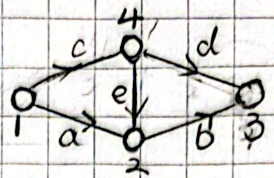
MATH 40007

An Introduction to Applied Mathematics

• incidence matrix

(1) 对于每一个图，我们均可以将其用matrix 表示，其中column表示node，row表示edge，-1表示每一个edge的起点，1表示终点。



$$\begin{pmatrix} -1 & 1 & 0 & 0 \\ 0 & -1 & 1 & 0 \\ -1 & 0 & 0 & 1 \\ 0 & 0 & 1 & -1 \\ 0 & 1 & 0 & -1 \end{pmatrix}$$

此时：
- matrix的right nullspace表示让graph的potential difference between nodes均为0的node potential的值
- matrix的left nullspace表示让graph的loop completed的row direction的值

注意：善用rank-nullity去计算其nullspace的dimension

(2) 如果A为incidence matrix，则：

别忘复习

- $A\vec{x}$ 表示potential differences across the edges
- $-A^T\vec{w}$ 表示net fluxes diverging out of each nodes.
- $\vec{w} = -CA\vec{x}$ where C is the diagonal matrix indicates conductors
- $\vec{f} = -A^T\vec{w}$, where $\vec{f}$ indicates the external current source
- to conclude $\vec{f} = A^TCA\vec{x}$. when $C = I$, $\vec{f} = A^TA\vec{x} = K\vec{x}$, K is the Laplacian matrix

其有如下特点，

$$\begin{pmatrix} 2 & -1 & 0 & -1 \\ -1 & 3 & -1 & -1 \\ 0 & -1 & 2 & -1 \\ -1 & -1 & -1 & 3 \end{pmatrix} = A^TA = K = D - W.$$

① diagonal elements represents how many edges are connected with the node

② if node i and j are connected then $a_{ij} = a_{ji} = (-1) \times$ (相连次数)

③ the only right nullspace basis is $\vec{x}_0$ if $A\vec{x}_0 = \vec{0}$

proof: by contradiction, if $K\hat{x} = \vec{0}$ and $\hat{x}$ is a linear independent new vector

$$K\hat{x} = \vec{0} \Rightarrow A^TA\hat{x} = \vec{0} \Rightarrow \hat{x}^TA^TA\hat{x} = \vec{0}$$
$$\Rightarrow (A\hat{x})^T(A\hat{x}) = 0, \text{ impossible}$$

注意：$\vec{f} = \vec{0}$ 时，$\vec{w}$ 既在column space又在left nullspace，$\vec{w} = \vec{0}$

$\vec{f} \neq \vec{0}$ 时，其必须要与right nullspace的basis orthogonal

④ 现在我们有 $\vec{f} = K\vec{x}$，已知 $\vec{f}$ 与 K，求解 $\vec{x}$

我们发现K的inverse并不存在，因此我们将其转化为invertible的matrix

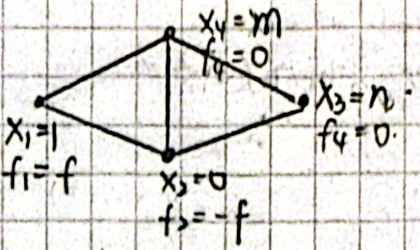
假设 $\vec{x}$ 中的某一个元素为0，WLOG $x_1 = 0$，如此，可得

$$K\vec{x} = \vec{f}, \quad \begin{pmatrix} P & \vec{Q}^T \\ \vec{Q} & R \end{pmatrix} \begin{pmatrix} 0 \\ \hat{x} \end{pmatrix} = \begin{pmatrix} f_1 \\ \hat{f} \end{pmatrix}$$

由此，可求出 $R\hat{x} = \hat{f}$，注意，此时的 $\hat{x}$ 一定满足 $q^T\hat{x} = f_1$，因为非接地节点流向接地节点的总电流，而 f_1 是外部施加在接地节点的电流

$$\vec{x} = \begin{pmatrix} 0 \\ \hat{x} \end{pmatrix} + c\vec{v}, \text{ where } c \in \mathbb{R}, \vec{v} \in \text{right nullspace of } A.$$

⑤ 有时我们会同时有多组unknown.
此时我们通过reordering，将K分成几块后再求解



$$\left(\begin{array}{cc|cc} 3 & -1 & 0 & -1 \\ -1 & 3 & -1 & -1 \\ \hline 0 & -1 & 2 & -1 \\ -1 & -1 & -1 & 3 \end{array}\right)\begin{pmatrix}1\\0\\m\\n\end{pmatrix}=\begin{pmatrix}f\\-f\\0\\0\end{pmatrix}$$

$$\begin{pmatrix}P & Q^T\\ Q & R\end{pmatrix}=\begin{pmatrix}\hat{x}_1\\ \hat{x}_2\end{pmatrix}=\begin{pmatrix}\hat{f}\\ \hat{0}\end{pmatrix}$$

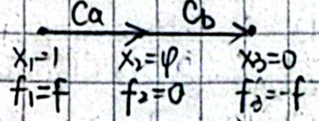
$$\begin{cases}P\hat{x}_1+Q^T\hat{x}_2=\hat{f}\\ Q\hat{x}_1+R\hat{x}_2=\hat{0}\end{cases}$$

• 事实上，按照右侧计算，则
$$\hat{f}=(P-Q^TR^{-1}Q)\hat{x}_1$$
永远成立，这里 $(P-Q^TR^{-1}Q)$ 为shur complement of R in K.

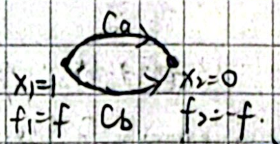
• 此时，in a 2-point source/sink circuit, the effective conductance in the current divergence at the source node (with unit voltage at source node; no voltage at sink node)
即 f is the effective conductance of edge 1.

[注意：在现实中，得知一组解后，若K不变，则可以用已知解求解别的问题
$K\lambda\vec{x}=\lambda\vec{f}$，$K(\vec{x}+\vec{x}_0)=\vec{f}$]

⑥ 有关于串并联的例子：

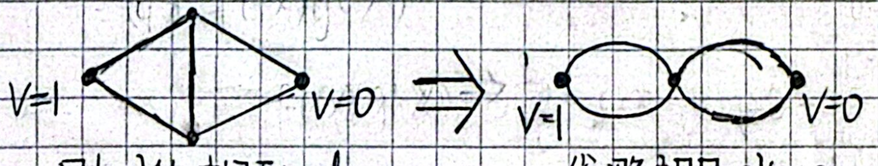


$$K\vec{x}=A^TCA\vec{x}=\vec{f}$$
$$\begin{pmatrix}-1 & 0\\ 1 & -1\\ 0 & 1\end{pmatrix}\begin{pmatrix}C_a & 0\\ 0 & C_b\end{pmatrix}\begin{pmatrix}-1 & 1 & 0\\ 0 & -1 & 1\end{pmatrix}\begin{pmatrix}1\\ \varphi\\ 0\end{pmatrix}=\begin{pmatrix}f\\0\\-f\end{pmatrix}$$
$$\begin{pmatrix}-C_a(\varphi-1)\\ C_a(\varphi-1)+C_b\varphi\\ -C_b\varphi\end{pmatrix}=\begin{pmatrix}f\\0\\-f\end{pmatrix}$$
$$\frac{1}{f}=\frac{1}{C_{eff}}=\frac{1}{C_a}+\frac{1}{C_b}$$



$$K\vec{x}=A^TCA\vec{x}=\vec{f}$$
$$\begin{pmatrix}-1 & -1\\ 1 & 1\end{pmatrix}\begin{pmatrix}C_a & 0\\ 0 & C_b\end{pmatrix}\begin{pmatrix}-1 & 1\\ -1 & 1\end{pmatrix}\begin{pmatrix}1\\0\end{pmatrix}=\begin{pmatrix}f\\-f\end{pmatrix}$$
$$\begin{pmatrix}C_a+C_b\\ -C_a-C_b\end{pmatrix}=\begin{pmatrix}f\\-f\end{pmatrix}$$
$$f=C_{eff}=C_a+C_b$$

⑦ 可以用上述定理，将一个图简化成更简易的形式后套公式求解
[注意：node with same voltage could be merged!]



因为对称，中间两node voltage一定相同

省略相同voltage的node间的电路，因为current = 0

⑧ 在得知 difference of voltage, conductance, current 三者任意之二后，我们可以用 $w=C\Delta x$ 求解第三个元素
example:

(1) 可知 $W=1$

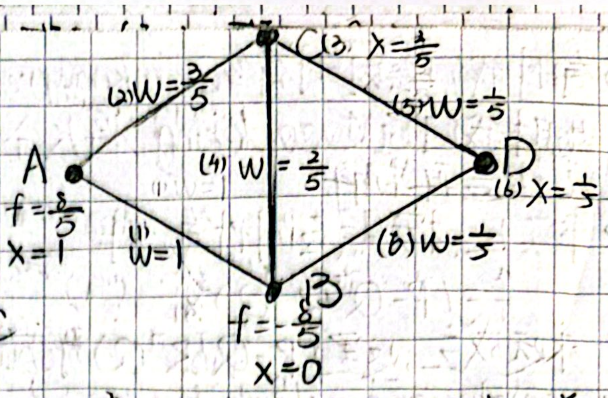
(2) 可知 $W=\frac{3}{5}$ 因为在A点共流出 $\frac{8}{5}$ 单位电流

(3) 可知 $X=\frac{2}{5}$

(4) 可知 $W=\frac{2}{5}$

(5) 可知 $W=\frac{1}{5}$, 因为共有 $\frac{3}{5}$ 单位电流流入C

(6) 可知 $x=\frac{1}{5}$ $W=\frac{1}{5}$



⑨ 有这样一类问题可以被用该模型解决：从A点出发，在到达B点之前到达C点的概率

这是因为 for interior nodes of both situations, their values are just the average of the values of adjacent nodes

proof 对于概率问题，用Bayes展开可得

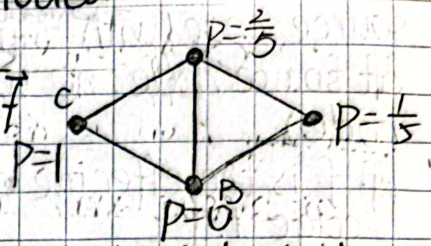
对于电路而言，考虑到 $K\vec{x}=(D-W)\vec{x}=\vec{f}$

对于KCL interior而言：

$$\sum_{j=1}^{n}(D_{ij}-W_{ij})x_j-f_i=0$$

$$\sum_{j=1}^{n}D_{ij}x_j=\sum_{j=1}^{n}W_{ij}x_j$$

$$D_{ii}x_i=\sum_{j\in \text{adjacent to } \boxed{i}}x_j$$



问题：在各点为A的情况下，在到达B之前到达C的概率

事实上, any set of potentials satisfying $x_i=\frac{1}{D_{ii}}\sum_{j\in \text{adjacent to } \boxed{i}}x_j$ is called harmonic potential at node $\boxed{i}$.

⑩ the maximum/minimum value of the potential is attained at a boundary points

let $\{x_i\}_i$ and $\{p_i\}_i$ be potentials on the same connected graph with same boundary values. then $x_i=p_i$ for all nodes.

⑪ define hopping probability as the probability of hopping from node $\boxed{i}$ to any adjacent node $\boxed{j}$, known as P_{ij}

if c_{ij} be the conductance of edge between node $\boxed{i}$ and $\boxed{j}$, then $P_{ij}=\frac{c_{ij}}{c_i}$, where $c_i=\sum_{j\in \text{adjacent to node } \boxed{i}}c_{ij}$

then $C_{eff}=\sum_{j\in \text{adjacent to node } \boxed{t}}-C_{tj}(x_j-x_t)=\sum_j C_{tj}(x_t-x_j)=\sum_j C_{tj}x_t-\sum_j C_{tj}x_j$

$=C_t-\sum_j C_{tj}x_j$ [因为根据定义，计算 C_{eff} 时 $x_t=1$]

$=C_t(1-\sum_j \frac{C_{tj}}{C_t}x_j)=C_tP$(从 $\boxed{t}$ 出发，在重回 $\boxed{t}$ 之前到达 $\boxed{s}$)

$=C_tP_{esc}$

$$P_{esc}=\frac{C_{eff}}{C_t}$$

⑫ 有此时候我们会通过 iteration 的方法计算各个 x_i
[注意：此时使用 harmonic potential 的性质求解]
对于 $x_{n+1}+ax_n=b$ where a, b constant 的问题：
- 求特解 let $x_i=k$ for all i, $k+ak=b$
- 求通解 let $x_i=B\lambda^i$ for all i, $B\lambda^{i+1}+aB\lambda^i=0$

⑬ VI = dissipation in the conductance
sum of dissipation in the conductance is $\underbrace{(-A\vec{x})^T}_{\Delta\vec{V}}\underbrace{(-CA\vec{x})}_{\vec{w}}$

let $\mathcal{E}(\vec{x})=\vec{x}^TK\vec{x}$, and according to Dirichlet's Principle, it is minimised when Ohm's law is satisfied and KCL at all free nodes

proof: $K\vec{x}=\vec{f}$ 我们设定 $x_{田}=1$, $x_{日}=0$:

$$\begin{pmatrix} P & Q^T \\ Q & R \end{pmatrix}\begin{pmatrix} \vec{e} \\ \vec{x}^* \end{pmatrix}=\begin{pmatrix} \vec{f} \\ \vec{0} \end{pmatrix}$$

$$\mathcal{E}(\hat{x})=(\vec{e}^T\ \ \hat{x}^T)\begin{pmatrix} P & Q^T \\ Q & R \end{pmatrix}\begin{pmatrix} \vec{e} \\ \hat{x} \end{pmatrix}$$

$$=(\hat{x}+R^{-1}Q\vec{e})^TR(\hat{x}-R^{-1}Q\vec{e})+\vec{e}^TP\vec{e}-(-R^{-1}Q\vec{e})^TR(-R^{-1}Q\vec{e})$$
$$=(\hat{x}-\vec{x}^*)^TR(\hat{x}-\vec{x}^*)+\vec{e}^T(P-Q^TR^{-1}Q)\vec{e}$$
$$=(\hat{x}-\vec{x}^*)^TR(\hat{x}-\vec{x}^*)+C_{eff}$$

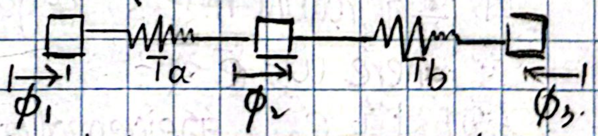
[注意：在现实中，常用偏导、配平的方法找最小值]

⑭ 同理，我们可以改写：$\mathcal{E}(\vec{w})=\vec{w}^TC^{-1}\vec{w}$，结果与上式一致
according to Thomson's Principle, pick any current $\vec{w}$ with unit current divergence at 田, and negative unit current divergence at 日, and all other nodes KCL
the minimised current can be determined by ohm's law and and a set of nodes voltage $\vec{x}$
[注意：在现实中，我们对 currency 存在限制：loop 内的 currency 总量为 0
因此，我们选取几个 currency 为元，剩下的可用已知表示，后求极限]

⑮ free node 少 → Dirichlet's
loop 少 → Thomson's

套公式时写矢号!

(B) ① 我们也可以用 incidence matrix 表示弹簧系统
$\vec{T}=CA\vec{x}$, $\vec{f_T}=-A^T\vec{T}$
where $\vec{f_T}$ is forces on each node



② 假设 everything is in equilibrium
ϕ: displacement, $T_a=C_a(\phi_2-\phi_1)$
if $\vec{f}$ represents external forces on each node
$\therefore$ $\vec{f}+\vec{f_T}=0$，其中 $\vec{f}$ 来源可能为重力、支持力等
[注意：没说考虑重力就不要考虑!]

求墙的支持力时注意受力平衡!

③ 假设everything not in equilibrium $\vec{f}+\vec{f}_T = M\frac{d^2}{dt^2}\vec{x}$
此时我们一般假设相关函数为f(t), x(t)等
- 如果两端未被固定：
  则我们求解一个system
- 如果两端被固定：
  (我们不考虑墙的受力等内容，只考虑可动点有关的submatrix)
  如果为free oscillation，则在可动点上的外力为0
  所以 $\vec{f}_T = M\frac{d^2}{dt^2}\vec{x}$，$\frac{d^2}{dt^2}\vec{x} = -M^{-1}K\vec{x}$. 为此，假设 $\vec{x}=\vec{x}_0 e^{i\omega t}$，所以，
  ω^2为$M^{-1}K$的eigenvalue
  根据ω^2找出eigenvectors $\vec{v}_1$，则答案为 $\vec{x}_{CF} = c_1\vec{v}_1\cos\omega_1 t + c_2\vec{v}_1\sin\omega_1 t + c_3\vec{v}_2\cos\omega_2 t + c_4\vec{v}_2\sin\omega_2 t + \cdots$
  此时，ω_i为natural frequency.

④ 假设PQ=QP, and suppose P has an eigen value λ with unit geometric multiplicity, then P and Q share the same eigenvectors
Proof: $QP\vec{x} = \lambda Q\vec{x} = PQ\vec{x}$, so $Q\vec{x} = \mu\vec{x}$

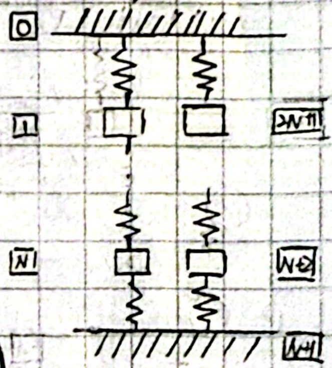
⑤ for matrix $M = c_0 I_N + c_1 S_N + c_2 S_N^2 + \cdots + c_{N-1}S_N^{N-1}$，其中$S_N = (\vec{e}_N\ \vec{e}_1\ \vec{e}_2 \cdots \vec{e}_{N-1})$，$S_N^N = I_N$
the matrix is a circulant matrix of size $N\times N$
因为$MS_N = S_N M$，S_N的eigenvalue与eigenvector为：

$$\omega_k = e^{\frac{2k\pi i}{N}},\ \vec{v}_k = \begin{pmatrix} 1 \\ \omega_k \\ \omega_k^2 \\ \vdots \\ \omega_k^{N-1} \end{pmatrix}$$

如此一来M的eigenvector也为$\vec{v}_k$，可求eigenvalue

求internal force时，T_i为负表示力的方向与我们规定的弹簧的正方向相反

⑥ if we consider a cycle graph:



$$C_{2N+2} = \begin{pmatrix} 2 & -1 & 2 & & -1 \\ -1 & 2 & -1 & \cdots & 0 \\ 0 & -1 & 2 & & 0 \\ & \vdots & & \ddots & \vdots \\ -1 & 0 & 0 & \cdots & 2 \end{pmatrix} = \begin{pmatrix} 2 & \vec{q}_1^T & 0 & \vec{q}_2^T \\ \vec{q}_1 & K_N & \vec{q}_2 & O_N \\ 0 & \vec{q}_2^T & 2 & \vec{q}_1^T \\ \vec{q}_2 & O_N & \vec{q}_1 & K_N \end{pmatrix}$$

因为其为circulant matrix，其eigenvector: $\vec{v}_k = \begin{pmatrix} 1 \\ \omega_k \\ \omega_k^2 \\ \vdots \\ \omega_k^{2N+1} \end{pmatrix} = \begin{pmatrix} 1 \\ \vec{\Phi}_k \\ (-1)^k \\ \hat{\Phi}_k \end{pmatrix}$
where $\omega = e^{\frac{2k\pi i}{2N+2}}$
经过计算，C_{2N+2}的eigenvalue为：

$$\lambda_k = 2-2\cos\left(\frac{2k\pi}{2N+2}\right)$$

随后我们发现，if $C_{2N+2}\vec{v}_k = \lambda_k\vec{v}_k$，则 $C_{2N+2}\bar{\vec{v}}_k = \lambda_k\bar{\vec{v}}_k$，因此，

$$C_{2N+2}\left(\frac{\vec{v}_k - \bar{\vec{v}}_k}{2i}\right) = C_{2N+2}(\mathrm{Im}(\vec{v}_k)) = \lambda_k'\left(\frac{\vec{v}_k - \bar{\vec{v}}_k}{2i}\right) = \lambda_k'(\mathrm{Im}(\vec{v}_k))$$

let $\vec{q}_1 = \begin{pmatrix} -1 \\ 0 \\ \vdots \\ 0 \end{pmatrix}$, $\vec{q}_2 = \begin{pmatrix} 0 \\ \vdots \\ 0 \\ -1 \end{pmatrix}$, we have:

$$\begin{pmatrix} 2 & \vec{q}_1^T & 0 & \vec{q}_2^T \\ \vec{q}_1 & K_N & \vec{q}_2 & O_N \\ 0 & \vec{q}_2^T & 2 & \vec{q}_1^T \\ \vec{q}_2 & O_N & \vec{q}_1 & K_N \end{pmatrix}\begin{pmatrix} 0 \\ Im(\Phi_k) \\ 0 \\ Im(\hat{\Phi}_k) \end{pmatrix} = \begin{pmatrix} \vec{q}_1^T Im(\Phi_k) + \vec{q}_2^T Im(\hat{\Phi}_k) \\ K_N Im(\Phi_k) \\ \vec{q}_2^T Im(\Phi_k) + \vec{q}_1^T Im(\hat{\Phi}_k) \\ K_N Im(\hat{\Phi}_k) \end{pmatrix} = \lambda_k \begin{pmatrix} 0 \\ Im(\Phi_k) \\ 0 \\ Im(\hat{\Phi}_k) \end{pmatrix}$$

so K_n shares the same same eigenvalues as C_{2N+2}, and its eigenvectors are

$$Im\,\Phi_k = \sqrt{\frac{2}{N+1}}\begin{pmatrix} \sin(\frac{k\pi}{N+1}) \\ \sin(\frac{2k\pi}{N+1}) \\ \vdots \\ \sin(\frac{Nk\pi}{N+1}) \end{pmatrix}, \quad Im(\Phi_n)\cdot Im(\Phi_n) = \delta_{mn} \text{ (orthogonal)}$$

$$\lambda_k = 2-2\cos(\tfrac{k\pi}{n+1})$$

计算时注意系数!

(1) given $\phi_{N+1}=0$, $\phi_0=1$. $\boxed{0}$ has external force f_0

$$K = \begin{array}{c} \begin{array}{cccccc} \boxed{0} & \boxed{N+1} & \boxed{1} & \boxed{2} & \cdots & \boxed{N} \end{array} \\ \left(\begin{array}{cc|cccc} 1 & 0 & -1 & 0 & \cdots & 0 \\ 0 & 1 & 0 & 0 & \cdots & -1 \\ -1 & 0 & 2 & -1 & \cdots & 0 \\ 0 & 0 & -1 & 2 & \cdots & 0 \\ & & & & \ddots & \vdots \\ 0 & -1 & 0 & 0 & \cdots & 2 \end{array}\right) \end{array} = \begin{pmatrix} I_2 & Q^T \\ Q & K_N \end{pmatrix}$$

$$K\begin{pmatrix} \vec{e} \\ \vec{x} \end{pmatrix} = \begin{pmatrix} \vec{f} \\ \vec{0} \end{pmatrix}, \quad Q\vec{e} + K_N\vec{x} = \vec{0}, \quad K_N\vec{x} = -Q\vec{e} = \begin{pmatrix} 1 \\ \vdots \\ 0 \end{pmatrix}$$

if we assume $\vec{x} = \sum_{k=1}^{N} a_k Im(\Phi_k)$, then $K_N\vec{x} = \sum_{k=1}^{N} a_k\lambda_k Im(\Phi_k) = \begin{pmatrix} 1 \\ \vdots \\ 0 \end{pmatrix}$

notice $Im(\Phi_k)^T K_N \vec{x} = \sum_{y=1}^{N} a_y\lambda_y Im(\Phi_k)^T Im(\Phi_y)$

$$= \sum_{y=1}^{N} a_y\lambda_y \delta_{my} = a_k\lambda_k = Im(\Phi_k)^T\begin{pmatrix} 1 \\ \vdots \\ 0 \end{pmatrix} = \sqrt{\frac{2}{N+1}}\sin(\frac{k\pi}{N+1})$$

$$\text{so } a_k = \sqrt{\frac{2}{N+1}}\,\frac{1}{2-2\cos(\frac{k\pi}{N+1})}\sin(\frac{k\pi}{N+1})$$

$$\text{thus } \vec{x} = \sum_{k=1}^{N}\sqrt{\frac{2}{N+1}}\,\frac{1}{2-2\cos(\frac{k\pi}{N+1})}\sin(\frac{k\pi}{N+1})\sqrt{\frac{2}{N+1}}\begin{pmatrix} \sin(\frac{k\pi}{N+1}) \\ \sin(\frac{2k\pi}{N+1}) \\ \vdots \\ \sin(\frac{Nk\pi}{N+1}) \end{pmatrix}$$

$$\phi_j = \sum_{k=1}^{N}\frac{1}{N+1}\,\frac{\sin(\frac{k\pi}{N+1})}{1-\cos(\frac{k\pi}{N+1})}\sin(\frac{jk\pi}{N+1}) = 1-\frac{j}{N+1}$$

其实跟电路的解是一样的！均为 $K\vec{x}=\vec{f}$!

we fix j, let $x = \frac{j}{N+1}$

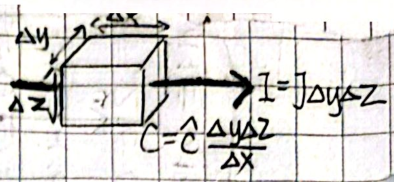
$$1-x = \sum_{k=1}^{N}\frac{1}{N+1}\,\frac{\sin(\frac{k\pi}{N+1})}{1-\cos(\frac{k\pi}{N+1})}\sin(j\pi x)$$

as $N\to\infty$, $\frac{k\pi}{N+1}\to 0$

$$1-x = \lim_{N\to\infty}\frac{1}{N+1}\sum_{k=1}^{N}\frac{(\frac{k\pi}{N+1})}{\frac{1}{2}(\frac{k\pi}{N+1})^2}\sin(k\pi x) = 2\sum_{k=1}^{\infty}\frac{\sin(k\pi x)}{k\pi}$$

(4) (1) 我们之前的想法是：discrete系统，先求解 $K\vec{x}=\vec{f}$，后使 $N\to\infty$ 以得到 continuum solution. 现在我们先将 discrete系统 $N\to\infty$，再求解 continuum system.

此时，我们要重新定义 $c_i^{(x)}=\hat{c}_i\frac{\Delta y\Delta z}{\Delta x}$，$w_i^{(x)}=J_i\Delta y\Delta z$，$f_i=\hat{f}_i\Delta x\Delta y\Delta z$. 其中 $\hat{c}$ 为 conductivity，J_i 为 current density，$\hat{f}_i$ 为 current source density



② 考虑 $\phi_0 \; w_1 \; \phi_1 \; w_2 \; \phi_2 \; w_3 \; \phi_3 \; w_4 \; \phi_4 \; \cdots \; \phi_{N-1} \; w_N \; \phi_N$　等分成N+1个node, $N\to\infty$

坐标: $x_0 \; x_1 \; x_2 \; x_3 \; x_4 \; \cdots \; x_{N-1} \; x_N$, $x_i=\frac{1}{N}$

则 $W_{i+1}=-c_{i+1}(\phi_{i+1}-\phi_i)$

$$J_{i+1}\Delta y\Delta z=-\hat{c}_{i+1}\frac{\Delta y\Delta z}{\Delta x}(\phi(x_{i+1})-\phi(x_i))$$

$$J_{i+1}=-\hat{c}_{i+1}\frac{\phi(x_{i+1})-\phi(x_i)}{\Delta x}$$

$$J(x_i+\tfrac{\Delta x}{2})=-C(x_i+\tfrac{\Delta x}{2})\frac{\phi(x_i+\Delta x)-\phi(x_i)}{\Delta x}$$　[J_i与C_i一般在$x_i-\frac{\Delta x}{2}$处测量]

$$J(x)=-C(x)\frac{d\phi}{dx}(x)$$

另外，$f_i=W_{i+1}-W_i$

$$\hat{f}_i\Delta x\Delta y\Delta z=J_{i+1}\Delta y\Delta z-J_i\Delta y\Delta z$$

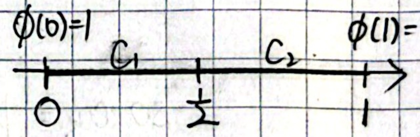
$$\hat{f}_i=\frac{J_{i+1}-J_i}{\Delta x}$$

$$F(x_i)=\frac{J(x_i+\frac{\Delta x}{2})-J(x_i-\frac{\Delta x}{2})}{\Delta x}$$

$$F(x)=\frac{dJ}{dx}(x)$$

因此 $F(x)=\frac{dJ}{dx}(x)=\frac{d}{dx}\left(-C(x)\frac{d\phi}{dx}(x)\right)$

通过带入boundary条件求解可得$\phi(x)$通式



example: assume $\phi(0)=0$, $\phi(1)=1$.

$\hat{f}(x)=0$ for $0<x<1$, and $\hat{c}(x)=\begin{cases}\hat{c}_1 & 0<x<\frac{1}{2}\\ \hat{c}_2 & \frac{1}{2}<x<1\end{cases}$

find the current density into the conductor at $x=1$

我们分段讨论：$\phi(x)=\begin{cases}\phi_1(x) & 0<x<\frac{1}{2}\\ \phi_2(x) & \frac{1}{2}<x<1\end{cases}$

for $\phi_1(x)$: $F(x)=\frac{dJ}{dx}(x)=-c\frac{d^2\phi}{dx^2}(x)=0$, $\phi(0)=0$

so $\phi_1(x)=Ax$

for $\phi_2(x)$: similarly, $\phi_2(x)=1+C(x-1)$

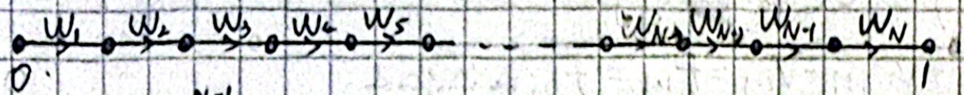
又因为 voltage and current are continuous at $x=\frac{1}{2}$.

$$\phi_1(\tfrac{1}{2})=\phi_2(\tfrac{1}{2}),\quad A+C=2$$

$$-\hat{c}_1\frac{d\phi_1}{dx}(\tfrac{1}{2})-\hat{c}_2\frac{d\phi_2}{dx}(\tfrac{1}{2})=0.\quad A=\frac{\hat{c}_2}{\hat{c}_1}C$$

so $C=\frac{2\hat{c}_1}{\hat{c}_1+\hat{c}_2}$, $F(1)=\hat{c}_2\frac{d\phi_2}{dx}(1)=\frac{2\hat{c}_1\hat{c}_2}{\hat{c}_1+\hat{c}_1}$

③ 在电路中，$\vec{x}_0^T A^T \vec{w} = 0$, where $\vec{x}_0$ is from the right nullspace of A
这是因为其结果为 the sum of diverges at all nodes



$w_1 - w_N + \sum_{i=1}^{N-1}(w_{i+1} - w_i) = 0$

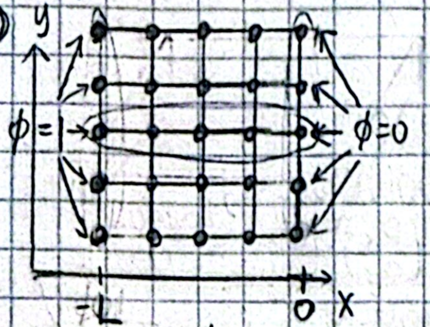
notice $w_1 = J_1 \Delta y \Delta z$, $w_N = J_N \Delta y \Delta z$, $w_i = J_i \Delta y \Delta z$

where $J_1 = J(\frac{\Delta x}{2})$, $J_N = J(1-\frac{\Delta x}{2})$, $J_{i+1} = J(x_i + \frac{\Delta x}{2})$, $J_i = J(x_i - \frac{\Delta x}{2})$

so $J(\frac{\Delta x}{2}) - J(1-\frac{\Delta x}{2}) + \sum_{i=1}^{N-1} J(x_i + \frac{\Delta x}{2}) - J(x_i - \frac{\Delta x}{2}) = 0$

$J(\frac{\Delta x}{2}) - J(1-\frac{\Delta x}{2}) + \sum_{i=1}^{N-1} \frac{dJ}{dx}(x_i)\Delta x = 0$

if $\Delta x \to 0$ $J(0) - J(1) = \int_0^1 \frac{dJ}{dx}(x)dx$

④ 

no external souces except for $x=-L, 0$; unit conductivity

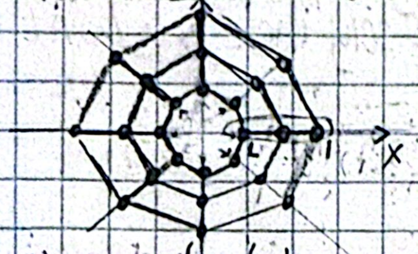
对于所有的两维 discrete 情况，我们均使用 symmetric 化为一维情况

图(1):

$w_{j+1} = w_j$ for $j = 1, 2, \ldots, N$

$-\hat{C}_{j+1}\frac{\Delta y \Delta z}{\Delta x}(\phi_{j+1} - \phi_j) = -\hat{C}_j \frac{\Delta y \Delta z}{\Delta x}(\phi_j - \phi_{j-1})$

$\phi_{j+1} - \phi_j = \phi_j - \phi_{j-1}$



inner nodes $\phi = 1$,
outer nodes: $\phi = 0$
no external sources elsewhere
unit conductivity

图(2):

$w_{j+1} = w_j$ for $j = 1, 2, \ldots, N$

$-\hat{C}_{j+1}\frac{r_{j+1}\Delta\theta\Delta z}{\Delta r}(\phi_{j+1} - \phi_j) = -\hat{C}_j \frac{r_j \Delta\theta \Delta z}{\Delta r}(\phi_j - \phi_{j-1})$

[注意 此时截面面积为 $r_j \Delta\theta$, where $r_j = L + \frac{(1-L)}{N} j$]

$r_{j+1}(\phi_{j+1} - \phi_j) = r_j(\phi_j - \phi_{j-1}) = D$

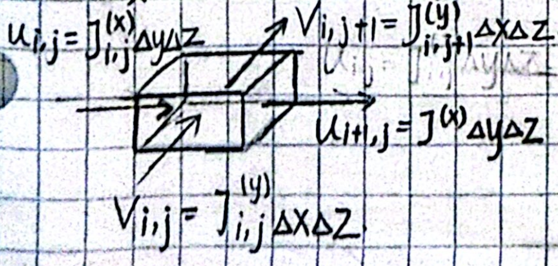
so $\phi_1 = 1 + \frac{D}{r_1}$, $\phi_2 = 1 + \frac{D}{r_1} + \frac{D}{r_2}$, ..., $\phi_j = 1 + D\sum_{n=1}^{j}\frac{1}{r_n}$

known $\phi_N = 0$, $D = -\frac{1}{\sum_{n=1}^{N}\frac{1}{r_n}}$

thus $\phi_j = \frac{\sum_{n=j+1}^{N}\frac{1}{r_n}}{\sum_{n=1}^{N}\frac{1}{r_n}} = \frac{\sum_{n=j+1}\frac{\Delta r}{r_n}}{\sum_{n=1}^{N}\frac{\Delta r}{r_n}} = \frac{\int_{r_j}^{1}\frac{1}{r}dr}{\int_L^1 \frac{1}{r}dr} = \frac{\ln r}{\ln L}$ for $L < r < 1$

so $\Phi(x,y) = \frac{\ln r}{\ln L} = \frac{\ln\sqrt{x^2+y^2}}{\ln L}$

⑤ 接下来考虑二维的 continuous 情况



so $u_{i,j} = -C_{i,j}(\phi_{i,j} - \phi_{i-1,j})$

$v_{i,j} = -C_{i,j}(\phi_{i,j} - \phi_{i,j-1})$

so $J^{(x)}(x_i, y_j) = -\hat{C}(x_i, y_j)\frac{\partial\Phi}{\partial x}(x_i, y_j)$

$$J^{(y)}(x_i,y_j) = -\hat{C}(x_i,y_j)\frac{\partial \Phi}{\partial y}(x_i,y_j)$$

so $\vec{j} = \begin{pmatrix} J^{(x)}(x,y) \\ J^{(y)}(x,y) \end{pmatrix} = -\hat{C}(x,y)\nabla\Phi(x,y)$

又因为 $U_{i+1,j} - U_{i,j} + V_{i,j+1} - V_{i,j} = f_{i,j} = \hat{f}_{i,j}\,\Delta x \Delta y \Delta z$.

$$\frac{\partial J^{(x)}}{\partial x}(x_i,y_j) + \frac{\partial J^{(y)}}{\partial y}(x_i,y_j) = \hat{f}_{i,j}(x_i,y_j)$$

so $\nabla\cdot\vec{j} = \frac{\partial J^{(x)}}{\partial x} + \frac{\partial J^{(y)}}{\partial y} = F(x,y)$,

combine them together, we get $\nabla(-\hat{C}(x,y)\nabla\Phi(x,y)) = F(x,y)$

⑯ 对于二维continuous情况: 在一个没有电荷的, 绝缘的, $\hat{C}(x,y) = C$ 的二维区域中, 此时拉普拉斯方程满足 ($\frac{\partial^2\Phi}{\partial x^2} + \frac{\partial^2\Phi}{\partial y^2} = 0$), 且电压 $\Phi(x,y)$ 可以被写作某个复函数(analytic function) $h(z)$ 的实部

$\Phi(x,y) = Re(h(z)) = \frac{1}{2}(h(z) + \overline{h(z)})$.

$J^{x}(x,y) - iJ^{(y)}(x,y) = -C\frac{\partial h}{\partial z}$.

total current entering the conductor through boundary: $\int_{boundary} \vec{j}\cdot\vec{n}\,ds$

[注意: if $a = a_x + ia_y$, $b = b_x + ib_y$, then $\vec{a}\cdot\vec{b} = \begin{pmatrix} a_x \\ a_y \end{pmatrix}\cdot\begin{pmatrix} b_x \\ b_y \end{pmatrix} = \bar{a}b$]

⑰ a point source of current for 二维continuous情况

if $h(z) = -\frac{m}{2\pi}\log(z-z_0) + \hat{h}(z)$, where the analytic function $\hat{h}(z)$ is not singular when evaluated at $z=z_0$, then we say there is a point source of current strength m at point z_0.

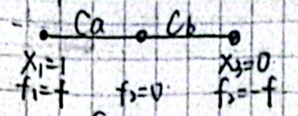
① 初步电路

- A为 incidence matrix：left nullspace：loop completed 的值
  right nullspace：potential difference 全0的值
- $\begin{cases}\vec{w} = -CA\vec{x} \\ \vec{f} = -A^T\vec{w}\end{cases}$ $\quad K\vec{x} = \vec{f}$

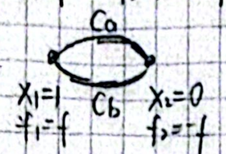
  $\begin{pmatrix} P & \vec{Q}^T \\ \vec{Q} & R \end{pmatrix}\begin{pmatrix} 0 \\ \hat{x} \end{pmatrix} = \begin{pmatrix} f_1 \\ \hat{f} \end{pmatrix}$ 设一个元素为0，则 $\vec{R}\hat{x} = \hat{f}$ 可求 $\hat{x}$
  $\vec{x} = \begin{pmatrix} 0 \\ \hat{x} \end{pmatrix} + c\vec{v}$, where $\vec{v} \in$ right nullspace of A

  $\begin{pmatrix} P & Q^T \\ Q & R \end{pmatrix}\begin{pmatrix} \vec{e}_1 \\ \hat{x} \end{pmatrix} = \begin{pmatrix} \hat{f} \\ \vec{0} \end{pmatrix}$ 得 $\hat{f} = \begin{pmatrix} f \\ \cdot \end{pmatrix}$：in a 2-point source/sink circuit, the effective conductance $= f$

[用已知的解通过变换 $K\lambda\vec{x} = \lambda\vec{f}$，$K(\vec{x}+\vec{x}_0) = \vec{f}$ 求解别的问题]

- 串联 

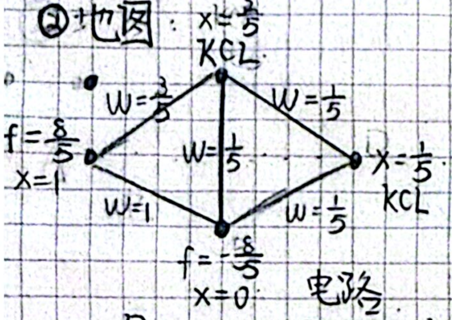
$\frac{1}{f} = \frac{1}{C_{eff}} = \frac{1}{C_a} + \frac{1}{C_b}$

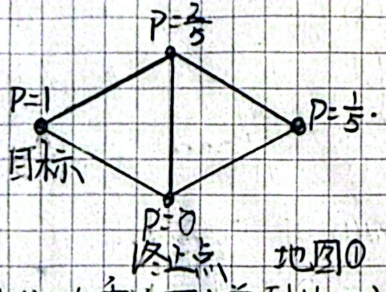
  并联 

$f = C_{eff} = C_a + C_b$

[通过 merge nodes with same voltage 化简图像]
[用 $w = C\Delta x$ 知二推三，注意 $f = -\sum w$]
[for interior nodes (KCL), their voltages are just the average of the voltages of adjacent nodes]

② 地图：





以各个点为起点，在到达终止点前到达目标的概率

- $P_{escape} = P$(从田点出发，在重返田之前到达日) $= \frac{C_{eff}}{C_{田}}$, $C_{田} = \sum_{j \in \text{adjecent to 田}} C_{田j}$
- 用 iteration + harmonic potential 性质求解 x_i
- 对于 $x_{n+1} + ax_n = b$, a, b are constant
  通解 $x_i = B\lambda^i$, $B\lambda^{n+1} + aB\lambda^n = 0$
  特解 $x_i = k$: $k + ak = b$
- Dirichlet's: $\mathcal{E}(\vec{x}) = \vec{x}^T K\vec{x}$, $\mathcal{E}(\vec{x}^*) = \min\mathcal{E}(\vec{x})$
  设定 $x_{田} = 1$, $x_{日} = 0$，当 free nodes 少时用
  Thomson's: $\hat{\mathcal{E}}(\vec{w}) = \vec{w}^T C^{-1}\vec{w}$, $\hat{\mathcal{E}}(\vec{w}^*) = \min\hat{\mathcal{E}}(\vec{w})$
  设定 unit current divergence at 田，日，当 loop 少时用

[找几个参数设为元，将别的参数用元表示，随后求偏导]

③ 弹簧系统

- $\begin{cases} \vec{T} = CA\vec{\Phi} \\ \vec{f}_I = -A^T\vec{T} \end{cases}$ $\quad \vec{f}_I = -K\vec{\Phi}$
- 静止 $\vec{f}_I + \vec{f} = \vec{0}$, $\vec{f} = K\vec{\Phi}$

  运动 $\vec{f}_I + \vec{f} = M\frac{d^2}{dt^2}\vec{\Phi}$

  此时我们不考虑墙的拉力等内容，只考虑与可动点有关的submatrix

  $\hat{f} - \hat{K}\hat{\Phi} = \hat{M}\frac{d^2}{dt^2}\hat{\Phi}$

  如果为free oscillation，则 $\hat{f} = \vec{0}$, $\frac{d^2}{dt^2}\hat{\Phi} = -\hat{M}^{-1}\hat{K}\hat{\Phi}$

  设 $\vec{x} = \vec{x}_0 e^{i\omega t}$，$\omega^2$为$M^{-1}K$的eigenvalue，$\omega$为natural frequency

  则 $\vec{x} = C_1\vec{v}_1\cos\omega_1 t + C_2\vec{v}_1\sin\omega_1 t + C_3\vec{v}_2\cos\omega_2 t + C_4\vec{v}_2\sin\omega_2 t$

- for circulant matrix $M = c_0 I_N + c_1 S_N + c_2 S_N^2 + \cdots + c_{N-1}S_N^{N-1}$, $S_N = (\vec{e}_N\ \vec{e}_1\ \vec{e}_2 \cdots \vec{e}_{N-1})$

  $\vec{v}_k = \frac{1}{\sqrt{n}}\begin{pmatrix} w_k \\ w_k^2 \\ \vdots \\ w_k^{N-1} \end{pmatrix}$, $w = e^{\frac{2k\pi i}{N}}$，M的eigenvector与$S_n$相同

  特殊的，$K_n = \begin{pmatrix} 2 & -1 & 0 & \cdots & 0 \\ -1 & 2 & -1 & \cdots & 0 \\ 0 & -1 & 2 & & 0 \\ \vdots & \vdots & & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & 2 \end{pmatrix}$, $\vec{v}_k = \sqrt{\frac{2}{n+1}}\begin{pmatrix} \sin\frac{k\pi}{n+1} \\ \sin\frac{2k\pi}{n+1} \\ \vdots \\ \sin\frac{nk\pi}{n+1} \end{pmatrix}$, $\lambda_k = 2-2\cos\left(\frac{k\pi}{n+1}\right)$

  [将特别的矩阵转化成包含K_n的分块矩阵]

  [对于 $K_n\vec{x} = \vec{y}$，设 $\vec{x} = c_1\vec{v}_1 + c_2\vec{v}_2 + \cdots + c_n\vec{v}_n$，因为eigenvectors orthonormal，$c_k = \vec{v}_k^T\vec{y}$]

④ 连续电路

- 1D conductor: $J(x) = -C(x)\frac{d\Phi(x)}{dx}$ $\quad \frac{dJ}{dx} = F(x)$ $\quad$ total current input $\int_{boundary} F(x)dx$
- 2D conductor: $\vec{j}(x,y) = \begin{pmatrix} j^{(x)}(x,y) \\ j^{(y)}(x,y) \end{pmatrix} = -\hat{c}(x,y)\nabla\vec{\Phi}(x,y)$, $\nabla\cdot\vec{j} = F(x,y)$
- the voltage potential $\Phi(x,y)$ in a 2D conductor of uniform conductivity $\hat{c}$ can be written in terms of an analytic function $h(z)$

  $\vec{\Phi}(x,y) = Re(h(z))$, $J^{(x)}(x,y) - iJ^{(y)}(x,y) = -\hat{c}\frac{dh}{dz}$

  the total current entering the conductor through boundary: $\int_{boundary}\vec{j}\cdot\vec{n}\,ds$

  [注意: $\vec{j}\cdot\vec{n} = Re(\bar{j}\cdot n)$]

  if near some points z_0, $h(z)$ has local form

  $$h(z) = -\frac{m}{2\pi}\log(z-z_0) + \tilde{h}(z)$$

  and $\tilde{h}(z)$ is non-singular at z_0, then we say there's a point source of current of strength m at z_0

  边缘不接地 $\longrightarrow J=0$

boundary条件写范围

求出的电流等朝向默认正方向，注意方向!