

$\Rightarrow \exists$ the least element $s \in S - \text{Im}(f)$, and $s \neq \min(S), s \neq f(0)$. (achievable due to well-ordering rule)

$\Rightarrow \exists$ the largest element $s' \in S, s' < s, s' \in \text{Im}(f)$. then by definition, if $s' = f(n_0), s = f(n_0 + 1)$

$\Rightarrow$ contradiction, so f is surjective

② $\mathbb{Q}^{>0}$ is countably infinite product of prime: 去要保
证 prime 个数 finite! (列出素数)

proof

我们固然可以用坐标轴证明其可以被列出, 但我们这里还有种别的方法 $f(\frac{a}{b}) = 2^n 3^m$ [值得学习!]

③ let S be an infinite set, then the following are equivalent:

(1) S is countable

(2) $\exists$ injection $f: S \rightarrow \mathbb{N}$

proof:

- prove " $\Rightarrow$ "

S is countable $\Rightarrow \exists$ bij $\Rightarrow \exists$ bijective $f: \mathbb{N} \rightarrow S$
 $\Rightarrow \exists$ bijective $f^{-1}: S \rightarrow \mathbb{N} \Rightarrow \exists$ injection $f^{-1}: S \rightarrow \mathbb{N}$

- prove " $\Leftarrow$ "

$\exists$ injection $f: S \rightarrow \mathbb{N}$, S is infinite

$\Rightarrow A = \{f(s) | s \in S\}$, $A \subseteq \mathbb{N}$, then it can be re-ordered as

$A = \{a_0, a_1, \dots\}$ such that $a_i < a_j$ if $i < j$

$\Rightarrow$ then there exist a bijection $h: \mathbb{N} \rightarrow A$, $h(n) = a_n$

$\Rightarrow \exists$ bijection $h \circ f: S \rightarrow \mathbb{N}$

$\Rightarrow S$ is countable

- completeness supremum: least upper bound
infimum: greatest lower bound

- sequences (1) a sequence is a map $f: \mathbb{N} \rightarrow \mathbb{R}$, equivalently, a sequence is an infinite list of real numbers.

$a_1, a_2, a_3, \dots$

with a real number a_n for each $n \in \mathbb{N}$

(2) $(\forall \varepsilon > 0, \exists N \in \mathbb{N}, \text{ such that } \forall n \geq N, |a_n - L| < \varepsilon)$

$\Leftrightarrow (\lim_{n \rightarrow \infty} a_n = L)$

注意: 当证明一个 sequence diverge 时, 除了用 contradiction, 也可以用 converse of convergence

$\forall L \in \mathbb{R}, \exists \varepsilon > 0, \forall N \in \mathbb{N}, \text{ such that } \exists n \geq N, |a_n - L| \geq \varepsilon$

(3) (a_n) diverges $\nRightarrow a_n$ goes to $\pm\infty$

(3) $(\forall R \in \mathbb{R}, \exists N \in \mathbb{N}, \text{ such that } \forall n \geq N, a_n > R) \Leftrightarrow (a_n \rightarrow \infty)$

(4) $L \in \mathbb{R}, s \in \mathbb{N}$; then $(a_n \rightarrow L) \Leftrightarrow (a_{n+s} \rightarrow L)$

2.

(5) if $a_n \rightarrow L$ and $a_n \rightarrow M$, then $L = M \iff (a_n \rightarrow \infty)$

(6) (a_n) is bounded above $\iff (\exists B \in \mathbb{R}, \forall n \in \mathbb{N}, a_n \leq B)$

(a_n) is bounded below $\iff (\exists B \in \mathbb{R}, \forall n \in \mathbb{N}, B \leq a_n)$

(a_n) is bounded $\iff (\exists B \in \mathbb{R}, \forall n \in \mathbb{N}, |a_n| \leq B)$

注意: bounded $\neq$ convergent

(6) if $a_n \rightarrow L$ and $a_n \rightarrow M$, then $L = M$

(7) if (a_n) is convergent, then it is bounded.

proof:

$\epsilon = 1, \exists N \in \mathbb{N}, \forall n \geq N, |a_n - L| < 1, |a_n| < 1 + |L|$

take $B = \max\{|a_1|, |a_2|, \dots, |1 + |L||\}$, then $\forall n \in \mathbb{N}, |a_n| < B$

(8) if $\forall n \in \mathbb{N}, a_n \geq b_n$, and $a_n \rightarrow L, b_n \rightarrow M$, then $L \geq M$

proof:

$\forall \epsilon > 0, \exists N_1, N_2 \in \mathbb{N}, \forall n \geq N_1, n \geq N_2, |a_n - L| < \epsilon, |b_n - M| < \epsilon$

let $N = \max\{N_1, N_2\}, \forall n \geq N, |a_n - L| < \epsilon, |b_n - M| < \epsilon$

so $L - \epsilon < a_n < L + \epsilon, M - \epsilon < b_n < M + \epsilon$

$M - \epsilon < b_n \leq a_n < L + \epsilon$

assume $L < M$, take $\epsilon = \frac{M-L}{2} > 0$, then

$b_n > \frac{M+L}{2}, a_n < \frac{M+L}{2}$, contradict

三明治定理

(9) if $a_n \rightarrow L, b_n \rightarrow L$, and $\forall n \in \mathbb{N}, a_n \leq c_n \leq b_n$, then $c_n \rightarrow L$

● assume $a_n \rightarrow L$ and $b_n \rightarrow M$

(1) $a_n + b_n \rightarrow L + M$

(2) $|a_n| \rightarrow |L|$

(3) $\forall \lambda \in \mathbb{R}, \lambda a_n \rightarrow \lambda L$

(4) $a_n b_n \rightarrow LM$

(5) if $M \neq 0, b_n \neq 0, \frac{a_n}{b_n} \rightarrow \frac{L}{M}$

proof:

注意:

$$\frac{a_n \rightarrow L, b_n \rightarrow M}{a_n b_n \rightarrow LM}$$

$$\begin{aligned}
 (4) \quad |a_n b_n - ML| &= |a_n b_n + M a_n - M a_n - L M| \\
 &= |a_n(b_n - M) + M(a_n - L)| \\
 &\leq |a_n| |b_n - M| + |M| |a_n - L| \\
 &< |a_n| \epsilon + |M| \epsilon \\
 &< (c + |M|) \epsilon, \text{ where } c \text{ is upper bound of } a_n
 \end{aligned}$$

(5) we first show that $\frac{1}{b_n} \rightarrow \frac{1}{M}$

$$\begin{aligned}
 \left| \frac{1}{b_n} - \frac{1}{M} \right| &= \left| \frac{M - b_n}{b_n M} \right| = \frac{|M - b_n|}{|b_n M|} < \frac{\epsilon}{|b_n| |M|} \\
 &< \frac{2\epsilon}{|M|^2} \text{ as for } n \text{ large enough, } |b_n| > \frac{|M|}{2}
 \end{aligned}$$

using when n large enough

• (1) if $a_n \rightarrow \infty$, b_n bounded below, then $a_n + b_n \rightarrow \infty$

(2) if $a_n \rightarrow \infty$, $\lambda \in \mathbb{R}$, $\lambda a_n \rightarrow \begin{cases} \infty & \text{if } \lambda > 0 \\ 0 & \text{if } \lambda = 0 \\ -\infty & \text{if } \lambda < 0 \end{cases}$

(3) if a_n bounded, $b_n \rightarrow \infty$, $b_n \neq 0$, $\frac{a_n}{b_n} \rightarrow 0$

(4) if $a_n \rightarrow \infty$, $\exists C > 0$, $b_n > C$, then $a_n b_n \rightarrow \infty$

注意: ① $(a_n \rightarrow \infty, b_n \rightarrow 0) \not\Rightarrow (a_n b_n \rightarrow 0)$

② $(a_n \rightarrow 0) \not\Rightarrow (\frac{1}{a_n} \rightarrow \infty)$, e.g. $a_n = \frac{(-1)^n}{n}$

(5) if $a_n \rightarrow \infty$, $m \in \mathbb{R}$, $a_n^m \rightarrow \begin{cases} \infty & \text{if } m > 0 \\ 0 & \text{if } m < 0 \end{cases}$

(2) if $\lambda \in \mathbb{R}$, $\lim_{n \rightarrow \infty} \lambda^n = \begin{cases} \infty & \text{if } \lambda > 1 \\ 0 & \text{if } |\lambda| < 1 \\ \text{diverge} & \text{if } \lambda \leq -1 \end{cases}$

(3) if $m > 0$, $\lambda > 1$, then $\lim_{n \rightarrow \infty} \frac{\lambda^n}{n^m} = \infty$

(4) if $m > 0$, $\lim_{n \rightarrow \infty} \frac{\ln n}{n^m} = 0$

• (1) monotonic: either increasing or decreasing sequence.

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注意 $\text{monotonic} \nRightarrow \text{convergent}$

$\text{convergent} \nRightarrow \text{monotonic}$. e.g. $a_n = \frac{(-1)^n}{n}$

定理常用!

(2) let (a_n) be monotonic and bounded, then (a_n) is convergent

proof:

let $A = \{a_n | n \in \mathbb{N}\}$, where a_n bounded, then $\sup A = M$ exists

assume (a_n) is increasing.

$\forall \varepsilon > 0$, $M - \varepsilon < M$. since M is supremum, $M - \varepsilon$ is not an upper bound. thus $\exists N \in \mathbb{N}$, such that $M - \varepsilon < a_N \leq M$

so $\forall n \geq N$, $M - \varepsilon \leq a_n \leq a_N \leq M < M + \varepsilon$, $|a_n - M| \leq \varepsilon$

• (1) let (a_n) be sequence, $F: \mathbb{N} \rightarrow \mathbb{N}$ strictly increasing function then a subsequence of (a_n) is $(a_{F(n)})$.

notation: $a_{n_1}, a_{n_2}, \dots$

(2) let $a_n \rightarrow L$, then any subsequence $a_{n_i} \rightarrow L$

proof:

assume $a_n \rightarrow L$, then $\forall \varepsilon > 0$, $\exists N \in \mathbb{N}$, $\forall n \geq N$, $|a_n - L| < \varepsilon$

since $F(n) \geq F(N) \geq N$, so $|a_{F(n)} - L| < \varepsilon$

(3) let $a_n \rightarrow \infty$, then any subsequence $a_{n_i} \rightarrow \infty$

定理常用!

注意

if (a_n) has two subsequences convergent to different limit, then it is divergent

• (1) any sequence (a_n) has monotonic subsequence

proof:

define $S := \{n | a_m > a_n, \forall m > n\} \subseteq \mathbb{N}$

now there are two possibilities:

① S is infinite, $S = \{n_1 < n_2 < \dots\} \subseteq \mathbb{N}$

then based on the definition, $a_{n_1} < a_{n_2} < \dots$

① S is finite, $S = \{n_1, n_2, \dots, n_k\} \subseteq \mathbb{N}$

since it is a finite nonempty subset of $\mathbb{N}$, it has an upper boundary.

$\exists N \in \mathbb{N}, \forall n > N, n \notin S$, so pick $n_1 > N$, then $\exists n_2 > n_1$, such that $a_{n_2} \leq a_{n_1}$.

since $n_2 \notin S$, then $\exists n_3 > n_2$, such that $a_{n_3} \leq a_{n_2} \leq a_{n_1}, \dots$

注意: any sequence contains either increasing or decreasing subsequence

(2) any bounded sequence has convergent subsequence.

注意: sequence 本身是否 convergent 并不重要! e.g. $(\sin n)$

• (1) cauchy sequence:

注意是 $\forall m, n \geq N$

$\forall \varepsilon > 0, \exists N \in \mathbb{N}, \forall m, n \geq N, |a_m - a_n| < \varepsilon$

(2) cauchy sequence is bounded.

proof:

for a cauchy sequence, let $\varepsilon = 1$, then $\exists N \in \mathbb{N}, \forall m > N, |a_m - a_N| < \varepsilon = 1$

so $|a_m| = |a_m - a_N + a_N| \leq |a_m - a_N| + |a_N| < 1 + |a_N|$

let the bound be $\max\{|a_1|, |a_2|, \dots, |a_N|, 1 + |a_N|\}$

(3) convergent sequence is cauchy.

proof:

$\forall \varepsilon > 0, \exists N \in \mathbb{N}, \forall n, m \geq N, |a_m - L| < \frac{\varepsilon}{2}, |a_n - L| < \frac{\varepsilon}{2}$

so $|a_m - a_n| = |a_m - L + L - a_n| \leq |a_m - L| + |L - a_n| < \varepsilon$

(4) cauchy sequence is convergent

proof:

since (a_n) is cauchy, it is bounded, and it has a monotonic

sequences.

$$\text{so } \forall \varepsilon > 0, \exists N_1 \in \mathbb{N}, \forall n, m \geq N_1, |a_m - a_n| < \frac{\varepsilon}{2}$$

$$\exists N_2 \in \mathbb{N}, \forall n_k \geq N_2, |a_{n_k} - L| < \frac{\varepsilon}{2}$$

$$\text{so } \forall \varepsilon > 0, \exists N = \max(N_1, N_2), \forall n, n_k \geq N$$

$$|a_n - L| = |a_n - a_{n_k} + a_{n_k} - L| \leq |a_n - a_{n_k}| + |a_{n_k} - L| \leq \varepsilon$$

注意: (Cauchy $\Leftrightarrow$ convergent) is true in $\mathbb{R}$ but not necessarily in other group.

Proof:

① consider in $\mathbb{R}$, $a_n = (1 + \frac{1}{n})^n$

$$\text{since } a_n = \sum_{k=0}^n \binom{n}{k} \frac{1}{n^k} = \sum_{k=0}^n \frac{n(n-1)\dots(n-k+1)}{k! n^k} \leq \sum_{k=0}^n \frac{n^k}{k! n^k} = \sum_{k=0}^n \frac{1}{k!}$$

when $k \geq 2$, $k! \geq 2^{k-1}$:

$$a_n \leq \sum_{k=0}^n \frac{1}{k!} = 2 + \sum_{k=2}^n \frac{1}{k!} \leq 2 + \sum_{k=2}^n \frac{1}{2^{k-1}} < \infty$$

show it is monotonous, notice that $(1-x)^m > 1-mx$

$$\begin{aligned} \frac{a_{n+1}}{a_n} &= \frac{(1 + \frac{1}{n+1})^{n+1}}{(1 + \frac{1}{n})^n} = (1 + \frac{1}{n}) \left(\frac{1 + \frac{1}{n+1}}{1 + \frac{1}{n}} \right)^{n+1} = (1 + \frac{1}{n}) \left(\frac{(n+1)^2 - 1}{(n+1)^2} \right)^{n+1} \\ &= (1 + \frac{1}{n}) \left(1 - \frac{1}{(n+1)^2} \right)^{n+1} > (1 + \frac{1}{n}) \left(1 - \frac{n+1}{(n+1)^2} \right) = 1 \end{aligned}$$

so (a_n) is increasing and bounded, it is convergent, with limit e .

② consider in $\mathbb{Q}$, $a_n = (1 + \frac{1}{n})^n$. it is Cauchy but not convergent as $e \notin \mathbb{Q}$.

(5) if $(a_n), (b_n)$ are Cauchy, then so do $(a_n) + (b_n)$ and (λa_n) for $\forall \lambda \in \mathbb{R}$

注意: this means Cauchy sequence $C(\mathbb{R})$ form a vector space.

(1) a complex sequence is a function $f: \mathbb{N} \rightarrow \mathbb{C}$, with notation $(z_n) = (f(n))$

(2) (z_n) is convergent if $\forall \varepsilon > 0, \exists N \in \mathbb{N}, \forall n \geq N, |z_n - z| < \varepsilon$, where z is the limit and $|\cdot|$ is the modulus for complex

(3) let (z_n) be a sequence in $\mathbb{C}$, then z_n convergent if and only if $(x_n) = (\operatorname{Re}(z_n))$ and $(y_n) = (\operatorname{Im}(z_n))$ convergent

(4) let (z_n) be a sequence in $\mathbb{C}$, then (z_n) is bounded if and only if $|z_n| \leq C$ for $\forall n \in \mathbb{N}$, $C \in \mathbb{R}$

(5) let (z_n) be a sequence in $\mathbb{C}$, then $((z_n))$ is convergent

$\Rightarrow ((z_n))$ is bounded) $\leftarrow$ (6) let (z_n) be a sequence in $\mathbb{C}$, then $((z_n))$ is bounded $\Rightarrow ((z_n))$ has convergent subsequence)

proof:

since (z_n) is bounded, so does (x_n) and (y_n)

then we can pick a monotonic subsequence (x_{n_k}) for (x_n)

since (y_{n_k}) is bounded, then we can pick a monotonic subsequence $(y_{n_{k_j}})$ for (y_{n_k})

since both $(x_{n_{k_j}})$ and $(y_{n_{k_j}})$ are monotonic and bounded they both converge, so does $(z_{n_{k_j}})$

(7) if (z_n) be a sequence in $\mathbb{C}$, then (z_n) is convergent if and only if $\forall \varepsilon > 0, \exists N \in \mathbb{N}, \forall n, m > N; |z_n - z_m| < \varepsilon$, which means (z_n) is Cauchy

proof:

$((z_n))$ is Cauchy

$$\Leftrightarrow \forall \varepsilon > 0, \exists N \in \mathbb{N}, \forall n, m > N, |x_n - x_m| < |z_n - z_m| < \frac{\varepsilon}{\sqrt{2}}$$

$$|y_n - y_m| < |z_n - z_m| < \frac{\varepsilon}{\sqrt{2}}, (x_n), (y_n) \text{ are Cauchy}$$

$$\Leftrightarrow (x_n), (y_n) \text{ are convergent} \Leftrightarrow (z_n) \text{ is convergent}$$

series

(1) let (a_n) be a sequence, the n -th partial sum S_n of (a_n) is $S_n = \sum_{i=1}^n a_i$. then (S_n) is called infinite series

(a) if (S_n) is convergent, then (a_n) is summable

(b) if (S_n) is convergent, then $a_n \rightarrow 0$

(4) if $(a_n), (b_n)$ be summable, $\sum_{i=1}^{\infty} a_i = L, \sum_{i=1}^{\infty} b_i = M$

5.

• $(a_n + b_n)$ is summable with limit $L + M$

• (λa_n) is summable with limit λL

(5) let (a_n) be summable, $b_n = a_{n+s}$, the (b_n) is summable

proof: $\sum_{i=1}^{\infty} b_i = \sum_{i=1+s}^{\infty} a_i = \sum_{i=1}^{\infty} a_i - (a_1 + a_2 + \dots + a_s)$

(6) let (S_n) be a series such that $\forall n \in \mathbb{N}, a_n \geq 0$. if the sequence of partial sums (S_n) is bounded, then (S_n) is summable. 此条常用

(7) let $(a_n), (b_n)$ be sequences such that $\forall n \in \mathbb{N}, 0 \leq a_n \leq b_n$, if (b_n) is summable, then (a_n) is summable. if (a_n) isn't summable, then (b_n) isn't summable

proof:

$\sum_{i=1}^{\infty} a_i, \sum_{i=1}^{\infty} b_i$ are increasing, and $0 \leq \sum_{i=1}^n a_i \leq \sum_{i=1}^n b_i \leq \sum_{i=1}^{\infty} b_i$

since $\sum_{i=1}^{\infty} b_i$ is bounded as well, it is summable.

(8) let (a_n) be a sequence such that $a_n \rightarrow 0$, and (a_n) is decreasing, then $\sum_{n=1}^{\infty} (-1)^{n+1} a_n$ is convergent

proof:

$\forall n \in \mathbb{N}, a_n \geq 0$

① show S_{2n+1} is decreasing and bounded

$\forall n \in \mathbb{N}, S_{2n+1} = (a_1 - a_2) + (a_3 - a_4) + \dots + a_{2n+1} \geq 0$

$S_{2(n+1)+1} - S_{2n+1} = a_{2n+3} - a_{2n+2} \leq 0$

so (S_{2n+1}) is convergent

② show S_{2n} is increasing and bounded

(S_{2n}) is convergent

③ assume $S_{2n} \rightarrow M, S_{2n+1} \rightarrow L$, show $L = M$

since $a_{n+1} = S_{2n+1} - S_{2n}$, when $n \rightarrow \infty, 0 = L - M$

thus (S_n) convergent.

(9) a sequence (a_n) is called absolutely summable if $(|a_n|)$

is summable. equivalently, (a_n) is absolutely convergent if $(|a_n|)$ is convergent

(2) if (a_n) is absolutely summable, then it is summable

此等式用

proof:

$$\sum_{i=1}^{\infty} a_i = \left(\sum_{i=1}^{\infty} a_i - \sum_{i=1}^{\infty} |a_i| \right) + \sum_{i=1}^{\infty} |a_i| \leq \left(2 \sum_{i=1}^{\infty} |a_i| \right) - \sum_{i=1}^{\infty} |a_i|$$

(3) a sequence is conditionally summable if it is summable but not absolutely summable.

(4) let (a_n) be a sequence which is never 0, and let $r_n = \left| \frac{a_{n+1}}{a_n} \right|$

(1) be the sequence of absolute ratios

此等式用

• suppose $r_n \rightarrow L$, where $L < 1$, then (a_n) is absolutely summable

• suppose $r_n \rightarrow L$, where $L > 1$, or $r_n \rightarrow \infty$, then (a_n) is not summable

proof

① case 1:

since $\forall \varepsilon > 0, \exists N \in \mathbb{N}, \forall n \geq N, |r_n - L| < \varepsilon$

pick $L < L + \varepsilon < 1$, such that $\exists N \in \mathbb{N}, \forall n \geq N, r_n \leq L + \varepsilon < 1$

so $|a_{n+1}| \leq (L + \varepsilon) |a_n|, |a_{n+2}| \leq (L + \varepsilon)^2 |a_n|, \dots$

② case 2:

similarly, $\forall \varepsilon > 0, \exists N \in \mathbb{N}, \forall n \geq N, |r_n - L| < \varepsilon$

pick $L > L - \varepsilon > 1$, such that $\exists N \in \mathbb{N}, \forall n \geq N, r_n > L - \varepsilon > 1$

so $|a_{n+1}| \geq (L - \varepsilon) |a_n|, |a_{n+2}| \geq (L - \varepsilon)^2 |a_n|, \dots$

(5) let (a_n) be a complex sequence. suppose the sequence

power series $p_n = \sum |a_n|$ is convergent to a limit L

• if $L < 1$, then (a_n) is absolutely summable

• if $L > 1$, or $p_n \rightarrow \infty$, then (a_n) is not summable

• power series (1) let (a_n) be any complex sequence, the power series

associated to (a_n) is the sequence of partial sums

$\sum_{k=0}^n a_k z^k$ of the sequence $(a_n z^n)$, where $z \in \mathbb{C}$.

- (2) let (a_n) be a complex sequence and $P(z)$ the power series associated to (a_n) . let $w \neq 0$ be a complex number such that $P(w)$ is absolute convergent. then $P(z)$ is convergent for all $z \in \mathbb{C}$ such that $|z| < |w|$

proof:

by assumption $P(w)$ is convergent, hence $a_n w^n$ has to tend to 0 and $(a_n w^n)$ is bounded, this mean there exists a real number C such that $|a_n w^n| \leq C$ for all $n \in \mathbb{N}$.

then $|a_n z^n| = |a_n w^n| \left| \frac{z}{w} \right|^n \leq C \left| \frac{z}{w} \right|^n$, $\left| \frac{z}{w} \right| < 1$

- (3) let $P(z) = \sum_{n=0}^{\infty} a_n z^n$, then convergence radius is the real number $r = \sup \{x \in \mathbb{R}^+ \mid P(x) \text{ is convergent}\}$

- $P(w)$ convergent for $|w| < r$

- $P(w)$ divergent for $|w| > r$

- r can be found using ratio tests and root tests.

- (4) Dirichlet's test: let $(a_n), (b_n)$ are complex sequences, if

- the n -th partial sum $A_n = \sum_{k=1}^n a_k$ are bounded

the $\sum a_n b_n$ is convergent $b_n \rightarrow 0$ is monotonic decreasing

proof:

by express $a_n = A_n - A_{n-1}$, we can show $\sum_{n=1}^N a_n b_n = A_N b_N + \sum_{n=1}^{N-1} A_n (b_n - b_{n+1})$

notice: for ①: $A_n b_n \leq M |b_n|$; where M is the upper bound of A_n .

$$\lim_{N \rightarrow \infty} A_N b_N = 0$$

$$\text{for } \textcircled{2}: \sum_{n=1}^{\infty} |A_n (b_n - b_{n+1})| \leq M \sum_{n=1}^{\infty} b_n - b_{n+1} = M b_1$$

so $|A_n b_n + \sum_{n=1}^N A_n (b_n - b_{n+1})|$ is absolute convergent since ② is absolute convergent

$$1) \cdot \left(\lim_{x \rightarrow \infty} f(x) = L \right) \Leftrightarrow (\forall \varepsilon > 0, \exists R \in \mathbb{R}, \forall x > R, |f(x) - L| < \varepsilon)$$

continuity

$$\bullet (\lim_{x \rightarrow \infty} f(x) = \infty) \Leftrightarrow (\forall M \in \mathbb{R}, \exists R \in \mathbb{R}, \forall x > R, f(x) > M)$$

$$\bullet (\lim_{x \rightarrow x_0^-} f(x) = L) \Leftrightarrow (\forall \varepsilon > 0, \exists \delta > 0, \forall x \in (x_0 - \delta, x_0), |f(x) - L| < \varepsilon)$$

$$(\lim_{x \rightarrow x_0^+} f(x) = L) \Leftrightarrow (\forall \varepsilon > 0, \exists \delta > 0, \forall x \in (x_0, x_0 + \delta), |f(x) - L| < \varepsilon)$$

$$(\lim_{x \rightarrow x_0} f(x) = L) \Leftrightarrow (\forall \varepsilon > 0, \exists \delta > 0, \forall 0 < |x - x_0| < \delta, |f(x) - L| < \varepsilon)$$

注意: $f(x_0)$ 不一定存在, 且 $f(x_0)$ 不一定为 $\lim_{x \rightarrow x_0} f(x)$

$$\bullet (\lim_{x \rightarrow x_0} f(x) = \infty) \Leftrightarrow (\forall M \in \mathbb{R}, \exists \delta > 0, \forall 0 < |x - x_0| < \delta, f(x) > M)$$

(2) if $\lim_{x \rightarrow x_0} f(x) = L_1$ and $\lim_{x \rightarrow x_0} f(x) = L_2$, then $L_1 = L_2$ (similar for other conditions)

(3) if $\lim_{x \rightarrow x_0} f(x) = L$, $\lim_{x \rightarrow x_0} g(x) = M \neq 0$

$$\bullet \lim_{x \rightarrow x_0} (f(x) + g(x)) = L + M$$

$$\bullet \lim_{x \rightarrow x_0} (f(x) \cdot g(x)) = L \cdot M$$

$$\bullet \lim_{x \rightarrow x_0} \frac{f(x)}{g(x)} = \frac{L}{M} \text{ if } M \neq 0$$

(4) let $f: (b, c) \rightarrow \mathbb{R}$ be a function, $a \in (b, c)$. then f is continuous at a if and only if, $\forall \varepsilon > 0, \exists \delta > 0, \forall |x - a| < \delta, |f(x) - f(a)| < \varepsilon$.

注意: 在证明 continuous 时, 我们要把 $|f(x) - f(a)|$ 用 $|x - a|$ 表示.

如果在转化时, 有单独的 $|x|$ 无法被消除, 我们先假设一个 δ 的上限, 以固定 $|x|$ 的上限

(5) ① 证明 discontinuous: $\exists \varepsilon > 0, \forall \delta > 0, \exists |x - a| < \delta, |f(x) - f(a)| \geq \varepsilon$.

(5) let $I \subset \mathbb{R}$ open, $f, g: I \rightarrow \mathbb{R}$ continuous

① $f + g: I \rightarrow \mathbb{R}$ continuous

② $f \cdot g: I \rightarrow \mathbb{R}$ continuous

③ if $f(x) \neq 0$ for $x \in I$, then $\frac{1}{f(x)}$ is continuous

(6) a function is Lipschitz continuous if: $\exists L > 0, \forall x, y \in I, |f(x) - f(y)| \leq L|x - y|$. in other words, $\frac{|f(x) - f(y)|}{|x - y|}$ is bounded.

注意: a function is Lipschitz $\Rightarrow$ it is continuous

a function is continuous $\nRightarrow$ it is Lipschitz.

7.

(7) let $f: I_1 \rightarrow I_2, g: I_2 \rightarrow I_3, a \in I_1, f(a) \in I_2$. let f be continuous at a , g be continuous at $f(a)$. then $g \circ f$ is continuous at a

proof:

$$\forall \varepsilon_1 > 0, \exists \delta_1 > 0, \forall |y - f(a)| < \delta_1, |g(y) - g(f(a))| < \varepsilon_1$$

$$\forall \varepsilon_2 > 0, \exists \delta_2 > 0, \forall |x - a| < \delta_2, |f(x) - f(a)| < \varepsilon_2$$

$$\text{let } \varepsilon_2 = \delta_1, \forall \varepsilon > 0, \exists \delta_1 = \varepsilon_2 > 0, \delta_2 > 0, \forall |x - a| < \delta_2, |f(x) - f(a)| < \varepsilon_2$$

$$|g(f(x)) - g(f(a))| < \varepsilon_1$$

(8) let $f: I \rightarrow \mathbb{R}$ be a function, then (f is continuous at a)

$$\Leftrightarrow (\forall \text{ sequence } (a_n), \text{ where } a_n \rightarrow a, f(a_n) \rightarrow f(a))$$

proof:

• assume f is continuous at a , and $a_n \rightarrow a$ for some sequence (a_n)

$$\forall \varepsilon > 0, \exists \delta > 0, \forall |x - a| < \delta, |f(x) - f(a)| < \varepsilon$$

$$\forall \varepsilon_2 > 0, \exists N \in \mathbb{N}, \forall n \geq N, |a_n - a| < \varepsilon_2$$

$$\text{let } \varepsilon_2 = \delta, \forall \varepsilon_1 > 0, \exists \delta_1 = \varepsilon_2 > 0, \exists N \in \mathbb{N}, \forall n \geq N, |a_n - a| < \delta_1$$

$$|f(a_n) - f(a)| < \varepsilon_1$$

$$\text{so } f(a_n) \rightarrow f(a)$$

• assume $\forall$ sequences (a_n) with $a_n \rightarrow a$, then $f(a_n) \rightarrow f(a)$ ↓

proof by contradiction, if f is not continuous at a

$$\exists \varepsilon > 0, \forall \delta > 0, \exists |x - a| < \delta, |f(x) - f(a)| \geq \varepsilon$$

in particular this holds for $\delta_n = \frac{1}{n}$ for all n , and $x(n) = a_n$

$\exists$ such that $|a_n - a| < \delta_n$

$$\exists \varepsilon > 0, \forall n \in \mathbb{N}, \exists x(n) = a_n, |a_n - a| < \delta_n, |f(a_n) - f(a)| \geq \varepsilon$$

so $f(a_n)$ not convergent to $f(a)$, impossible

注意 可用于证明 continuous. 若要证明 uncontinuous, 只需证

$$(a_n), (b_n) \rightarrow K \text{ 但 } f(a_n) \rightarrow M, f(b_n) \rightarrow N, M \neq N \text{ 即 } \neg$$

(9) let $K = [b, c] \subset \mathbb{R}$ be a closed bounded interval. let $f: K \rightarrow \mathbb{R}$ be a function, then f is continuous at a on K if and only if

- f is continuous on (b, c)
- $\lim_{x \rightarrow b^+} f(x) = f(b)$, $\lim_{x \rightarrow c^-} f(x) = f(c)$

- (10) let $S \subset \mathbb{R}$ be any subset, $f: S \rightarrow \mathbb{R}$ be a function. then f is bounded if and only if $f(S) = \{f(x) | x \in S\}$ is bounded.
- (11) let K be a compact interval, $f: K \rightarrow \mathbb{R}$, if f is continuous, then f is bounded.

proof.

assume f is not bounded. $\forall B \in \mathbb{R}, \exists x \in K, |f(x)| > B$

$\forall n \in \mathbb{N} \subset \mathbb{R}, \exists x_n \in K, |f(x_n)| > n$

this gives us a sequence $(x_n) \in K$, which is bounded.

so there exists a convergent subsequence (x_{n_i}) such that $x_{n_i} \rightarrow L$.

since $x_{n_i} \in K$, $L \in K$.

now f is continuous, so $f(x_{n_i}) \rightarrow f(L)$, so $f(x_{n_i})$ is bounded since convergent, contradict to assumption.

- (12) if $S \subset \mathbb{R}$ is a subset, $f: S \rightarrow \mathbb{R}$ bounded.

- $\sup f = \sup \{f(x) | x \in S\}$
- $\inf f = \inf \{f(x) | x \in S\}$

- (13) if K is a compact interval, $f: K \rightarrow \mathbb{R}$ is a continuous function:

- $\exists x_1 \in K, f(x_1) = \sup f$
- $\exists x_2 \in K, f(x_2) = \inf f$

注意:

• open interval:

$\forall x \in U, \exists \varepsilon > 0, (x - \varepsilon, x + \varepsilon) \subset U$

• closed interval

its complement is open

• compact interval

closed + bounded.

proof: 另证明第一部分

we can prove it by using contradiction. assume $\forall x \in K, f(x) \neq \sup f$. define $g: K \rightarrow \mathbb{R}, g(x) = \sup f - f(x)$. it is well-defined since $f(x) \neq \sup f$. g is continuous on K , so g is bounded.

since $\sup f$ is the supremum of f , $\forall n \in \mathbb{N}, \sup f - \frac{1}{n}$ is not an upper bound.

$\forall n \in \mathbb{N}, \exists x_n \in K, f(x_n) > \sup f - \frac{1}{n}, g(x_n) = \sup f - f(x_n) < \frac{1}{n}$

thus g is not bounded. contradict!

- (14) let $f: K = [b, c] \rightarrow \mathbb{R}$ continuous, then if $f(b) \leq f(c)$, $A \in \mathbb{R}$, such that $f(b) \leq A \leq f(c)$, then $\exists a \in K$, such that $f(a) = A$.

proof

- ① let $S \subset K, S = \{x \in K | f(x) \leq A\}$

since S is non-empty ($f(b) \leq A$) and bounded ($f(b) \leq A \leq f(c)$) then S has a supremum $\sup S = \alpha$.

- ② if $f(\alpha) < A$,

f is continuous, so let $\varepsilon_0 = A - f(\alpha)$, $\exists \delta_0 > 0, |x - \alpha| < \delta_0, |f(x) - f(\alpha)| < \varepsilon_0$

$\exists \delta_0 > 0, \forall x \in (\alpha - \delta_0, \alpha + \delta_0), f(x) < \varepsilon_0 + f(\alpha) = A$

since $\alpha \leq c, f(\alpha) < A \leq f(c)$, then $\alpha < c$

if we pick x such that $\alpha < x < \alpha + \delta$, for some δ , $f(x) < A, x \in S$

impossible. α is $\sup S$.

8.

③ if $f(a) > A$.

let $\varepsilon_0 = f(a) - A > 0$. f is continuous, so $\exists \delta_0 > 0$, $|x - a| < \delta_0$, $|f(x) - f(a)| < \varepsilon_0$.

$\exists \delta_0 > 0$, $\forall x \in (a - \delta_0, a + \delta_0)$, $f(x) > f(a) - \varepsilon_0 = A$

since $f(b) \leq A < f(a)$, $b < a$

if we pick x such that $a - \delta < x < a$ for some δ , $f(x) > A$, $x \notin S$.

impossible, since a is sup.

④ so $f(a) = A$.

(1) let $\Omega \subseteq \mathbb{C}$, $f: \Omega \rightarrow \mathbb{C}$ is a function.

(f is continuous at $z_0 \in \mathbb{C}$) $\iff (\forall \varepsilon > 0, \exists \delta > 0, \forall z \in \mathbb{C}, |z - z_0| < \delta, |f(z) - f(z_0)| < \varepsilon)$

(2) a set $S \subseteq \mathbb{C}$ is closed $\iff (\forall$ convergent sequences (a_n) in S , $\lim_{n \rightarrow \infty} a_n \in S$).

注意: 这与 "complement to open interval" 等价

(3) a set $K \subseteq \mathbb{C}$ is compact $\iff$ any sequence (z_n) of elements in K has a convergent subsequence with $\lim_{n \rightarrow \infty} z_{n_i} \in K$

Proof:

$\Rightarrow$: since K is compact, sequence (z_n) in K is bounded.

since any bounded sequence of complex numbers has convergent subsequence (z_{n_i})

(z_{n_i}) is closed S , so $\lim_{n \rightarrow \infty} z_{n_i} \in K$.

$\Leftarrow$ ① K is bounded: assume K is not bounded.

$\forall n \in \mathbb{N}, \exists z_n \in K, |z_n| > n$, so (z_n) may not have

convergent subsequence.

contradict

② K is closed: let (z_n) converge, $z_n \rightarrow z_0$.

since (z_n) has a convergent subsequence (z_{n_i}) , and $\lim_{n \rightarrow \infty} z_{n_i} \in K$

since $z_{n_i} \rightarrow z_0$, $\lim_{n \rightarrow \infty} z_n \in K$

(4) let $K \subseteq \mathbb{C}$ compact, $f: K \rightarrow \mathbb{C}$ continuous, then $f(K)$ is compact

Proof: let (w_n)

let (w_n) be sequence in $f(K)$, we want to prove we can always construct a convergent subsequence with limit in $f(K)$

$\exists$ sequence (z_n) , $\forall n \in \mathbb{N}$, $w_n = f(z_n)$

since K is compact, (z_n) will have a convergent subsequence (z_{n_i}) , and $z_{n_i} \rightarrow z_0 \in K$

f is continuous, $w_{n_i} = f(z_{n_i})$, so $w_{n_i} \rightarrow f(z_0) \in f(K)$

(5) let $D \subseteq \mathbb{C}$ be compact, $f: D \rightarrow \mathbb{R}$ continuous, then $\exists z_1, z_2 \in D$, $\forall z \in D$, $f(z_1) \leq f(z) \leq f(z_2)$

Proof:

$f(D)$ is bounded and closed, so it has a maximum $M = \sup f(D)$ and minimum.

$m = \inf f(D)$

by definition of $\inf$ (same for $\sup$), we can find the values of the function

continuous close to $\inf$: $\forall n \in \mathbb{N}$, $\exists f(z_n) \in f(D)$, $f(z_n) < m + \frac{1}{n}$

in such a way, we can find a sequence z_n in D , such that $f(z_n) \rightarrow m$

$z_n \rightarrow z_0$, $z_0 \in D$, $f(z_0) = m$.

$$f(a(x+h)) - f(a(x)) \quad f(a(x+h)) - f(a(x)) \quad a(x+h) - a(x)$$

(6) a function $f: D \rightarrow \mathbb{C}$ is uniformly continuous if and only if $\forall \epsilon > 0, \exists \delta > 0, \forall z, w \in D, |z-w| < \delta, |f(z) - f(w)| < \epsilon$

注意: Lipschitz continuous $\Rightarrow$ uniformly continuous, 不可逆!

(7) let $D \subseteq \mathbb{C}$ compact, $f: D \rightarrow \mathbb{C}$ continuous. then f is uniformly continuous.

proof:

assume D compact, $f: D \rightarrow \mathbb{C}$ continuous but not uniformly continuous

$\exists \epsilon > 0, \forall \delta > 0, \exists x, y \in D, |x-y| < \delta, |f(x) - f(y)| \geq \epsilon$

$\forall \delta_n = \frac{1}{n}, \exists x_n, y_n \in D$ such that $|x_n - y_n| < \delta_n$ and $|f(x_n) - f(y_n)| \geq \epsilon$

since D is compact, then (x_n) has convergent subsequence $(x_{n_j}) \rightarrow L \in D$

so $|x_{n_j} - y_{n_j}| < \delta_{n_j} = \frac{1}{n_j} < \frac{1}{j}$, so $(y_{n_j}) \rightarrow L \in D$ as well. (can be proved)

thus $f(x_{n_j}), f(y_{n_j}) \rightarrow f(L), |f(x_{n_j}) - f(y_{n_j})| \rightarrow 0$

so $|f(x_{n_j}) - f(y_{n_j})| < \epsilon$ for big enough j .

sequence of functions $\{f_n\}$

lots of functions in analysis are defined as limits as series

(pointwise convergence: let $D \subseteq \mathbb{C}$, (f_n) is a sequence of functions $f_n: D \rightarrow \mathbb{C}$ (sequence of functions $\{f_1, f_2, \dots\}$, $f_n: D \rightarrow \mathbb{C}$ are

maps from N to space of functions $D \rightarrow \mathbb{C}$ = $\mathbb{C}^D$ (or $\mathcal{F}(D, \mathbb{C})$)

we say that (f_n) converges pointwise on D if for every $z \in D$, the

sequence of complex numbers $(f_n(z))$ converges.

注意: f_n continuous $\nRightarrow$ f continuous, f_n continuous $\nRightarrow$ f continuous

example: $f_n(z) = e^{-nz^2}$

$$f(z) = \begin{cases} 0 & \text{for } z \neq 0 \\ 1 & \text{for } z = 0 \end{cases}$$

(2) let $D \subseteq \mathbb{C}$, $f: D \rightarrow \mathbb{C}$ be a function. we call $\|f\|_D = \sup\{|f(z)| : z \in D\}$ the supremum norm.

(3) if f is bounded, $\|f\|_D < \infty$. $\|f\|_D < M \Leftrightarrow |f(z)| < M \forall z \in D$

(4) let $D \subseteq \mathbb{C}$, $f, g: D \rightarrow \mathbb{C}$, then

$$\|f\|_D = 0 \Leftrightarrow (f(z) = 0 \forall z \in D)$$

$$\forall \lambda \in \mathbb{C}, \|\lambda f\|_D = |\lambda| \|f\|_D$$

$$\|f + g\|_D \leq \|f\|_D + \|g\|_D$$

(5) we define the distance between functions f and g as $\|f - g\|_D$

(6) let (f_n) be a sequence of functions $f_n: D \rightarrow \mathbb{C}$. then (f_n) is uniformly convergent with limit $f: D \rightarrow \mathbb{C}$ if and only if $\forall \epsilon > 0, \exists N \in \mathbb{N}$ such that $\forall n \geq N, \forall z \in D, |f_n(z) - f(z)| < \epsilon$

(7) (f_n) uniformly converges to $f: D \rightarrow \mathbb{C}$ if and only if $\forall \epsilon > 0, \exists N \in \mathbb{N}$ such that $\forall n \geq N, \|f_n - f\|_D < \epsilon$

(8) let (f_n) be a convergent sequence of continuous functions $f_n: D \rightarrow \mathbb{C}$ and (f_n) uniformly convergent, then the limit $f: D \rightarrow \mathbb{C}$ is continuous.

9.

proof

With $\forall \epsilon > 0, \exists \delta > 0, \forall z \in D, |z - z_0| < \delta, |f(z) - f(z_0)| < \epsilon$
 since f_n is continuous $\forall \epsilon > 0, \exists \delta > 0, \forall z \in D, |z - z_0| < \delta, |f_n(z) - f_n(z_0)| < \frac{\epsilon}{3}$
 since f_n is uniformly convergent $\forall \epsilon > 0, \exists N \in \mathbb{N}, \forall n > N, \forall z \in D, |f_n(z) - f(z)| < \frac{\epsilon}{3}$
 so $\forall \epsilon > 0, \exists \delta > 0, \forall n > N, \forall z \in D, |z - z_0| < \delta, |f(z) - f(z_0)| < \epsilon$
 $|f(z) - f_n(z)| + |f_n(z) - f_n(z_0)| + |f_n(z_0) - f(z_0)| < \epsilon$

• some of functions (2.1.14-15.3)

(1) consider the series $\sum_{k=0}^{\infty} f_k$ which is a limit of sequence of partial sum $S_n := \sum_{k=0}^n f_k, k \in \mathbb{N}$. let $D \subset \mathbb{C}, f_n: D \rightarrow \mathbb{C}$. then the series $\sum_{k=0}^{\infty} f_k$ converges normally if and only if
 ① $\forall n \in \mathbb{N}, \|f_n\|_D < \infty$ ② $\sum_{k=0}^{\infty} \|f_k\|_D < \infty$

normal convergence is stronger than absolute convergence

举个例子 $\sum_{k=0}^{\infty} z^k$ it is absolute convergent at $(0,1)$
 but $\|f_n\|_{(0,1)} = 1, \sum_{k=0}^{\infty} \|f_k\|_{(0,1)} = \sum_{k=0}^{\infty} 1 \rightarrow \infty$, not normally convergent
 但是 normally convergent $\rightarrow$ absolute convergent

(2) assume the series $\sum_{k=0}^{\infty} f_k$ convergent normally, then

- ① $\sum_{k=0}^{\infty} f_k = S_n$ uniformly convergent on D .
- ② $\forall z \in D, \sum_{k=0}^{\infty} f_k(z)$ absolutely convergent
- ③ if $\forall n \in \mathbb{N}, f_n$ is continuous, then the limit $\sum_{k=0}^{\infty} f_k = f$ is continuous.

proof

① $\forall z \in D, |f(z) - S_N(z)| = \left| \sum_{k=N+1}^{\infty} f_k(z) \right| \leq \sum_{k=N+1}^{\infty} |f_k(z)| \leq \sum_{k=N+1}^{\infty} \|f_k\|_D$
 so $\|f - S_N\|_D = \sup_{z \in D} |f(z) - S_N(z)| \leq \sum_{k=N+1}^{\infty} \|f_k\|_D \rightarrow 0$
 due to it is the tail of convergent sum
 ② $\forall z \in D, \sum_{k=0}^{\infty} |f_k(z)| \leq \sum_{k=0}^{\infty} \|f_k\|_D < \infty, |f(z)| \leq \sum_{k=0}^{\infty} |f_k(z)|$
 ③ from ②, limit of a uniformly convergent sequence of continuous function is continuous.

• differentiable

(1) if $f(x)$ is differentiable, it is continuous.

(2) let $I = [a, b]$, we denote:

$C^0(I) := \{f: [a, b] \rightarrow \mathbb{R} : f \text{ is continuous on } [a, b]\}$

$C^1(I) := \{f: [a, b] \rightarrow \mathbb{R} : f' \text{ is continuous on } [a, b]\}$

then $\frac{d}{dx}: C^1(I) \rightarrow C^0(I)$

(3) some proofs about differentiation

• $[f(g(x))]' = f'(g(x)) \cdot g'(x)$

proof:

$$\frac{f(g(x+h)) - f(g(x))}{h} = \frac{f(g(x+h)) - f(g(x))}{g(x+h) - g(x)} \cdot \frac{g(x+h) - g(x)}{h} \quad \text{for } g(x+h) \neq g(x)$$

then we may define

$$\frac{f(g(x+h)) - f(g(x))}{h} = \phi(h) \frac{g(x+h) - g(x)}{h}$$

$$\text{where } \phi(h) = \begin{cases} \frac{f(g(x+h)) - f(g(x))}{g(x+h) - g(x)} & \text{for } g(x+h) \neq g(x) \\ f'(g(x)) & \text{for } g(x+h) = g(x) \end{cases}$$

[注意: 此时带入情况2发现 RHS 已经为 0, 因此等式永远成立]

因此我们只需证明 $\lim_{h \rightarrow 0} \phi(h) = f'(g(x)) = \phi(0)$ 即可

① by f is differentiable at $g(x)$:

$$\forall \varepsilon > 0, \exists \delta' > 0, \forall 0 < |k| < \delta', \left| \frac{f(g(x)+k) - f(g(x))}{k} - f'(g(x)) \right| < \varepsilon$$

② by g is continuous at x :

$$\forall \delta' > 0, \exists \delta > 0, \forall |h| < \delta, |g(x+h) - g(x)| < \delta'$$

由此得

$$\text{当 } g(x+h) = g(x) \text{ 时, } |\phi(h) - f'(g(x))| = 0 < \varepsilon$$

$$\text{当 } g(x+h) \neq g(x) \text{ 时, } |\phi(h) - f'(g(x))| = \left| \frac{f(g(x+h)) - f(g(x))}{g(x+h) - g(x)} - f'(g(x)) \right|$$

$$= \left| \frac{f(g(x) + g(x+h) - g(x)) - f(g(x))}{g(x+h) - g(x)} - f'(g(x)) \right|$$

$$= \left| \frac{f(g(x)+k) - f(g(x))}{k} - f'(g(x)) \right| < \varepsilon$$

• (1) if f is differentiable on (a, b) and $f'(x) > 0$ for all $x \in (a, b)$, then f is strictly monotonic increasing on (a, b)

proof: by contradiction: we assume then for some $x \in (a, b)$, there exists $y \in (x, b)$ such that $f(x) \geq f(y)$

矛盾!

then consider $A = \{y \in (x, b) \mid f(y) \leq f(x)\}$

by assumption, A is nonempty, and A is bounded as $A \subset (a, b)$

then there is $\alpha = \inf A \in \mathbb{R}$, and $f(\alpha) \leq f(x)$ by considering a sequence of A with limit α ,

also, we know that $x < \alpha$ by considering the local comparison: since $f'(x) > 0$, $\exists \delta > 0$, $f(y) > f(x)$ for all $y \in (x, x+\delta)$, α doesn't touch x

so $f'(\alpha) > 0$ implies $\exists \delta > 0$, $y \in (\alpha - \delta, \alpha)$, $f(y) < f(\alpha) \leq f(x)$, $y \in A$, impossible since $y < \alpha = \inf A$.

[注意: $(f'(\alpha) = 0) \nRightarrow (\alpha \text{ is local maximum/minimum})$, 考虑 saddle point
critical points 是 $f'(x) = 0$ 的所有点]

local maximum/minimum is only critical points if they are interior, differentiable + local maximum/minimum interior $\Rightarrow f'(x)=0$

(2) let $f: [a, b] \rightarrow \mathbb{R}$ continuous on $[a, b]$ and differentiable on (a, b)
if $f(a) = f(b)$, then there exists $t \in (a, b)$ such that $f'(t) = 0$

proof: there is a $t \in [a, b]$ such that $f(t) = \sup\{f(x) \mid x \in [a, b]\}$

- if t is interior, we are done
- if t isn't interior, then there is a $u \in [a, b]$ such that $f(u) = \inf\{f(x) \mid x \in [a, b]\}$
 - if u is interior, we are done
 - if u isn't interior, f is constant for $x \in [a, b]$, $f'(x) = 0$ $\forall x \in (a, b)$

(3) let $f: (a, b) \rightarrow \mathbb{R}$ be differentiable on (a, b) , assume $x < y \in (a, b)$ are such that $f'(x) > 0 > f'(y)$, then there is $t \in (x, y)$ such that $f'(t) = 0$

proof:

by local comparison, since $f'(x) > 0$, there is $\delta > 0$ ($x + \delta < y$) such that for all $p \in (x, x + \delta)$, $f(p) > f(x)$. this implies that f doesn't attain its maximum over $[x, y]$ at x .

similarly, f doesn't attain its maximum over $[x, y]$ at y .

on the other hand, continuous + compact domain guarantees there is $t \in [x, y]$ such that $f(t) = \text{maximum of } f \text{ over } [x, y]$, so $f'(t) = 0$

(4) let $f: [a, b] \rightarrow \mathbb{R}$ be continuous on $[a, b]$ and differentiable on (a, b)
then, there is $t \in (a, b)$ such that $f'(t) = \frac{f(b) - f(a)}{b - a}$

(5) let $f: (a, b) \rightarrow \mathbb{R}$ be differentiable on (a, b) , then $\forall x < y$, and any c between $f'(x)$ and $f'(y)$, $\exists t \in (x, y)$ with $f'(t) = c$

proof: assume WLOG, that $f'(x) < f'(y)$, then $c \in (f'(x), f'(y))$

let $g(s) = c \cdot s - f(s) \quad \forall s \in (a, b)$

so $g'(x) > 0$, $g'(y) < 0$, in such a situation, we have an interior maximum at a critical point $t \in (a, b)$

$$g'(t) = c - f'(t) = 0$$

[注意: 在没有定义 integration 的前提下, 证明 $f'(x) = g'(x) + c$ 是唯一解
设另外的解为 $h(x)$, 证明 $\frac{d}{dx}(g(x) + c - h(x)) = 0$, 相差常数]

(6) if f and g are continuous on $[a, b]$ and differentiable on (a, b) , then there exists $x \in (a, b)$ such that

$$[f(b) - f(a)] g'(x) = [g(b) - g(a)] f'(x)$$

* proof: consider $h(t) = [f(b) - f(a)] g(t) - [g(b) - g(a)] f(t)$

[注意: 这可以用来证明洛比达, when $\lim_{x \rightarrow a} f(x) = \lim_{x \rightarrow a} g(x) = 0$]

(7) assume that $\lim_{x \rightarrow a} f(x) = \lim_{x \rightarrow a} g(x) = 0$

if $\lim_{x \rightarrow a} \frac{f'(x)}{g'(x)}$ exists, then $\lim_{x \rightarrow a} \frac{f(x)}{g(x)}$ exists and both limits are equal

proof:

• by ' $\lim_{x \rightarrow a} \frac{f'(x)}{g'(x)}$ exists', we are assuming:

① f and g are differentiable on $(a-\delta, a+\delta) \setminus \{a\}$

② $g'(x) \neq 0$ on $x \in (a-\delta, a+\delta) \setminus \{a\}$.

• it makes sense to set $f(a) = g(a) = 0$ (we can redefine f and g !)

• firstly show $g(x) \neq 0$ for all $x \in (a-\delta, a+\delta) \setminus \{a\}$.

since $g'(x) \neq 0$ for $x \in (a, a+\delta)$ and $x \in (a-\delta, a)$, g is strictly monotone in both interval.

since $g(a) = 0$, then for all $x \in (a-\delta, a+\delta) \setminus \{a\}$, $g(x) \neq 0$

• for any x , there exists $a_x \in (a, x)$, by extended mean value theorem.

$$\frac{f'(a_x)}{g'(a_x)} = \frac{f(x) - f(a)}{g(x) - g(a)} = \frac{f(x)}{g(x)}$$

• set $L = \lim_{x \rightarrow a} \frac{f(x)}{g(x)}$, then $\forall \varepsilon > 0, \exists \delta > 0, \forall 0 < |x-a| < \delta, \left| \frac{f'(x)}{g'(x)} - L \right| < \varepsilon$

then notice $0 < |a_x - a| < |x - a| < \delta, \left| \frac{f'(a_x)}{g'(a_x)} - L \right| = \left| \frac{f(x)}{g(x)} - L \right| < \varepsilon$

so $\forall \varepsilon > 0, \exists \delta > 0, \forall 0 < |x-a| < \delta, \left| \frac{f(x)}{g(x)} - L \right| < \varepsilon$

(8) let a be a critical point of f , say $f'(a) = 0$

• if $f''(a) > 0$, then f has a local minimum at a .

• if $f''(a) < 0$, then f has a local maximum at a

proof: prove first statement:

let $g(x) = f'(x)$, so $g(a) = 0, g'(a) > 0$, by local comparison:

$$\begin{cases} g(x) < g(a) = 0 & \text{for } x \in (a-\delta, a) \\ g(x) > g(a) = 0 & \text{for } x \in (a, a+\delta) \end{cases} \text{ for some } \delta > 0.$$

$$\begin{cases} g(x) < g(a) = 0 & \text{for } x \in (a-\delta, a) \\ g(x) > g(a) = 0 & \text{for } x \in (a, a+\delta) \end{cases}$$

so:

$f'(x) < 0$, f monotone decreasing at $(a-\delta, a)$

$f'(x) > 0$, f monotone increasing at $(a, a+\delta)$

so for $x < a < y \in (a-\delta, a+\delta)$, $f(x) > f(a)$, $f(y) > f(a)$

[注意] (f has local minimum at a) $\nRightarrow f''(a) > 0$

比如说其可能根本不-differentiable $y = |x|$

或者是其可能为常数, 又 $f''(a) = 0$ ($y = x^4$)

(9) assume $f''(a)$ exists and f has a local minimum at a , then $f''(a) \geq 0$

proof: by contradiction

a is local minimum, $f'(a) = 0$

$f''(a) < 0$

then $f(a)$ is local maximum, so it is both a maximum and minimum
thus f is constant, but in such a way, $f''(a) = 0$, contradict!

判断函数最大/小时, 用 f'' 检验! 考虑边界!

(10) • a set $U \subset \mathbb{R}^2$ is convex if for any two points $p, q \in U$, the straight line segment joining p and q is contained in U .

[注意: convex doesn't imply open/closed interval]

• for convex f on I : for all $a < b \in I$, and $t \in [0, 1]$, we have

$$f((1-t)a + tb) \leq (1-t)f(a) + tf(b)$$

similarly, for all $a < x < b \in I$, we have

$$f(x) \leq \frac{f(b)-f(a)}{b-a}(x-a) + f(a)$$

• f is strictly convex on I if for all $a < b \in I$, $t \in (0, 1)$, we have,

$$f((1-t)a + tb) < (1-t)f(a) + tf(b)$$

similarly, for all $a < x < b \in I$, we have

$$f(x) < \frac{f(b)-f(a)}{b-a}(x-a) + f(a)$$

[注意: convex 不代表 $f'' \geq 0$, 因为不一定存在 f'' !]

(11) if f is differentiable on I and f' is strictly monotone increasing, then f is convex

proof ① case $f(a) = f(b)$

$$f_{ab}(x) = \text{constant} = f(a) = f(b)$$

prove by contradiction, assume $f(x) \leq f(a) = f(b)$ won't happen for all

$x \in [a, b]$, then $f(x) > f(a) = f(b)$ for some $x \in (a, b)$

then we conclude that f attains an interior maximum $x_0 \in (a, b)$,

such that $f'(x_0) = 0$

then, for $x \in (x_0 - \delta, x_0)$, $f'(x) > 0$; for $x \in (x_0, x_0 + \delta)$, $f'(x) < 0$,
contradict to conditions!

② general case: $f(a) \neq f(b)$

$$\text{let } g(x) = f(x) - f_{ab}(x) = f(x) - \frac{f(b)-f(a)}{b-a}(x-a) - f(a)$$

$$\text{then } g(a) = g(b) = 0$$

$$g'(x) = f'(x) - \frac{f(b)-f(a)}{b-a} \text{ (monotone increasing)}$$

in particular, apply the previous case to get $g(x) < 0 \forall x \in (a, b)$

so $f(x) < f_{ab}(x)$ for all $x \in (a, b)$

(12) I is an open interval, and if f is convex on I , then f is continuous on I .

proof: 我们分两个部分证明:

① for all $a < b \in I$, for all $x \in I \setminus (a, b)$: $f(x) \geq \frac{f(b)-f(a)}{b-a}(x-a) + f(a)$

WLOG, consider $a < b < 0 \in I$.

if $f(c) < l_{ab}(c)$, then $l_{ac}(x) < l_{ab}(x)$ for $x \in (a, c)$

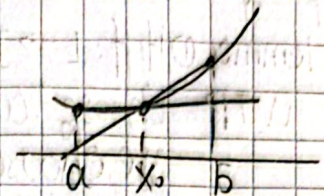
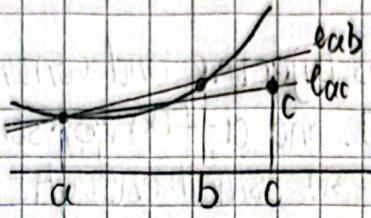
whereas, at $x=b$, $l_{ac}(b) < l_{ab}(b) = b$, contradict to definition of convex

① for $x \in [x_0, b]$

$$l_{ax_0}(x) \leq f(x) \leq l_{x_0b}(x)$$

when $x \rightarrow x_0^+$, by squeeze theorem, $f(x) \rightarrow f(x_0)$

similarly for $x \in [a, x_0]$



[注意: 对于 close interval, 该条不成立 考虑 $y = \begin{cases} x^2 & \text{for } x \in (-1, 1) \\ 2 & \text{for } x = \pm 1 \end{cases}$]

(13) we say $f: I \rightarrow \mathbb{R}$ is convex on I if the epigraph of f

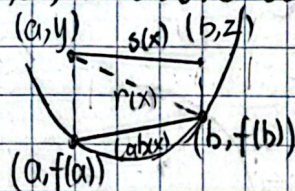
$$\text{Epi}(f) = \{(x, y) \in \mathbb{R}^2 \mid x \in I, y \geq f(x)\}$$

is a convex set.

(14) (for $x \in [a, b]$, $f(x) \leq l_{ab}(x)$) $\Leftrightarrow$ (epi(f) is convex)

proof: $\Rightarrow$: pick $(a, f(a)), (b, f(b))$ on epi(f) then l_{ab} is in epi(f) by definition of convex, so $l_{ab}(x) \geq f(x)$ for $x \in [a, b]$

$\Leftarrow$: pick two points $(a, y), (b, z)$ such that $y \geq f(a), z \geq f(b)$ we just need to show the straight line segment joining (a, y) with (b, z) is above $l_{ab}(x)$



用中间做桥梁, 证明 $l_{ab}(x) \leq r(x) \leq s(x)$

(15) let $f: I \rightarrow \mathbb{R}$ be convex, then for any $a < b < c \in I$, we have $\frac{f(b)-f(a)}{b-a} \leq \frac{f(c)-f(b)}{c-b}$

proof: for $x \in [b, c]$, $l_{ab}(x) \leq f(x) \leq l_{cb}(x)$

$$\frac{f(b)-f(a)}{b-a}(x-b) + f(b) \leq \frac{f(c)-f(b)}{c-b}(x-b) + f(b)$$

$$\frac{f(b)-f(a)}{b-a} \leq \frac{f(c)-f(b)}{c-b}$$

(16) let $f: I \rightarrow \mathbb{R}$ be convex, then for any $a < b < c < d \in I$, we have $\frac{f(b)-f(a)}{b-a} \leq \frac{f(d)-f(c)}{d-c}$

let $f: I \rightarrow \mathbb{R}$ be convex and differentiable at the two points $s < t \in I$, then $f'(s) \leq f'(t)$

inverse

(1) assume $f: I \rightarrow \mathbb{R}$ satisfies:

① f is differentiable on I ② $f': I \rightarrow \mathbb{R}$ is continuous

③ if $f'(a) \neq 0$ at $a \in I$ then f is bijection near a and its inverse is differentiable

[注意: 如果 f^{-1} 是 differentiable, f 必须满足 $f'(a) \neq 0$! etc $y=x^3$]

f is bijection between $f(a) = f(1) = 1$

proof: 10m

lemma ① if $f: I \rightarrow J$ is bijective and continuous on I , then f^{-1} is continuous
choose $\alpha \in J$, and $\alpha = f(x)$ for some $x \in I$

we choose $\varepsilon > 0$ small enough such that $(x - \varepsilon, x + \varepsilon) \subset I$
since f is bijective and continuous, by lemma 2 we know f is
monotone. WLOG, f is monotone increasing, so.

$$x - \varepsilon < x < x + \varepsilon$$

$$f(x - \varepsilon) < f(x) < f(x + \varepsilon)$$

$$f(x - \varepsilon) < \alpha < f(x + \varepsilon)$$

$$\text{let } \delta = \min\{f(x + \varepsilon) - \alpha, \alpha - f(x - \varepsilon)\} > 0$$

$$\text{then } (\alpha - \delta, \alpha + \delta) \subset (f(x - \varepsilon), f(x + \varepsilon))$$

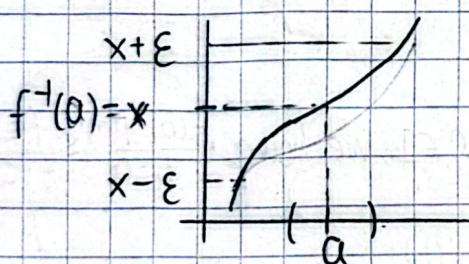
finally, f is monotone increasing, so is f^{-1} is also monotone increasing

if $|y - \alpha| < \delta$, then

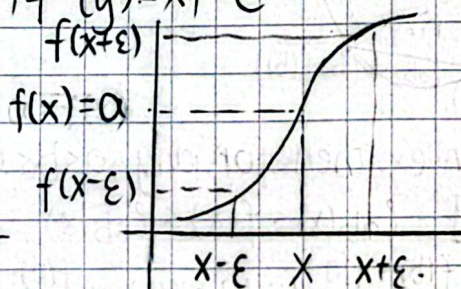
$$f(x - \varepsilon) \leq \alpha - \delta < y < \alpha + \delta \leq f(x + \varepsilon)$$

$$x - \varepsilon \leq f^{-1}(y) \leq x + \varepsilon$$

$$|f^{-1}(y) - x| < \varepsilon$$



f^{-1} graph



f graph

lemma ② if $f: I \rightarrow J$ is bijective and continuous on I , then f is strictly monotone

prove by contradiction, WLOG, there exists $a < b < c \in I$
such that $f(a) < f(c) < f(b)$

by intermediate value theory, there exists a $t \in (a, b)$ such
that $f(t) = f(c)$, no bijective

lemma ③ let $f: I \rightarrow J$ be continuous and injective. assume f is differentiable
at $x_0 \in I$ with $f'(x_0) \neq 0$, then f^{-1} is differentiable at $f(x_0)$
and $(f^{-1})'(f(x_0)) = \frac{1}{f'(x_0)}$

$$\text{let } y_0 = f(x_0), \text{ w.i.p: } \lim_{h \rightarrow 0} \frac{f^{-1}(y_0 + h) - f^{-1}(y_0)}{h} = \frac{1}{f'(x_0)}$$

since f is bijective, given small h , there exists $k = k(h)$ that
such that $y_0 + h = f(x_0 + k)$, $f^{-1}(y_0 + h) = x_0 + k$

$$k(h) = k = f^{-1}(y_0 + h) - x_0$$

since f^{-1} is continuous, $\lim_{h \rightarrow 0} k(h) = 0$, therefore

$$\text{therefore } \lim_{h \rightarrow 0} \frac{f^{-1}(y_0 + h) - f^{-1}(y_0)}{h} = \lim_{h \rightarrow 0} \frac{x_0 + k - x_0}{f(x_0 + k) - f(x_0)}$$

$$= \lim_{k \rightarrow 0} \frac{k}{f(x_0 + k) - f(x_0)}$$

since denominator $\neq 0$, LHS = $\frac{1}{f'(x_0)}$

• integral

(1) let $f: [a, b] \rightarrow \mathbb{R}$ be a bounded function, given a partition P of $[a, b]$, let

$$m_i = \inf_{t_{i-1} \leq t \leq t_i} f, M_i = \sup_{t_{i-1} \leq t \leq t_i} f$$

then the lower sum of f for P : $L(f, P) = \sum_{i=1}^n m_i (t_i - t_{i-1})$

the upper sum of f for P : $U(f, P) = \sum_{i=1}^n M_i (t_i - t_{i-1})$

注意定义是
sup与inf

(2) given P and Q partitions of $[a, b]$, we have $L(f, P) \leq U(f, Q)$

proof: ① if $P \leq P'$, so P' is a refinement of P , then

$$L(f, P) \leq L(f, P') \leq U(f, P') \leq U(f, P)$$

证明时考虑 $P \leq P_1 \leq P_2 \leq \dots \leq P_n \leq P'$, 每次增加一个 t_i

② let $R = P \cup Q$, then done

(3) a bounded function $f: [a, b] \rightarrow \mathbb{R}$ is integrable if $\sup \{L(f, P) \mid P \text{ partition of } [a, b]\} = \inf \{U(f, P) \mid P \text{ partition of } [a, b]\}$, and we denote this number as $\int_a^b f(x) dx$

(4) if f is bounded on $[a, b]$ then f is integrable on $[a, b]$ if and only if for every $\epsilon > 0$, there is a partition P of $[a, b]$ such that $U(f, P) - L(f, P) < \epsilon$

proof: ① $\Rightarrow$

since integrable, $\sup_P \{L(f, P)\} = \inf_{P'} \{U(f, P')\}$, then $\exists P', P''$ such that $\sup_P \{L(f, P)\} - \frac{\epsilon}{2} \leq L(f, P') \leq U(f, P') \leq \inf_{P'} \{U(f, P')\} + \frac{\epsilon}{2}$

then define $P = P' \cup P''$, done

$$\textcircled{2} \leftarrow \forall \epsilon > 0, \inf_{P''} \{U(f, P'')\} - \sup_{P'} \{L(f, P')\} \leq U(f, P) - L(f, P) < \epsilon$$

(5) [注意: 常用此判断是否 integrable]

(5) if f is continuous on $[a, b]$, then it is integrable on $[a, b]$

proof: recall that f is uniformly continuous in this case

$$\forall \epsilon > 0, \exists \delta > 0, \forall x, y, \text{ such that } |x - y| < \delta, |f(x) - f(y)| < \epsilon$$

choose a partition $P = \{a = t_0 < t_1 < \dots < t_n = b\}$ such that $|t_k - t_{k-1}| < \delta$ for $k = 1, \dots, n$

then for all $s, t \in [t_{k-1}, t_k]$, $|f(s) - f(t)| < \epsilon$

$$\text{assume } f(s) = \max_{t_{k-1} \leq t \leq t_k} f = M_k, f(t) = \min_{t_{k-1} \leq t \leq t_k} f = m_k$$

$$U(f, P) - L(f, P) = \sum_{k=1}^n M_k (t_k - t_{k-1}) - \sum_{k=1}^n m_k (t_k - t_{k-1}) \\ = \sum_{k=1}^n (M_k - m_k) (t_k - t_{k-1}) \leq \epsilon \sum_{k=1}^n (t_k - t_{k-1}) = \epsilon (b - a)$$

13

(6) f is integrable on $[a, c]$ if and only if f is integrable on $[a, b]$ and $[b, c]$ for $a < b < c$

• if f and g are integrable on $[a, b]$, then $f + g$ is integrable on $[a, b]$

• if f is integrable on $[a, b]$ and $c \in \mathbb{R}$, then $c \cdot f$ is integrable on $[a, b]$

proof: ① $\Leftarrow$ Join the partitions all together

$\Rightarrow$ there is a partition R of $[a, c]$, such that $U(f, R) - L(f, R) < \epsilon$

let $R' = R \cup \{b\}$, then R' is a refinement of R , so $U(f, R') - L(f, R') < \epsilon$

let $P = R' \cap [a, b]$, $Q = R' \cap [b, c]$

so $(U(f, P) - L(f, P)) + (U(f, Q) - L(f, Q)) < \epsilon$

since they are all positive, done

[证明 $\int_a^b f + g dx = \int_a^b f dx + \int_a^b g dx$ 时, 考虑 $U(f, P) - L(f, P) < \epsilon$, $U(f, P) > \int_a^b f dx$

$> L(f, P)$, 得到 $|\int_a^b f dx - U(f, P)| < \epsilon$... 最后证明 $|\int_a^b f + g dx - (\int_a^b f dx +$

$\int_a^b g dx)| < 2\epsilon$]

[注意: $f(x) \geq 0$ 且 $\int_a^b f(x) dx = 0$ 并不说明 $f(x) = 0 \forall x$!

考虑 $f(x) = \begin{cases} 0 & \text{for } x \neq 0 \\ 1 & \text{for } x = 0 \end{cases}$

事实上 integrable 并不要求 continuous. 有有限个有限间断点的话, 函数仍可积]

(7) if f is continuous, $f(x) \geq 0$ for $\forall x \in [a, b]$, then if $\int_a^b f(x) dx = 0$, then $f(x) = 0$

proof say $\exists x_0 \in [a, b]$ such that $f(x_0) > 0$

by continuity, $\exists \delta > 0$ such that $f(t) > \frac{f(x_0)}{2}$ for $\forall t \in [x_0 - \delta, x_0 + \delta]$

so $\int_a^b f(x) dx \geq L(f, [a, x_0 - \delta, x_0 + \delta, b]) > 0$.

(8) if f is integrable on $[a, b]$, then it is also integrable on $[a, x], \forall x \in [a, b]$

therefore let $F(x) = \int_a^x f(t) dt$, then $F(x)$ is continuous on $[a, b]$

proof: pick $x < y \in [a, b]$

$|F(y) - F(x)| = |\int_a^y f(t) dt - \int_a^x f(t) dt| = |\int_x^y f(t) dt| \leq \int_x^y |f(t)| dt$

since integration function is always bounded, $\exists M > 0$ such that

$|F(y) - F(x)| \leq \int_x^y |f(t)| dt \leq M|y - x|$

(9) if f is continuous at x , then $F'(x) = f(x)$

proof: ① $\frac{F(x+h) - F(x)}{h} = \frac{1}{h} \int_x^{x+h} f(t) dt$

so $m \cdot h \leq \int_x^{x+h} f(t) dt \leq M \cdot h$, where m, M are minimum and maximum on $[a, a+h]$

when $h \rightarrow 0$, $F'(x) \rightarrow f(x)$

② $|\frac{F(x+h) - F(x)}{h} - f(x)| = |\frac{1}{h} \int_x^{x+h} f(t) dt - f(x)|$

$= |\frac{1}{h} \int_x^{x+h} f(t) dt - \frac{1}{h} \int_x^{x+h} f(x) dt| = |\frac{1}{h} \int_x^{x+h} (f(t) - f(x)) dt|$

$\leq \frac{1}{h} \int_x^{x+h} |f(t) - f(x)| dt \leq \frac{1}{h} \int_x^{x+h} \epsilon dt = \epsilon$

that there exists $\delta > 0$ such that $|t - x| < \delta$ leads to $|f(t) - f(x)| < \epsilon, h \in (0, \delta)$

(10) if f is continuous on $[a, b]$ and $f = g'$ for some function g , then

$$\int_a^b f(x) dx = g(b) - g(a)$$

proof: let $F(x) = \int_a^x f(t) dt$, notice that $F'(x) = f(x) = g'(x)$
 so $H(x) = F(x) - g(x)$ is a constant

$$H(a) = F(a) - g(a) = -g(a)$$

$$H(b) = F(b) - g(b) = -g(a)$$

$$F(b) = \int_a^b f(x) dx = g(b) - g(a)$$

[注意: $\frac{d}{dx} \int_x^b f(t) dt \Big|_{x=c} = -f(c)$]

(11) if g' is integrable on $[a, b]$ then $\int_a^b g'(t) dt = g(b) - g(a)$

proof: let $P: t_0 = a < t_1 < \dots < t_n = b$ be any partition on $[a, b]$, then by MVT there exists a point $x_i \in [t_{i-1}, t_i]$ such that

$$g(t_i) - g(t_{i-1}) = g'(x_i)(t_i - t_{i-1})$$

since $m_i(t_i - t_{i-1}) \leq g'(x_i)(t_i - t_{i-1}) \leq M_i(t_i - t_{i-1})$, where m_i, M_i are the minimum and maximum of $g'(x)$ over $[t_{i-1}, t_i]$

sum together, $L(f, P) \leq g(b) - g(a) \leq U(f, P)$

(12) let $\phi: [a, b] \rightarrow \mathbb{R}$ and $f: \mathbb{R} \rightarrow \mathbb{R}$ assume f and ϕ' are continuous everywhere, then $\int_a^b (f(\phi(t))\phi'(t)) dt = \int_{\phi(a)}^{\phi(b)} f(t) dt$

proof: define $F(x) = \int_a^x f(\phi(t))\phi'(t) dt$

$$G(x) = \int_{\phi(a)}^{\phi(x)} f(t) dt$$

then show $F(a) = G(a)$, $F'(x) = G'(x)$

• log and exp.

(1) let $L: (0, +\infty) \rightarrow \mathbb{R}$ be the function defined by $L(x) = \int_1^x \frac{1}{t} dt$, $x > 0$.

• then L is differentiable and monotonic increasing since fundamental theorem of calculus, $\frac{1}{t}$ is continuous for $t \in (0, +\infty)$

• for $x, y > 0$, we have $L(xy) = L(x) + L(y)$

proof: $L(xy) = \int_1^{xy} \frac{1}{t} dt = \int_1^x \frac{1}{t} dt + \int_x^{xy} \frac{1}{t} dt = \int_1^x \frac{1}{t} dt + \int_1^y \frac{1}{xt} x dt = L(x) + L(y)$

• $L(1) = 0$, and $L(\frac{1}{x}) = -L(x)$ for $x > 0$

proof: $L(x) = \int_1^x \frac{1}{t} dt = \int_{\frac{1}{x}}^1 \frac{1}{x \cdot t} x dt = \int_{\frac{1}{x}}^1 \frac{1}{t} dt = -L(\frac{1}{x})$

• $\lim_{x \rightarrow \infty} L(x) = \infty$, $\lim_{x \rightarrow 0^+} L(x) = -\infty$

proof: since $L(x) = -L(\frac{1}{x})$ for $x > 0$, it's enough to just show $\lim_{x \rightarrow \infty} L(x) = \infty$
 pick $x_0 > 1$, since $\frac{1}{t} > 0$ and continuous, $L(x_0) > 0$

since $L(xy) = L(x) + L(y)$, $L(x_0^n) = nL(x_0)$, $\lim_{n \rightarrow \infty} L(x_0^n) = \infty$

since L is monotonic increasing, $\lim_{x \rightarrow \infty} L(x) = \infty$

(2) since $L: (0, \infty) \rightarrow \mathbb{R}$ is bijective, it has an inverse $E: \mathbb{R} \rightarrow (0, \infty)$

by inverse function theorem, it satisfies:

- E is differentiable.

- $E'(y) = \frac{1}{L'(E(y))} = \frac{1}{\frac{1}{E(y)}} = E(y)$

- $E(xy) = E(x)E(y)$, as $E(L(a)+L(b)) = E(L(ab)) = ab$

• Taylor Polynomial and Taylor Series

given f has $(n+1)^{\text{th}}$ derivatives, its Taylor polynomial of order n at a is

$$P_{n,a} = \sum_{k=0}^n \frac{f^{(k)}(a)}{k!} (x-a)^k$$

we also define the remainder of f at a as $R(x) = f(x) - P_{n,a}(x)$

assume $f^{(n+1)}$ are defined on $[a, x]$, there exists $\theta \in (a, x)$ such that

$$R(x) = \frac{f^{(n+1)}(\theta)}{(n+1)!} (x-a)^{n+1}, \quad f(x) = P_{n,a}(x) + \frac{f^{(n+1)}(\theta)}{(n+1)!} (x-a)^{n+1} \text{ for } \theta \in (a, x)$$

proof: we can let $G(x) = (x-a)^{n+1}$, we all know that

$$\begin{cases} R^{(k)}(a) = 0 \\ G^{(k)}(a) = 0 \end{cases} \text{ for } k = 1, 2, \dots, n$$

$$\begin{aligned} \text{so } \frac{R(x)}{G(x)} &= \frac{R(x) - R(a)}{G(x) - G(a)} = \frac{R'(\theta_1)}{G'(\theta_1)} = \frac{R'(\theta_1) - R'(a)}{G'(\theta_1) - G'(a)} = \frac{R''(\theta_2)}{G''(\theta_2)} \\ &= \dots = \frac{R^{(n+1)}(\theta_n)}{G^{(n+1)}(\theta_n)} = \frac{R^{(n+1)}(\theta_n)}{(n+1)!} \end{aligned}$$

$$\text{for } a \leq \theta_n \leq \dots \leq \theta_2 \leq \theta_1 \leq x$$

$$f \in o(|x|^k) \text{ as } x \rightarrow a \text{ if } \lim_{x \rightarrow a} \frac{f(x)}{|x|^k} = 0$$

if $f^{(n+1)}$ is bounded, then $f - P_{n,a} = R \in o(|x|^{n+1})$

[注意: there are important function such that $P_{n,a} \equiv 0$ for $\forall n$ without f being 0. consider $f(x) = \begin{cases} e^{-\frac{1}{x}} & \text{if } x > 0 \\ 0 & \text{if } x \leq 0 \end{cases}$]

(3) assume $f, f', \dots, f^{(n+1)}$ are all continuous near a , then

$$R(x) = \int_a^x f^{(n+1)}(t) \frac{(x-t)^n}{n!} dt$$

proof: integration by part

$$f(x) = P_{0,a}(x) + \int_a^x f^{(1)}(t) dt$$

$$\int_a^x f^{(1)}(t) dt = \int_a^x f'(t) \frac{d}{dt} (t-x) dt = f'(a)(x-a) + \int_a^x f^{(2)}(t) (x-t) dt = \dots$$

(4) Taylor series is derived by building the Taylor polynomial incrementally

$$P_{0,a}(x), P_{1,a}(x), P_{2,a}(x), \dots$$

a function f is real analytic at a if there is $r > 0$ such that for all $x \in (a-r, a+r)$, $f(x) = \lim_{n \rightarrow \infty} P_{n,a}(x)$

the maximal $r > 0$ for which this property holds is called the radius of convergence of Taylor series around a .

[注意: if the Taylor series of $f(x)$ converges, it may not converge

to $f(x)$ itself. consider $f(x) = \begin{cases} e^{-\frac{1}{x}} & \text{if } x > 0 \\ 0 & \text{if } x \leq 0 \end{cases}$

(5) let f have derivative of all orders in $|x-a| < p$, assume there are constants $N, C, R > 0$ and $p > 0$ such that $|f^{(n)}(x)| \leq C \cdot \frac{n!}{R^n}$ for all $|x-a| < p$ and $n \geq N$, then $f(x) = \lim_{n \rightarrow \infty} P_{n,a}(x)$ for all $|x-a| < r$, where $r = \min\{R, p\}$

proof: in such a way,

$$|R_{n,a}(x)| \leq C \cdot \frac{(n+1)!}{R^{n+1}} \cdot \frac{(x-a)^{n+1}}{(n+1)!} = C \cdot \left(\frac{x-a}{R}\right)^{n+1}$$

example: for $f(x) = e^x$, $f^{(n)}(x) = e^x$
consider $e^x \leq C \cdot \frac{n!}{R^n}$ for $|x-a| < p$

pick $C = e^{10+p}$ and N, R such that $N! \geq R^N$ (can be achieved since $\frac{1}{N} \sum_{k=1}^N \ln k \rightarrow \ln x$ as $\ln x \rightarrow \infty$)

therefore the Taylor series of e^x centered around any points $a \in \mathbb{R}$ converges in all of $\mathbb{R}$.

• convergence of function

(1) let $f_n: [a, b] \rightarrow \mathbb{R}$ be a sequence of functions. we say that f_n converge to f pointwise on $[a, b]$ if for all $x \in [a, b]$, the sequence formed by the values $f_n(x)$ converges to $f(x)$

$$\lim_{n \rightarrow \infty} f_n(x) = f(x) \text{ for all } x \in [a, b]$$

[注意] 即使 $f_n(x)$ 均 continuous, $f(x)$ 也不一定 continuous ($f_n(x) = x^n$)
even if f_n 's are all infinitely differentiable, pointwise continuous isn't strong enough to pass continuity to limit.

即使 $f_n(x)$ 均 integrable, $f(x)$ 也不一定 integrable ($f_n = \sqrt[n]{T(x)}$ where

$$T(x) = \begin{cases} 0 & \text{if } x \notin \mathbb{Q} \\ \frac{1}{p} & \text{if } x \in \mathbb{Q} \text{ and } x = \frac{a}{p} \text{ in lowest form} \end{cases}$$

$$\lim_{n \rightarrow \infty} f_n(x) = D = \begin{cases} 0 & \text{if } x \notin \mathbb{Q} \\ 1 & \text{if } x \in \mathbb{Q} \end{cases}$$

即使 $f(x)$ integrable, $\lim_{n \rightarrow \infty} \int f_n dx$ 也不一定为 $\int f dx$. ($f_1 = \text{dis}(x, \{0, 1\})$,

$$f_n = n f_1(n x), \text{ then } f_n \rightarrow 0, \int f_n dx = \frac{1}{4} \forall n)$$

(2) a sequence of functions $f_n: [a, b] \rightarrow \mathbb{R}$ is uniformly convergent if, given $\epsilon > 0$, $\exists N \in \mathbb{N}$, $\forall n \geq N$, $|f_n(x) - f(x)| < \epsilon$ for $\forall x \in [a, b]$

[注意]: (f_n uniformly convergent to f) $\Rightarrow$ (f_n pointwisely convergent to f), the converse isn't correct

(3) let $f_n: [a, b] \rightarrow \mathbb{R}$ be a sequence of functions uniformly convergent to $f: [a, b] \rightarrow \mathbb{R}$. if f_n are continuous then f is continuous.

Proof: let $x, y \in [a, b]$

$$|f(x) - f(y)| = |f(x) - f_n(x) + f_n(x) - f_n(y) + f_n(y) - f(y)| \\ \leq |f(x) - f_n(x)| + |f_n(x) - f_n(y)| + |f_n(y) - f(y)|$$

notice that f_n is continuous and $[a, b]$ is compact, so f_n is uniformly continuous.

due to uniformly convergent, $\exists N \in \mathbb{N}, \forall n > N, |f_n(x) - f(x)| < \varepsilon, |f_n(y) - f(y)| < \varepsilon$.

due to uniformly continuous, $\exists \delta > 0, \forall x, y$ s.t. $|x - y| < \delta, |f_n(x) - f_n(y)| < \varepsilon$

so if we fixed $n > N, \exists \delta > 0, \forall x, y$ s.t. $|x - y| < \delta, |f(x) - f(y)| < 3\varepsilon$

(4) let $f_n: [a, b] \rightarrow \mathbb{R}$ be a sequence of functions uniformly convergent to a function $f: [a, b] \rightarrow \mathbb{R}$. if f_n are integrable, then their limit f is also integrable and

$$\lim_{n \rightarrow \infty} \int_a^b f_n(t) dt = \int_a^b f(t) dt$$

Proof: we firstly check that $f(x)$ is bounded

$$|f(x)| \leq |f(x) - f_n(x)| + |f_n(x)| \leq \sup_{[a, b]} |f(x) - f_n(x)| + \sup_{[a, b]} |f_n|$$

we can choose $n > N$ so $|f(x) - f_n(x)| \leq \varepsilon$ by uniformly convergent and $|f_n(x)|$ is bounded

• we then show $f(x)$ is integrable

$\exists N \in \mathbb{N}, \forall n > N, |f_n(x) - f(x)| < \varepsilon$. if $n > N$, let $C_n = \sup_{[a, b]} |f_n(x) - f(x)|$

then for each $x \in [a, b]$, we have

$$|f_n(x) - f(x)| \leq C_n$$

$$f_n(x) - C_n \leq f(x) \leq f_n(x) + C_n$$

$$m_i^n - C_n \leq m_i \leq M_i \leq M_i^n + C_n$$

where $M_i = \sup_{[t_{i-1}, t_i]} f, m_i = \inf_{[t_{i-1}, t_i]} f, M_i^n = \sup_{[t_{i-1}, t_i]} f_n, m_i^n = \inf_{[t_{i-1}, t_i]} f_n$

therefore $M_i - m_i \leq M_i^n - m_i^n + 2C_n$

it follows, for any partition P of $[a, b]$

$$U(f, P) - L(f, P) \leq U(f_n, P) - L(f_n, P) + 2C_n(b-a)$$

given $\varepsilon > 0$, choosing $n \in \mathbb{N}$ such that $C_n < \frac{\varepsilon}{b-a}$, and then P such

$$U(f_n, P) - L(f_n, P) < \varepsilon, \text{ we obtain } U(f, P) - L(f, P) \leq 3\varepsilon$$

so f is integrable

• finally, show the equality

$$|\int_a^b f_n(t) dt - \int_a^b f(t) dt| \leq \int_a^b |f_n(t) - f(t)| dt \leq C_n(b-a)$$

as $C_n \rightarrow 0$, the limit of integral exists

(5) let $f_n: [a, b] \rightarrow \mathbb{R}$ be a sequence of continuously differentiable functions. (if f and f' are all continuous)

assume (i) f_n uniformly convergent to g on $[a, b]$

(ii) $\exists x_0 \in [a, b]$ such that $f_n(x_0)$ convergent.

then there exists a function $f: [a, b] \rightarrow \mathbb{R}$ such that

(a) f_n uniformly convergent to f on $[a, b]$

(b) $f'(x) = \lim_{n \rightarrow \infty} f'_n(x)$ for all $x \in (a, b)$

proof: ① g is continuous

② let $L = \lim_{n \rightarrow \infty} f_n(x_0)$, define $f(x)$ as $f(x) = L + \int_{x_0}^x g(t) dt$

then $f'(x) = g(x) = \lim_{n \rightarrow \infty} f'_n(x)$

$$\begin{aligned} \textcircled{3} |f_n(x) - f(x)| &= |f_n(x_0) - f(x_0) + \int_{x_0}^x f'_n(t) - g(t) dt| \\ &\leq |f_n(x_0) - f(x_0)| + \left| \int_{x_0}^x f'_n(t) - g(t) dt \right| \rightarrow 0 \end{aligned}$$

[注意: 一般的 uniformly continuous 并不能保证 differentiable]

考虑 $f_n(x) = \sqrt{\frac{1}{n} + x^2} \rightarrow f(x) = |x|$

(6) let $\phi_k: [a, b] \rightarrow \mathbb{R}$ be a sequence of functions. we denote $f_n(x) = \sum_{k=1}^n \phi_k(x)$
the sequence of partial sum

we say the series $\sum_{k=1}^{\infty} \phi_k$ converges uniformly on $[a, b]$ to a function f
if the sequence of partial sums f_n converges to f uniformly on $[a, b]$

(7) assume ϕ_k are integrable on $[a, b]$ and $\sum_{k=1}^{\infty} \phi_k$ converges uniformly to f on $[a, b]$, then $\int_a^b f(x) dx = \sum_{k=1}^{\infty} \int_a^b \phi_k(x) dx$

(8) assume ϕ_k are continuous by differentiable, suppose

- $\sum_{k=1}^n \phi'_k$ converges uniformly on $[a, b]$

- $\exists x_0 \in [a, b]$ such that $\sum_{k=1}^{\infty} \phi_k(x_0)$ converges

then:

- $\sum_{k=1}^{\infty} \phi_k$ converges uniformly to a differentiable function f on $[a, b]$

- $\sum_{k=1}^{\infty} \phi'_k(x)$ converges uniformly to $f'(x)$

in other words:

$$\frac{d}{dx} \left(\sum_{k=1}^{\infty} \phi_k(x) \right) = \sum_{k=1}^{\infty} \left(\frac{d}{dx} \phi_k(x) \right)$$

(9) let $f_n(x)$ be the partial sums of a series of functions

assume there is a sequence of real numbers $M_n \geq 0$ such that

- $|\phi_k(x)| \leq M_k$ for all $x \in [a, b]$,

- $\sum_{k=1}^{\infty} M_k < \infty$

then the series $\sum_{k=1}^{\infty} \phi_k(x)$ uniformly converges on $[a, b]$

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Proof: use Cauchy to prove it. let $n < m \in \mathbb{N}$

$$|f_m(x) - f_n(x)| = \left| \sum_{k=n+1}^m \phi_k(x) \right| \leq \sum_{k=n+1}^m |\phi_k(x)| \leq \sum_{k=n+1}^m M_k$$

in particular, since $\sum_k M_k$ is a convergent series, the final value is smaller than some $\varepsilon > 0$ if n is large enough.

so for $x \in [a, b]$, $f_n(x)$ is Cauchy

let $m \rightarrow \infty$, $|f(x) - f_n(x)| \leq \sum_{k=n+1}^{\infty} M_k$, which can be arbitrary small, and the bound is uniform for $x \in [a, b]$ (不取决于 x).

注意 (M-test 通过) $\rightarrow$ (uniformly convergent)

(M-test 不通过) $\rightarrow$ (not uniformly convergent)]

(10) let f_n be the partial sums of a series of function if ϕ_k are integrable and there are $M_k \geq 0$ such that

- $|\phi_k(x)| \leq M_k$ for all $x \in [a, b]$

- $\sum_{k=1}^{\infty} M_k < \infty$

then f_n converges uniformly, in particular, $\int_a^b \sum_{k=1}^{\infty} \phi_k(x) dx = \sum_{k=1}^{\infty} \int_a^b \phi_k(x) dx$

(11) let f_n be the partial sums of a series of function

if ϕ_k are continuously differentiable and there are numbers $M_k \geq 0$ such that

- $|\phi'_k(x)| \leq M_k$ for all $x \in [a, b]$

- $\sum_{k=1}^{\infty} M_k < \infty$

- $\sum_{k=1}^{\infty} \phi_k(x_0)$ converges for some $x_0 \in [a, b]$

then f_n converges uniformly to a differentiable function and we can differentiate them by terms.

Curves

(1) a curve $C(t) = (u(t), v(t))$ is continuous if and only if u, v are continuous. it is differentiable if and only if u, v are differentiable, and $\frac{dC}{dt}(t) = \left(\frac{du}{dt}(t), \frac{dv}{dt}(t) \right)$

(2) $\hat{C}: I' \rightarrow \mathbb{R}^2$ is a reparametrisation of $C: I \rightarrow \mathbb{R}^2$ if there is a map $\Phi: I' \rightarrow I$ such that $\hat{C} = C \circ \Phi$ and Φ is strictly monotone increasing.

(3) a differentiable curve $C: I \rightarrow \mathbb{R}^2$ is parametrised with unit speed if $|C'(t)| = 1 \forall t \in I$. we also call it parametrised by arc length.

(4) let $\phi: (A, B) \rightarrow (a, b)$, $\bar{\phi}: (\bar{A}, \bar{B}) \rightarrow (a, b)$ be monotone increasing and differentiable such that $C(\phi(t_0)) = C(\bar{\phi}(t_0))$ for some $t_0 \in I = (\bar{A}, \bar{B}) \cap (A, B)$. if $\left| \frac{dC(\phi(t))}{dt} \right| = 1 \forall t \in (A, B)$, $\left| \frac{dC(\bar{\phi}(t))}{dt} \right| = 1 \forall t \in (\bar{A}, \bar{B})$, then $\phi = \bar{\phi}$ on I .

proof:

since $|c'(\phi(t))||\phi'(t)| = 1$, $|c'(\bar{\phi}(t))||\bar{\phi}'(t)| = 1$

so $\phi'(t) = \frac{1}{|c'(\phi(t))|}$, $\bar{\phi}'(t) = \frac{1}{|c'(\bar{\phi}(t))|}$ as they are monotonic increasing

$$L = \int_{t_0}^t |(c(\phi(x)))'| dx = \int_{\phi(t_0)}^{\phi(t)} |c'(u)| du = t - t_0$$

$$L = \int_{t_0}^t |(c(\bar{\phi}(x)))'| dx = \int_{\bar{\phi}(t_0)}^{\bar{\phi}(t)} |c'(u)| du = t - t_0$$

since $\phi(t_0) = \bar{\phi}(t_0)$, $\phi(t) = \bar{\phi}(t)$

so reparametrisation by arc length is unique!

(5) let $c: [a, b] \rightarrow \mathbb{R}^2$ be a continuously differentiable curve, we define its length as $\ell(c) = \ell(c|_{[a, b]}) = \int_a^b |c'(t)| dt$

(6) let $c: (a, b) \rightarrow \mathbb{R}^2$ be a continuously differentiable curve assume $c'(t) \neq 0$ for $\forall t \in (a, b)$, then there exists an open interval $(A, B) \subset \mathbb{R}$ and a differentiable function $\phi: (A, B) \rightarrow (a, b)$ such that $\tilde{c} = c \circ \phi$ has unit speed

proof:

fix $t_0 \in (a, b)$. let $f(t) = \int_{t_0}^t |c'(s)| ds$

by the fundamental theorem of calculus, the function $f: (a, b) \rightarrow \mathbb{R}$ is differentiable and $f'(t) = |c'(t)| \neq 0 \quad \forall t \in (a, b)$

by the inverse function theorem, there is an interval (A, B) and an inverse map $\phi: (A, B) \rightarrow (a, b)$, $f(\phi(t)) = t$, $\phi'(t) = \frac{1}{|c'(\phi(t))|}$ for $\forall t \in (A, B)$

in particular, $|(c(\phi(t)))'| = |c'(\phi(t))||\phi'(t)| = 1 \quad \forall t \in (A, B)$

[注意: 任何图像均不能被光滑可导函数表示, 但并不一定被弧长参数化]

example: $C(t) = (t^3, |t|^2)$

• a nowhere differentiable function

firstly introduce $\psi(x) = \text{dist}(x, \mathbb{Z})$, which mean $\psi(x) = \begin{cases} x-k & \text{if } x \in [k, k+\frac{1}{2}] \\ k+1-x & \text{if } x \in [k+\frac{1}{2}, k+1] \end{cases}$

define the takagi function: $T(x) = \sum_{n=0}^{\infty} 2^{-n} \psi(2^n x)$

① by proving uniformly convergent, as $|2^{-n} \psi(2^n x)| \leq 2^{-(n+1)}$, so $T(x)$ is continuous.

② we then prove that it isn't differentiable by lemmas

• lemma 1: fix $N \geq 1$, then $T_N(x) = \sum_{n=0}^N 2^{-n} \psi(2^n x)$ is differentiable

17.

- on every interval $(\frac{k}{2^N}, \frac{k+1}{2^N})$, $k \in \mathbb{Z}$ and its slope is a gradient
- lemma 2: if f is differentiable at x , and $a_n < x < b_n$, satisfying $b_n - a_n \rightarrow 0$ then $f'(x) = \lim_{n \rightarrow \infty} \frac{f(b_n) - f(a_n)}{b_n - a_n}$
 - lemma 3: if $x = \frac{m}{2^n}$, where $m \in \mathbb{Z}$, then for all $N \geq n \in \mathbb{Z}$, $2^N \psi(2^N x) = 0$
let $a = \frac{k}{2^N}$, $b = \frac{k+1}{2^N}$, where $k, l \in \mathbb{Z}$, $T(b) - T(a) = T_N(b) - T_N(a)$

• then for not dyadic points:

fix $x \in (0, 1)$, for $\forall N \geq 1$, there exists a unique dyadic interval (a_N, b_N) of the form $a_N = \frac{k}{2^N}$, $b_N = \frac{k+1}{2^N}$, where $k = k(N) \in \mathbb{Z}$ containing x .

if T is differentiable at x , then $q_N = \frac{T(b_N) - T(a_N)}{b_N - a_N} \rightarrow T'(x)$ by lemma 2.

by lemma 1, q_N is an integer

notice that $T_{N+1}(x) = T_N(x) + 2^N \psi(2^N x)$

while T_N has constant slope on $(\frac{k}{2^N}, \frac{k+1}{2^N})$, the function $2^N \psi(2^N x)$ has 2 regions of constant slope at $(\frac{k}{2^N}, \frac{k}{2^N} + \frac{1}{2^{N+1}})$ and $(\frac{k}{2^N} + \frac{1}{2^{N+1}}, \frac{k+1}{2^N})$

since x is not dyadic, so it is in one of these intervals.

so $q_{N+1} = q_N \pm 1$, $|q_{N+1} - q_N| = 1$, so q_N never convergent.

• for dyadic points:

let $x = \frac{k}{2^n}$, $k \in \mathbb{Z}$, then for $N \geq n$,

$$q_N = \frac{T(x - 2^{-N}) - T(x)}{2^{-N}}, q_{N+1} = q_N + 1, q_N \rightarrow \infty$$

① countability

- injective $f: S \rightarrow \mathbb{N}$ • 列出元素后定义 • product of finite number of primes

② sequence

(1) 证明 convergent

- 老实找定义 (考虑 when n large enough)
- monotonic + bounded
- bounded sequence has convergent subsequence
- cauchy sequence is convergent, convergent sequence is cauchy.
- complex sequence, Re, Im convergent separately

(2) 证明 divergent

- 老实找定义
- two subsequences convergent to different limits

③ series

(1) 证明 summable

- 老实找通式与定义
- $a_n > 0, (S_n)$ bounded
- $a_n \rightarrow 0, \sum (-1)^n a_n$
- absolutely summable
- 3个 comparison test ($L=1?$)
- dirichlet's test: $\sum a_n b_n$ where $\sum a_n$ bounded and $b_n \rightarrow 0$ monotonic decreasing

(2) 证明 not summable

- 老实找通式与定义
- $a_n \rightarrow 0$

④ function

(1) 证明 continuous

- 老实找定义 (设个 δ 上限?)
- Lipschitz continuous, uniformly continuous
- $(a_n \rightarrow a) \Rightarrow (f(a_n) \rightarrow f(a))$

(2) 证明 not continuous

- 老实找定义
- $a_n \rightarrow M, b_n \rightarrow M, f(a_n) \rightarrow A, f(b_n) \rightarrow B, A \neq B$

(3) 证明 Lipschitz continuous

- 老实找定义
- differentiable and derivative bounded

(4) 证明 uniformly continuous

- 老实找定义
- continuous + compact domain

$\mathbb{Q}$ is dense

(a_n) divergent $\nRightarrow (a_n) \rightarrow \infty$

(a_n) bounded $\nRightarrow (a_n)$ convergent

$a_n \rightarrow \infty, b_n \rightarrow 0 \nRightarrow a_n b_n \rightarrow 0$

$a_n \rightarrow 0 \nRightarrow \frac{1}{a_n} \rightarrow \infty$

(a_n) monotonic $\nRightarrow (a_n)$ convergent

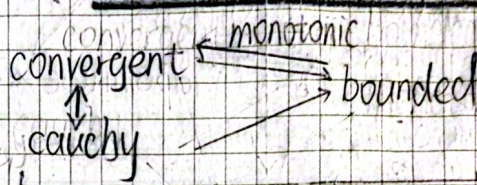
(a_n) convergent $\nRightarrow (a_n)$ monotonic

cauchy = convergent works in $\mathbb{R}$ but not necessarily in other fields.

supremum & infimum

maximum & minimum

写定义时点出 $\mathbb{R}/\mathbb{C}$

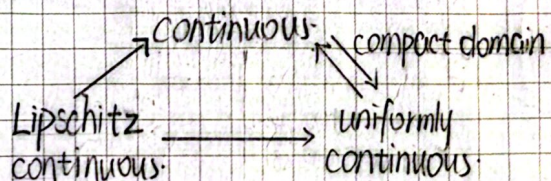


all have monotonic subsequences

用结论时写条件

function 极限的定义不包含本点

极限两证法



- continuous + compact domain $\Rightarrow$ bounded
- $\Rightarrow$ image (f) compact
- $\Rightarrow \exists x_1, x_2 \in K, f(x_1) = \sup f, f(x_2) = \inf f$
- continuous $\Rightarrow$ 介值定理
- closed $\Leftrightarrow$ sequence convergent in K
- compact $\Leftrightarrow$ every sequence has convergent subsequence in K
- $\Leftrightarrow$ closed + bounded

证明时找性质!

open 的逆否: 存在一个 z , such that no open disc around z lies

⑤ 证明 not Lipschitz continuous.

• find a sequence $|f(x_n) - f(y_n)| \rightarrow \infty$

• 证明 not uniformly continuous. $\rightarrow$ pointwise convergent

(6) 证明 not uniformly continuous

• find a sequence $|x_n - y_n| \rightarrow 0, |f(x_n) - f(y_n)| \not\rightarrow 0$

f continuous/integrable

pointwise convergent + continuous $f_n \Rightarrow$ continuous f

⑥ curves.

• reparametrisation: $\hat{c}: I \rightarrow \mathbb{R}^2, c: I \rightarrow \mathbb{R}^2, \hat{c} = c \circ \Phi, \Phi$ monotonic increasing

• parametrised by arc length (which is unique). Φ 为 $\int_{t_0}^t |c'(s)| ds$ 的逆

其 unique. 对于 continuously differentiable curve assuming $c'(t) \neq 0$ for all t .

让 $\Phi(x)$ 为 $\int_{t_0}^x |c'(s)| ds$ 的逆即可

• $\ell(c|_{[a,b]}) = \int_a^b |c'(t)| dt$

pointwise convergence

任何图像均可被光滑导函数表示, 但不一定可以被弧长参数化 ($c'(t) \neq 0$!)

⑦ nowhere differentiable function.

$$T(x) = \sum_{n=0}^{\infty} 2^{-n} \psi(2^n x), \quad \psi(x) = \begin{cases} x-k & x \in [k, k+\frac{1}{2}] \\ k+1-x & x \in [k+\frac{1}{2}, k+1] \end{cases}, k \in \mathbb{Z}.$$

构造

尤其是 ① $a_n \rightarrow a$ 时, 设 $a_n = a + \frac{1}{n}$

② $a_n \rightarrow a$ 时, 设 ε 为定值.

③ $a_n \rightarrow a$ 时, 大胆设含 n 式的上下限去化简

反证

不等式

$f(x)$

$$(f(x) > 0 \text{ 且 } \int_a^b f(x) dx = 0) \not\Rightarrow (f(x) = 0)$$

$$(P_{n,a} = 0 \quad \forall n) \not\Rightarrow (f = 0)$$

$$[\text{考虑 } f(x) = \begin{cases} e^{-\frac{1}{x^2}} & x > 0 \\ 0 & \text{otherwise} \end{cases}]$$

pointwise convergent

① 与 continuous 有关的复习

• 判断 continuous (定义)

• continuous + compact domain

- bounded image

- $\exists x, x \in \text{image}(f), f(x) = \sup f, f(x) = \inf f$

• intermediate value theorem

若 f 在 $[a, b]$ 上连续, 且 $f(a) \neq f(b)$, 则对任意介于 $f(a)$ 与 $f(b)$ 之间的实数 A , 必存在至少一点 $\xi \in (a, b)$ 使得 $f(\xi) = A$

• closed set K : sequence convergent in K

compact set K : every sequence has convergent subsequence in K

• bounded + closed

open set K : $\forall x \in K, \exists \delta > 0, \forall y \in (x - \delta, x + \delta), y \in K$

② differentiable

• differentiable $\rightarrow$ continuous

• find out $f'(x) = 0$ by considering:

- local interior max/min (特例/作题设即可) + differentiable

- differentiable on (a, b) , continuous at $[a, b], f(a) = f(b)$

- differentiable on $(a, b), x < y \in (a, b)$ and $f'(x) > 0 > f'(y)$

• 一定存在某个值使得 --

- intermediate value theorem

- mean value theorem: differentiable on (a, b) , continuous on $[a, b]$

$$\frac{f(a) - f(b)}{a - b} = f'(t), t \in (a, b); [f(a) - f(b)]g'(t) = [g(a) - g(b)]f'(t), t \in (a, b)$$

- differentiable on (a, b) , then $\forall x < y$ and c between $f'(x)$ and $f'(y)$,

$$f'(t) = c, t \in (x, y); [g(b) - g(a)]f'(t) = [f(b) - f(a)]g'(t), t \in (a, b)$$

• 证明 convex over I :

$$f((1-t)a + tb) \leq (1-t)f(a) + tf(b) \quad \text{for } t \in [0, 1]$$

$$f((1-t)a + tb) > (1-t)f(a) + tf(b) \quad \text{for } t \notin [0, 1]$$

$$f(x) \leq \frac{f(b) - f(a)}{b - a}(x - a) + f(a) \quad \text{for } x \in [a, b]$$

$$f(x) > \frac{f(b) - f(a)}{b - a}(x - a) + f(a) \quad \text{for } x \in I \setminus [a, b]$$

- differentiable and $f'(x)$ strictly monotonic increasing

• convex on open interval $\rightarrow$ continuous

• convex + critical point $\rightarrow$ global minimum

• convex $\rightarrow \forall a < b < c < d, \frac{f(b) - f(a)}{b - a} < \frac{f(d) - f(c)}{d - c}$

• convex + differentiable $\rightarrow f(x) \geq f(x_0) + f'(x_0)(x - x_0)$

$f(x) > 0$ 且 $\lim_{x \rightarrow a} \frac{f(x)}{g(x)} = \lim_{x \rightarrow a} \frac{f'(x)}{g'(x)}$ if $\lim_{x \rightarrow a} f(x)$ (strictly increase + differentiable) $\rightarrow f'(x) > 0$

$$= \lim_{x \rightarrow a} g(x) = 0$$

$(f'(x) = 0) \rightarrow$ (local max/min)

(local max/min) $\rightarrow (f'(x) = 0)$

(local max/min) $\rightarrow (f''(x) \geq 0 / \leq 0)$

(convex) $\rightarrow (f''(x) \geq 0)$

(convex + closed domain) $\rightarrow$ (continuous)

we assume that for $(\alpha - \delta, \alpha + \delta) \setminus \{\alpha\}$,

f, g differentiable, $g(x) \neq 0, g'(x) \neq 0$

③ inverse

- $f: I \rightarrow \mathbb{R}$ differentiable over I , f' continuous over I , $f'(a) \neq 0$.
 $\Rightarrow f$ is bijective near a and f^{-1} continuous. $\frac{d}{dx}[f^{-1}(f(x))] = \frac{1}{f'(x)}$

④ integral

- 判断可integrable.

$$- \forall \epsilon > 0, \exists P, U(f, P) - L(f, P) < \epsilon.$$

$$- \inf\{U(f, P)\} = \sup\{L(f, P)\}$$

$$- \text{常用方法: } L(f, P) \leq L(f, P \cup Q) \leq U(f, P \cup Q) \leq U(f, Q)$$

$$- \text{continuous over } [a, b]$$

- if f is integrable over $[a, b]$, $\forall x \in [a, b]$, $F(x) = \int_a^x f(t) dt$ is continuous.

$$\text{if } f \text{ is continuous, } F'(x) = f(x)$$

- 证明两式相等: $f(a) = g(a)$, $f'(x) = g'(x)$.

证明了和先证有界
可积一定存在
通过 sup/inf 证明某个区间有元素
注意: 定义是 sup/inf

f integrable, 改变有限个意得到 g
 g integrable. consider $f - g$.

⑤ Taylor Polynomial

- if f has $(n+1)$ th derivative at a :
$$f(x) = \sum_{k=0}^n \frac{f^{(k)}(a)}{k!} (x-a)^k + \frac{f^{(n+1)}(\theta)}{(n+1)!} (x-a)^{n+1}$$
 for θ between a and x .

- assume $f, f', \dots, f^{(n)}$ are all continuous near a , $R(x) = \int_a^x f^{(n+1)}(t) \frac{(x-t)^n}{n!} dt$

- taylor series: $P_{0,a}(x), P_{1,a}(x), P_{2,a}(x) \dots$ is real analytic at a if radius of convergence exists. though it may not converge to original function.

- let f have derivative of all orders in $|x-a| < \rho$ for $\rho > 0$. assume $\exists M, C, R$ such that $|f^{(n)}(x)| \leq C \cdot \frac{n!}{R^n}$ for all $|x-a| < \rho, n \geq N$. so $f(x) = \lim_{n \rightarrow \infty} P_{n,a}(x)$ for all $|x-a| < \min\{\rho, R\}$.

⑥ convergence of function

- pairwisely convergent: 不保存 continuous, differentiable ($f_n(x) = x^n$), integrable ($f_n = \sqrt[n]{x}$).

- uniformly convergent over compact interval:

保存 continuous, integrable, 不保存 differentiable ($f_n(x) = \sqrt{x + \frac{1}{n}}$).

- let $f_n: [a, b] \rightarrow \mathbb{R}$ be a sequence of continuously differentiable function, and f'_n uniformly convergent to g , $\exists x_0 \in [a, b]$, $f_n(x_0)$ convergent, then f_n uniformly convergent to f , and $\lim_{n \rightarrow \infty} f'_n = f'$.

⑦ sequence of partial sum

- uniformly convergent over compact interval

- M-test: assume $\exists$ a sequence of real numbers $M_k \geq 0$ such that $|\phi_k(x)| < M_k$ for all x , $\sum_{k=1}^{\infty} M_k < \infty$. then $\sum_{k=1}^{\infty} \phi_k(x)$ uniformly convergent.