

MATH 40004

CALCULUS & APPLICATION

• limit

① let f be a function defined at all points near x_0 , except possibly at x_0 , and let l be a real number. l is the limit of $f(x)$ as x approaches x_0 , if for every $\varepsilon > 0$, there exists $\delta > 0$ such that $|f(x) - l| < \varepsilon$ whenever $|x - x_0| < \delta$ and $x \neq x_0$.

② let $f(x)$ be defined on a domain containing the interval (a, ∞) . a real number l is the limit of $f(x)$ as x approaches ∞ if, for every $\varepsilon > 0$, there exists a $A > a$, such that $|f(x) - l| < \varepsilon$ whenever $x > A$.

[similar for $\lim_{x \rightarrow -\infty} f(x) = l$]

③ let $f(x)$ be defined in an interval containing x_0 , except possibly at $x = x_0$. $f(x)$ approaches ∞ as x if given any real number $B > 0$, there exists $\varepsilon > 0$, such that $f(x) > B$ whenever $|x - x_0| < \varepsilon$ and $x \neq x_0$.

[similar for $\lim_{x \rightarrow x_0} f(x) = -\infty$]

注意: 极限存在 $\Leftrightarrow$ (左极限 = 右极限)

④ let $f(x)$ be defined for all x in an interval (x_0, a) . $f(x)$ approaches l as x approaches x_0 from the right if, for any $\varepsilon > 0$, there exist $\delta > 0$, such that $|f(x) - l| < \varepsilon$ for all $x_0 < x < x_0 + \delta$.

[similar for $\lim_{x \rightarrow x_0^-} f(x) = l$]

assume $\lim_{x \rightarrow x_0} f(x)$ and $\lim_{x \rightarrow x_0} g(x)$ both exist:

(1) $\lim_{x \rightarrow x_0} [f(x) + g(x)] = \lim_{x \rightarrow x_0} f(x) + \lim_{x \rightarrow x_0} g(x)$

(2) $\lim_{x \rightarrow x_0} [f(x)g(x)] = (\lim_{x \rightarrow x_0} f(x))(\lim_{x \rightarrow x_0} g(x))$

(3) if $\lim_{x \rightarrow x_0} f(x) \neq 0$: $\lim_{x \rightarrow x_0} \frac{1}{f(x)} = \frac{1}{\lim_{x \rightarrow x_0} f(x)}$

(4) if $\lim_{x \rightarrow x_0} g(x) \neq 0$: $\lim_{x \rightarrow x_0} \left[\frac{f(x)}{g(x)} \right] = \frac{\lim_{x \rightarrow x_0} f(x)}{\lim_{x \rightarrow x_0} g(x)}$

(5) if $h(x)$ is continuous at $\lim_{x \rightarrow x_0} f(x)$:
 $\lim_{x \rightarrow x_0} h(f(x)) = h(\lim_{x \rightarrow x_0} f(x))$

comparison test for limits:

(1) if $\lim_{x \rightarrow x_0} f(x) = 0$ and $|g(x)| \leq |f(x)|$ for all x near x_0 with $x \neq x_0$, then $\lim_{x \rightarrow x_0} g(x) = 0$

(2) if $\lim_{x \rightarrow \infty} f(x) = 0$ and $|g(x)| \leq |f(x)|$ for all large enough x , then $\lim_{x \rightarrow \infty} g(x) = 0$

• basic trigonometric limit (1) $\lim_{h \rightarrow 0} \frac{\sinh}{h} = 1$
 (2) $\lim_{h \rightarrow 0} \frac{\cosh - 1}{h} = 0$

proof:

$$\frac{\cosh - 1}{h} = \frac{(\cosh - 1)(\cosh + 1)}{h(\cosh + 1)} = \frac{-\sin^2 h}{h(\cosh + 1)}$$

$$\lim_{h \rightarrow 0} \frac{\cosh - 1}{h} = \left(\lim_{h \rightarrow 0} \frac{-\sin h}{h} \right) \left(\lim_{h \rightarrow 0} \sinh \right) \left(\lim_{h \rightarrow 0} \frac{1}{\cosh + 1} \right) = 0$$

continuity f is continuous at x_0 if $\lim_{x \rightarrow x_0} f(x) = f(x_0)$
 a totally equivalent definition is: $f(x)$ is continuous at a point x_0 if for every $\epsilon > 0$ there exists a number $\delta > 0$ such that $|f(x) - f(x_0)| < \epsilon$ for all x in the domain of f for which $|x - x_0| < \delta$

derivative

并不是(左斜率=右斜率) $\Rightarrow$ 导数存在!

counterexample: $f(x) = \begin{cases} x & 0 < x \leq 1 \\ x-1 & 1 < x \leq 2 \end{cases}$, $f'(1)$ not exists

if $f(x)$ is differentiable at $x = x_0$, then it is also continuous here.

proof:

$$\begin{aligned}\lim_{x \rightarrow x_0} f(x) - f(x_0) &= \lim_{x \rightarrow x_0} \left(\frac{f(x) - f(x_0)}{x - x_0} \cdot (x - x_0) \right) \\ &= \lim_{x \rightarrow x_0} \left(\frac{f(x) - f(x_0)}{x - x_0} \right) \cdot \lim_{x \rightarrow x_0} (x - x_0) \\ &= f'(x_0) \cdot 0 = 0\end{aligned}$$

- 1) let $f(x)$ be continuous on closed interval $[a, b]$, then max/min $f(x)$ has a maximum and a minimum on this interval.
- 2) let $f(x)$ be continuous over $[a, b]$ and differentiable over (a, b) . assume $f(a) = f(b) = 0$, then there exists $c \in (a, b)$ such that $f'(c) = 0$

suppose f is continuous on $[a, b]$ and differentiable mean value theorem on (a, b) , there exists $c \in (a, b)$ such that

$$f'(c) = \frac{f(b) - f(a)}{b - a}$$

proof:

consider $y = \frac{f(b) - f(a)}{b - a} (x - a) + f(a)$

let f be continuous on $[a, b]$. given any intermediate value theorem number y^* between $f(a)$ and $f(b)$, there exists a point $x^* \in [a, b]$ such that $f(x^*) = y^*$

let $f(x)$ be differentiable on (a, b) and $f'(x) > 0$ or $f'(x) < 0$ for all x in (a, b) , then the inverse function g exists and derivative of inverse

$$g'(y) = \frac{dg}{dy} = \frac{1}{f'(x)}$$

proof:

let $f(x) = y$, $f(x+h) = y+k$, then $g(y) = x$, $g(y+k) = x+h$

$$\frac{dg}{dy} = \lim_{k \rightarrow 0} \frac{g(y+k) - g(y)}{k} = \lim_{h \rightarrow 0} \frac{h}{f(x+h) - f(x)} = \frac{1}{f'(x)}$$

• logarithm ① the quantity $\ln x$ is the area under the curve $\frac{1}{x}$ from $x=1$ and x .

② $\ln(x)$ is differentiable and $\frac{d}{dx} \ln(x) = \frac{1}{x}$.

proof:

考虑面积: $\frac{1}{x+h} \cdot h < \ln(x+h) - \ln(x) < \frac{1}{x} \cdot h$.

$$\frac{1}{x+h} < \frac{\ln(x+h) - \ln(x)}{h} < \frac{1}{x}$$

$$\text{so } \lim_{h \rightarrow 0} \frac{\ln(x+h) - \ln(x)}{h} = \frac{1}{x}$$

③ for $a, b > 0$, $\ln(ab) = \ln a + \ln b$

proof:

consider $\ln(ax)$: $\frac{d}{dx} \ln(ax) = \frac{1}{ax} \cdot a = \frac{1}{x}$

so $\ln ax = \ln x + C$

when $x=1$, $C = \ln a$, so $\ln ab = \ln a + \ln b$.

④ $\lim$

exponential ① $x = \exp(y)$ is the inverse function of $y = \ln(x)$, and define $e = \exp(1)$

② $\exp(x)$ is differentiable and $\frac{d}{dx} \exp(x) = \exp(x)$

proof:

$$\begin{aligned} \frac{d}{dx} \exp(x) &= \lim_{h \rightarrow 0} \frac{e^{x+h} - e^x}{h} = e^x \lim_{h \rightarrow 0} \frac{e^h - 1}{h} \\ &= e^x \frac{d}{dx} \exp(x) \Big|_{x=0} = e^x \frac{d}{dy} \ln(y) \Big|_{y=1} = e^x \end{aligned}$$

function estimation ① let a be any positive number, then $\frac{(1+a)^n}{n} \rightarrow \infty$ as $n \rightarrow \infty$

② proof:

$(1+a)^n = 1 + na + \frac{n(n-1)}{2} a^2 + b$, where $b \geq 0$ for large n

so $\frac{(1+a)^n}{n} = \frac{1}{n} + a + \frac{n-1}{2} a^2 + \frac{b}{n}$, so $\lim_{n \rightarrow \infty} \frac{(1+a)^n}{n} = \infty$

3

$$(2) x - \ln(x) \rightarrow \infty \text{ as } x \rightarrow \infty$$

proof:

$$\ln\left(\frac{e^x}{x}\right) = x - \ln(x)$$

$$(3) \frac{x}{\ln(x)} \rightarrow \infty \text{ as } x \rightarrow \infty.$$

proof:

$$\text{let } x = e^y, \frac{x}{\ln(x)} = \frac{e^y}{y}$$

$$(4) x^{\frac{1}{x}} \rightarrow 1 \text{ as } x \rightarrow \infty$$

proof:

$$x^{\frac{1}{x}} = e^{\frac{\ln(x)}{x}}$$

$$(5) \text{ for all } a > 0, \frac{e^x}{x^a} \rightarrow \infty \text{ as } x \rightarrow \infty$$

proof:

$$\frac{e^x}{x^a} = \frac{e^x}{e^{a \ln(x)}} = e^{x - a \ln(x)}$$

$$\ln\left(\frac{e^x}{x^a}\right) = x - a \ln(x) = \ln(x) \left(\frac{x}{\ln(x)} - a\right)$$

let $f(x)$ and $g(x)$ be differentiable on an open interval containing x_0 . $g(x) \neq 0$ and $g'(x) \neq 0$ for x in an interval about x_0 but with $x \neq x_0$. f and g are continuous at x_0 with $f(x_0)$ and $g(x_0)$ both be 0 or infinite, then

$$\lim_{x \rightarrow x_0} \frac{f(x)}{g(x)} = \lim_{x \rightarrow x_0} \frac{f'(x)}{g'(x)}$$

note that x_0 may be replaced by $+\infty$ or x_0^-

proof:

我们只证明 $\frac{0}{0}$ 型, 这涉及 Cauchy mean value theorem:

let f, g be continuous on $[a, b]$ and differentiable on (a, b) with $g(a) \neq g(b)$, and g' doesn't vanish on (a, b) , then there exists c in (a, b) such that

洛比达

$$g'(c) \frac{f(b)-f(a)}{g(b)-g(a)} = f'(c)$$

这一步的证明考虑 $h(x) = f(a) + (g(x) - g(a)) \frac{f(b)-f(a)}{g(b)-g(a)}$ $h(a) = h(b)$

那么利用这个结论,

$$\lim_{x \rightarrow x_0} \frac{f(x)}{g(x)} = \lim_{x \rightarrow x_0} \frac{f(x)-f(x_0)}{g(x)-g(x_0)} = \lim_{x \rightarrow x_0} \frac{f'(c)}{g'(c)} = \lim_{x \rightarrow x_0} \frac{f'(x)}{g'(x)}$$

注意: 在实际中, 我们常用: 取ln, 取倒数等方法以洛比达

在求解不定积分时, 若无法计算但知 $f(x_0) = y$,
可将 $\int f(t) dt$ 化为 $\int_{x_0}^x f(t) dt + k$.

improper integral (1) suppose f and g satisfy

① $|f(x)| \leq g(x)$ for all $x \geq a$

comparison

② $\int_a^b f(x) dx$ and $\int_a^b g(x) dx$ exist for every $b > a$

可用 substitution then

by part/partial

① if $\int_a^\infty g(x) dx$ is convergent, so is $\int_a^\infty f(x) dx$

② if $\int_a^\infty f(x) dx$ is divergent, so is $\int_a^\infty g(x) dx$

先行预处理

注意: 考试中可能涉及将一个 integral 分成几个部分

(2) $\int_0^1 \frac{1}{x^p} dx$ { converges if $p < 1$

{ diverges if $p \geq 1$

$\int_1^\infty \frac{1}{x^p} dx$ { converges if $p > 1$

{ diverges if $p \leq 1$

mean value theorem for integrals

let f be continuous on $[a, b]$, then there exists a point $x_0 \in (a, b)$ such that $f(x_0) = \frac{1}{b-a} \int_a^b f(x) dx$

proof:

define $F(x) = \int_a^x f(t) dt$; and $F'(x) = f(x)$. since F is continuous on $[a, b]$, then there exists $x_0 \in (a, b)$ such that 4

$$F'(x_0) = \frac{F(b) - F(a)}{b - a}$$

$$f(x_0) = \frac{\int_a^b f(t) dt - \int_a^a f(t) dt}{b - a} = \frac{\int_a^b f(t) dt}{b - a}$$

$$(1) \sin(x)\cos(y) = \frac{1}{2}(\sin(x+y) + \sin(x-y))$$

$$\sin(x)\sin(y) = \frac{1}{2}(\cos(x-y) - \cos(x+y))$$

$$\cos(x)\cos(y) = \frac{1}{2}(\cos(x+y) + \cos(x-y))$$

integral technique

(2) if $\sqrt{a^2 - x^2}$ appears, try $x = a\sin(\theta)$, $dx = a\cos(\theta)d\theta$, $\sqrt{a^2 - x^2} = a\cos(\theta)$

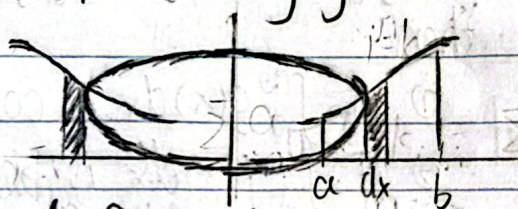
(3) if $\sqrt{x^2 - a^2}$ appears, try $x = a\sec(\theta)$, $dx = a\tan(\theta)\sec(\theta)d\theta$, $\sqrt{x^2 - a^2} = a\tan(\theta)$

(4) if $\sqrt{x^2 + a^2}$ or $x^2 + a^2$ occur, try $x = a\tan(\theta)$, $dx = a\sec^2(\theta)d\theta$, $\sqrt{x^2 + a^2} = a\sec(\theta)$

(5) find the volume of a revolved $f(x)$ along y-axis

$$V = \int_a^b 2\pi x f(x) dx$$

转一圈一层的壳



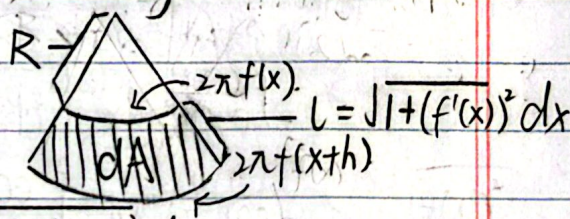
(6) find the area of a revolved $f(x)$ along x-axis

$$A = \int_a^b 2\pi f(x) \sqrt{1 + (f'(x))^2} dx$$

proof:

$$dA = \lim_{h \rightarrow 0} 2\pi f(x+h)(R+L) - 2\pi f(x)R$$

$$= \lim_{h \rightarrow 0} 2\pi f(x+h)L = 2\pi f(x) \sqrt{1 + (f'(x))^2} dx$$



(7) find the moment of a curve $f(x)$

$$M_x = \int_a^b x f(x) dx \quad M_y = \int_a^b \frac{1}{2} (f(x))^2 dx$$

Riemann Sum 证明不可积分: Upper Sum $\neq$ Lower Sum when $n \rightarrow \infty$

(1) if $a > 1$ is a rational number, then $\sum_{n=1}^{\infty} \frac{1}{na}$ convergent. • test series

(2) if $\sum_{i=1}^{\infty} a_i$ and $\sum_{i=1}^{\infty} b_i$ convergent, then $\sum_{i=1}^{\infty} \lambda a_i + \mu b_i$ convergent

• 判断 series 是否 converge 时, 考虑 $R_n \rightarrow 0$
 • 考虑 重写 series 以应对 $(-1)^n / \cos \frac{\pi n}{2}$ e.g.

- (3) every cauchy/absolute convergent series is convergent
- (4) suppose $\{a_n\}$ is a decreasing sequence of positive number with $a_n \rightarrow 0$ as $n \rightarrow \infty$, then $\sum_{n=1}^{\infty} (-1)^{n+1} a_n$ convergent
- (5) let $\sum_{n=1}^{\infty} b_n$ convergent with b_n non-negative. if $|a_n| \leq b_n$, then $\sum_{n=1}^{\infty} a_n$ convergent
- (6) let $f(x)$ be a function which is defined for all $x \geq 1$, and is positive and decreasing. then the series $\sum_{n=1}^{\infty} f(n)$ convergent if and only if $\int_1^{\infty} f(x) dx$ convergent.
- (7) let $\sum_{n=1}^{\infty} a_n$ be a series satisfying $\lim_{n \rightarrow \infty} \left| \frac{a_{n+1}}{a_n} \right| = L$. then if $L < 1$ the series convergent. if $L > 1$ then the series is divergent.
- (8) for the given series $\sum_{n=1}^{\infty} a_n$, suppose that $\lim_{n \rightarrow \infty} |a_n|^{\frac{1}{n}} = L$, then if $L < 1$, the series convergent. if $L > 1$, then series divergent

test power series (1) assume that there is a number $R > 0$ such that $\sum_{n=0}^{\infty} |a_n| R^n$ convergent, then for all $|x| < R$, $\sum_{n=0}^{\infty} a_n x^n$ convergent.

(2) the greatest value of R is called the radius of convergence. if $|x| = R$, series behaves indifferently

- for $|x| < R$, the series convergent
- for $|x| > R$, the series divergent.
- for $|x| = R$, the series behaviour is unsure

(2) let $\sum_{n=0}^{\infty} a_n x^n$ be a power series and assume $\lim_{n \rightarrow \infty} \left| \frac{a_{n+1}}{a_n} \right| = L$ exists. then $R = \frac{1}{L}$ (if $L = 0$, $R = \infty$; if $L = \infty$, $R = 0$)

(3) let $\sum_{n=0}^{\infty} a_n x^n$ be a power series and assume $\lim_{n \rightarrow \infty} |a_n|^{\frac{1}{n}} = L$ exists. then $R = \frac{1}{L}$ (if $L = 0$, $R = \infty$; if $L = \infty$, $R = 0$)

$$L = \infty, R = 0)$$

- (1) let $f(x) = \sum_{n=0}^{\infty} a_n x^n$ be a power series which convergent for $|x| < R$, then $f(x)$ is differentiable for $|x| < R$, and $f'(x) = \sum_{n=1}^{\infty} n a_n x^{n-1}$ differentiation and integration of power series
- (2) let $f(x) = \sum_{n=0}^{\infty} a_n x^n$ be a power series which convergent for $|x| < R$, then in the interval $|x| < R$, $\int_0^x f(t) dt = \sum_{n=0}^{\infty} \frac{a_n x^{n+1}}{n+1}$ 注意积分上下限
- (3) if a function $f(x)$ has a power series expansion $f(x) = \sum_{n=0}^{\infty} a_n (x-x_0)^n$ in the neighborhood of the point $x=x_0$, then it is said to be analytic at x_0 . in addition $a_n = \frac{f^{(n)}(x_0)}{n!}$
- (4) let $f(x) = \sum_{n=0}^{\infty} a_n x^n$ be a power series with radius of convergence R_1 ; $g(x) = \sum_{n=0}^{\infty} b_n x^n$ be another power series with radius of convergence R_2 . let $R = \min(R_1, R_2)$, then

$$\bullet f(x) + g(x) = \sum_{n=0}^{\infty} (a_n + b_n) x^n \quad \text{for } |x| < R$$

$$\bullet \lambda f(x) = \sum_{n=0}^{\infty} (\lambda a_n) x^n \quad \text{for } |x| < R_1$$

$$\bullet f(x)g(x) = \sum_{n=0}^{\infty} \left(\sum_{m=0}^n a_m b_{n-m} \right) x^n \quad \text{for } |x| < R$$

- (1) let f be a function defined on a closed interval between two numbers x and x_0 . assume the function has $(n+1)$ derivatives on the interval and they are all continuous, then Taylor expansion

$$f(x) = \sum_{i=0}^n \frac{f^{(i)}(x_0)}{i!} (x-x_0)^i + R_n$$

$$\text{where } R_n = \int_{x_0}^x \frac{(x-t)^n}{n!} f^{(n+1)}(t) dt$$

proof:

$$\begin{aligned} \text{using integration by part. } f(x) &= f(x_0) + \int_{x_0}^x f'(t) dt \\ &= f(x_0) + \int_{x_0}^x f'(t) d(t-x) = f(x_0) + (x-x_0)f'(x_0) + \int_{x_0}^x (x-t)f''(t) dt \\ &= \dots \end{aligned}$$

注意: R_n 还有别的写法

$$\text{Since } \frac{(x-t)^n}{n!} \geq 0, R_n = f^{(n+1)}(c) \int_{x_0}^x \frac{(x-t)^n}{n!} dt \text{ where } x_0 \leq c \leq x$$

$$= \frac{(x-x_0)^{n+1}}{(n+1)!} f^{(n+1)}(c) \quad (*)$$

(*) 的证明: for f, g continuous on $[a, b]$ with $g(x) \geq 0$, we can find $m \leq f(x) \leq M$, so:

$$mg(x) \leq f(x)g(x) \leq Mg(x)$$

$$m \int_a^b g(x) dx \leq \int_a^b f(x)g(x) dx \leq M \int_a^b g(x) dx$$

$$m \leq \frac{\int_a^b f(x)g(x) dx}{\int_a^b g(x) dx} \leq M$$

• 套用 Taylor expansion 时
注意 $|x|$ 限定

• 通过 Taylor expansion 判断

$f^{(n)}(x)$ 时一定包含 so there exists $c \in [a, b]$ such that $f(c) =$

$$\text{公式 } \sum \frac{f^{(n)}(x_0)}{n!} (x-x_0)^n$$

$$\frac{\int_a^b f(x)g(x) dx}{\int_a^b g(x) dx}$$

• 巧用 binomial expansion 证明 $[f(x)]^{-1}$

(2) 一些关于 remainder 项可以被忽略的证明

$$\bullet e^x = 1 + x + \frac{x^2}{2!} + \dots + \frac{x^n}{n!} + \frac{e^c}{(n+1)!} x^{n+1}$$

$$\text{if } x < 0, \text{ then } c < 0, \text{ and } |R_n| \leq \frac{|x|^{n+1}}{(n+1)!} \rightarrow 0$$

$$\text{if } x > 0, \text{ then there exists } x \leq b; \text{ and } |R_n| \leq e^b \frac{b^{n+1}}{(n+1)!} \rightarrow 0$$

$$\bullet \ln(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \dots + (-1)^{n-1} \frac{x^n}{n} + R_{n+1}$$

$$\text{since } (1-t+t^2-\dots+(-1)^{n-1}t^{n-1}) + (-1)^n \frac{t^n}{1+t} = \frac{1}{1+t}$$

$$\int \frac{1}{1+t} dx = \ln(1+x), R_n = (-1)^n \int_0^x \frac{t^n}{1+t} dt$$

$$\text{if } 0 \leq x \leq a \leq 1, \text{ then } \frac{t^n}{1+t} \leq t^n, |R_n| \leq \frac{x^{n+1}}{n+1} \leq \frac{a^{n+1}}{n+1} \rightarrow 0$$

$$\text{if } -1 \leq a \leq x \leq 0, \text{ then } \left| \frac{t^n}{1+t} \right| \leq \left| \frac{t^n}{1+a} \right|, |R_n| \leq \left| \frac{a^{n+1}}{(n+1)(1+a)} \right| \leq \frac{|a|^{n+1}}{(n+1)(1+a)} \rightarrow 0$$

注意: 在证明 remainder 项可以被忽略时一定要分类讨论!

(3) consider $F(x) = \frac{f(x)}{g(x)}$ and consider $\lim_{x \rightarrow a} F(x)$ in case where $f(a) = g(a) = 0$. assume that the first $(k-1)$ derivatives of f and g also vanish at $x=a$

$$\text{then } f(a+h) = f(a) + f'(a)h + \dots + f^{(k-1)}(a) \frac{h^{k-1}}{(k-1)!} + f^{(k)}(c_1) \frac{h^k}{k!}$$

$$g(a+h) = g(a) + g'(a)h + \dots + g^{(k-1)}(a)\frac{h^{k-1}}{(k-1)!} + g^{(k)}(c_2)\frac{h^k}{k!}$$

and $a < c_1, c_2 < a+h$, when $h \rightarrow 0, c_1, c_2 \rightarrow a$

$$F(a+h) = \frac{f^{(k)}(a)}{g^{(k)}(a)}$$

(1) if f, g are real valued functions that are Riemann integrable Fourier on $[a, b]$, then we define the inner product of f and g , denoted by (f, g) , by $(f, g) = \int_a^b f(x)g(x)dx$.
note $\|f\| = (f, f)^{\frac{1}{2}} = \left(\int_a^b (f(x))^2 dx\right)^{\frac{1}{2}}$

(2) let $S = \{\phi_0, \phi_1, \phi_2, \dots\}$ be a collection of functions that are Riemann integrable on $[a, b]$. if $(\phi_n, \phi_m) = 0$ whenever $n \neq m$, then S is an orthogonal system on $[a, b]$. if in addition, if $\|\phi_n\| = 1$, then S is orthonormal.

$$\begin{aligned} (3) S_n(x) &= \frac{1}{2}a_0 + \sum_{k=1}^n (a_k \cos(kx) + b_k \sin(kx)) \\ &= \frac{1}{2}a_0 + \sum_{k=1}^n \left(\frac{a_k(e^{ikx} + e^{-ikx})}{2} + b_k \frac{(e^{ikx} - e^{-ikx})}{2i} \right) \\ &= \frac{1}{2}a_0 + \sum_{k=1}^n \left(\frac{(a_k - ib_k)}{2} e^{ikx} + \frac{(a_k + ib_k)}{2} e^{-ikx} \right) \\ &= \sum_{k=-n}^n \gamma_k e^{ikx} \quad \text{where } \gamma_0 = \frac{1}{2}a_0 \\ &\quad \gamma_k = \frac{1}{2}(a_k - ib_k) \quad k=1, 2, \dots \\ &\quad \gamma_{-k} = \frac{1}{2}(a_k + ib_k) \end{aligned}$$

notice, $S_n(x)$ is real, then $S_n(x) = \overline{S_n(x)}$

$$\text{so } \sum_{k=-n}^n \gamma_k e^{ikx} = \sum_{k=-n}^n \overline{\gamma_k} e^{-ikx} = \sum_{k=-n}^n \overline{\gamma_{-k}} e^{ikx}$$

so $\gamma_k = \overline{\gamma_{-k}}, \overline{\gamma_k} = \gamma_{-k}$, then $\gamma_k + \overline{\gamma_{-k}} = \gamma_k + \gamma_k$ is real

$\gamma_k - \overline{\gamma_{-k}} = \gamma_k - \gamma_k$ is imaginary.

(4) for any piecewise continuous function $f(x)$ of period 2π which has continuous derivatives of first and second order, $f(x)$ can be expressed as $\frac{1}{2}a_0 + \sum_{n=1}^{\infty} (a_n \cos nx + b_n \sin nx)$

proof:

我们要证明, 当 $n \rightarrow \infty, S_n - f(x) = 0$

前情提要: $\frac{1}{2} + \cos x + \cos 2x + \dots + \cos nx = \frac{1}{2} \sum_{k=-n}^n e^{ikx} = \frac{\sin((n+\frac{1}{2})x)}{\sin \frac{1}{2}x}$

$$\int_{-\pi}^{\pi} \frac{\sin((n+\frac{1}{2})t)}{2\sin \frac{1}{2}t} dt = \pi$$

$$\int_0^{\pi} \frac{\sin((n+\frac{1}{2})t)}{2\sin \frac{1}{2}t} dt = \int_{-\pi}^0 \frac{\sin((n+\frac{1}{2})t')}{2\sin \frac{1}{2}t'} dt' = \frac{\pi}{2} \text{ by substitution } t = -t'$$

因为 $S_n = \frac{1}{2}a_0 + \sum_{k=1}^n a_k \cos kx + b_k \sin kx$

$$= \frac{1}{\pi} \int_{-\pi}^{\pi} f(t) \left[\frac{1}{2} + \sum_{k=1}^n (\cos kt \cos kx + \sin kt \sin kx) \right] dt$$

$$= \frac{1}{\pi} \int_{-\pi}^{\pi} f(t) \left[\frac{1}{2} + \sum_{k=1}^n \cos(k(t-x)) \right] dt = \frac{1}{\pi} \int_{-\pi}^{\pi} f(t) \frac{\sin[(n+\frac{1}{2})(t-x)]}{2\sin \frac{1}{2}(t-x)} dt$$

$$= \frac{1}{\pi} \int_{-\pi-x}^{\pi-x} \frac{f(x+\xi) \sin(n+\frac{1}{2})\xi}{2\sin \frac{1}{2}\xi} d\xi \quad \text{let } \xi = t-x$$

$$= \frac{1}{\pi} \int_{-\pi}^{\pi} \frac{f(x+\xi) \sin(n+\frac{1}{2})\xi}{2\sin \frac{1}{2}\xi} d\xi \quad \text{due to periodical}$$

since $f(x) = \frac{f(x^+) + f(x^-)}{2}$

$$= \frac{1}{\pi} \int_0^{\pi} f(x^+) \frac{\sin(n+\frac{1}{2})t}{2\sin \frac{1}{2}t} dt + \frac{1}{\pi} \int_{-\pi}^0 f(x^-) \frac{\sin(n+\frac{1}{2})t}{2\sin \frac{1}{2}t} dt$$

then $f(x) - S_n = \frac{1}{\pi} \int_0^{\pi} [f(x+\xi) - f(x^+)] \frac{\sin(n+\frac{1}{2})\xi}{2\sin \frac{1}{2}\xi} d\xi + \frac{1}{\pi} \int_{-\pi}^0 [f(x+\xi) - f(x^-)] \frac{\sin(n+\frac{1}{2})\xi}{2\sin \frac{1}{2}\xi} d\xi$

notice that, if the function $g(x)$ is integrable on $[0, b]$,

$$\lim_{\lambda \rightarrow \infty} \int_a^b g(x) \sin \lambda x dx = \lim_{\lambda \rightarrow \infty} \left[-\frac{\cos \lambda x}{\lambda} g(x) \right]_a^b + \int_a^b \frac{\cos \lambda x}{\lambda} g'(x) dx = 0$$

so $\lim_{\lambda \rightarrow \infty} \int_a^b g(x) \sin \lambda x dx = 0$

因此 我们只需证明 $\frac{f(x+\xi) - f(x^+)}{2\sin \frac{1}{2}\xi}$ 与 $\frac{f(x+\xi) - f(x^-)}{2\sin \frac{1}{2}\xi}$ 有界即可

以第一个式子为例:

$$\lim_{\xi \rightarrow 0} \frac{f(x+\xi) - f(x^+)}{\sin \frac{1}{2}\xi} = \lim_{\xi \rightarrow 0} \frac{f(x^+) + \xi f'(x^+) + o(\xi^2) - f(x^+)}{\frac{1}{2}\xi + o(\xi^2)} = 2f'(x^+)$$

因此, $f(x) - S_n = 0$

注意: 根据 Fourier Series 可知: $\frac{\pi}{8} = 2(1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \dots)$

$$\frac{\pi}{8} = 1 - \frac{1}{3} + \frac{1}{5} - \dots$$

① 可以利用 Fourier Series 对函数进行求导, 但要确保结果 convergent.

(5) function $f(x)$ of period 2π . Fourier Series in other forms

$$f(x) = \sum_{n=-\infty}^{\infty} \gamma_n e^{inx}, \text{ where } \gamma_n = \frac{1}{2\pi} \int_{-\pi}^{\pi} f(x) e^{-inx} dx$$

(6) function $f(x)$ of period $2L$ can be written as:

$$f(x) = \frac{1}{2}a_0 + \sum_{n=1}^{\infty} a_n \cos \frac{n\pi x}{L} + b_n \sin \frac{n\pi x}{L}, \text{ where } a_n = \frac{1}{L} \int_{-L}^L f(x) \cos \frac{n\pi x}{L} dx$$

$$b_n = \frac{1}{L} \int_{-L}^L f(x) \sin \frac{n\pi x}{L} dx$$

$$= \sum_{n=-\infty}^{\infty} \gamma_n e^{\frac{in\pi x}{L}}, \text{ where } |x| \leq L, \gamma_n = \frac{1}{2L} \int_{-L}^L f(x) e^{-\frac{in\pi x}{L}} dx$$

(7) function $f(x)$ of period 2π can be expressed as:

$f(x) = \frac{1}{2}a_0 + \sum_{n=1}^{\infty} a_n \cos nx + b_n \sin nx$, then $\frac{1}{\pi} \int_{-\pi}^{\pi} f^2 dx = \frac{1}{2}a_0^2 + \sum_{n=1}^{\infty} (a_n^2 + b_n^2)$
proof:
 $f(x) = \sum_{n=-\infty}^{\infty} \gamma_n e^{inx}$
 $\int_{-\pi}^{\pi} f^2(x) dx = \int_{-\pi}^{\pi} \sum_{n=-\infty}^{\infty} \sum_{m=-\infty}^{\infty} \gamma_n \gamma_m e^{i(n+m)x} dx = \int_{-\pi}^{\pi} \sum_{n=-\infty}^{\infty} \gamma_n \gamma_{-n} e^0 dx$
 $= 2\pi \sum_{n=-\infty}^{\infty} |\gamma_n|^2 = \frac{1}{2}a_0^2 + \sum_{n=1}^{\infty} (a_n^2 + b_n^2)$

(5) any function can be written as

$$f(x) = \lim_{L \rightarrow \infty} \sum_{n=-\infty}^{\infty} \left(\frac{1}{2L} \int_{-L}^L f(t) e^{-\frac{in\pi t}{L}} dt \right) e^{\frac{in\pi x}{L}}$$

let $\frac{n\pi}{L} = \omega_n$, $\omega_n - \omega_{n-1} = \Delta\omega$, when $L \rightarrow \infty$, $\Delta\omega \rightarrow 0$

$$= \lim_{\Delta\omega \rightarrow 0} \frac{1}{2\pi} \sum_{n=-\infty}^{\infty} \left(\int_{-L}^L f(t) e^{-i\omega_n t} dt \right) e^{i\omega_n x} \Delta\omega = \frac{1}{2\pi} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} f(t) e^{-i\omega t} dt e^{i\omega x} d\omega$$

因此: $\hat{f}(\omega) = \int_{-\infty}^{\infty} f(t) e^{-i\omega t} dt$, $f(x) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \hat{f}(\omega) e^{i\omega x} d\omega$

use power series to find the solution of differential equations

(1) 一般来说, 我们 firstly assume the power series exists. 然后用 $y = \sum_{n=0}^{\infty} a_n x^n$ 代入, 然后得出 y 的通项公式

最后通过 ratio test $\lim_{n \rightarrow \infty} \left| \frac{a_{n+1} x^{n+1}}{a_n x^n} \right|$ 验证 radius of convergence 不为 0

注意: 有时 ratio test 的结果与 x 有关, 这说明了 x 的取值

(2) let x_0 be an ordinary point ($P(x)$ and $Q(x)$ to be nice well behaved functions near x_0 ; i.e. they are analytic at $x=x_0$, meaning all derivatives exist and bounded) of $y'' + P(x)y' + Q(x)y = 0$, and let a_0, a_1 be arbitrary constants, then there exists a unique function $y(x)$ such that is analytic at x_0 and solves the differential equation with initial conditions $y(x_0) = a_0$, $y'(x_0) = a_1$.

furthermore, if the power series expansions of $P(x)$ and $Q(x)$ are valid for $|x-x_0| < R$ with $R > 0$, then the power series expansion of $y(x)$ is also valid on the same interval

(3) a single point x_0 of equation $y'' + P(x)y' + Q(x)y = 0$ is said to be regular if the functions $(x-x_0)P(x)$ and $(x-x_0)^2Q(x)$ are analytic.

assume $x=0$ is a singular point x_0 of the equation $y'' + P(x)y' + Q(x)y = 0$, and the power series expansions of $xP(x)$ and $x^2Q(x)$ have radius of convergence $R > 0$, i.e. the power series are valid on the interval $|x| < R$, if the roots of the indicial equation are real and $\lambda_2 \leq \lambda_1$, then the equation has at least one solution:

$$y_1 = x^{\lambda_1} \sum_{n=0}^{\infty} a_n x^n \quad (a_0 \neq 0)$$

on the interval $0 < x < R$ with the coefficients a_n determined by recursion formulas and the series $\sum_{n=0}^{\infty} a_n x^n$ converges for $|x| < R$.

if $\lambda_1 - \lambda_2$ is not zero or a positive integer, then the equation has a second independent solution

$$y_2 = x^{\lambda_2} \sum_{n=0}^{\infty} b_n x^n \quad (b_0 \neq 0)$$

where the coefficients b_n are again found from the recursion formulas and $\sum_{n=0}^{\infty} b_n x^n$ converges for $|x| < R$.

Proof:

since $P(x)$ and $Q(x)$ are singular at $x=0$ but $xP(x)$ and $x^2Q(x)$ are analytic

$$P(x) = \frac{C_{-1}}{x} + C_0 + C_1 x + \dots \quad Q(x) = \frac{d_{-2}}{x^2} + \frac{d_{-1}}{x} + d_0 + d_1 x + \dots$$

$$\text{so } y'' + \left(\frac{C_{-1}}{x} + C_0 + C_1 x + \dots\right) y' + \left(\frac{d_{-2}}{x^2} + \frac{d_{-1}}{x} + d_0 + d_1 x + \dots\right) y = 0$$

$$x^2 y'' + x(C_{-1} + C_0 x + C_1 x^2 + \dots) y' + (d_{-2} + d_{-1} x + d_0 x^2 + \dots) y = 0$$

因为 x 在 0 附近, 我们目前先考虑最低阶项近似: $x^2 y'' + C_{-1} x y' + d_{-2} y = 0$

by substitution $z = \ln x$ [注意: $x > 0$] $\frac{d^2 y}{dz^2} + (C_{-1} - 1) \frac{dy}{dz} + d_{-2} y = 0$

然后用 $y = e^{\lambda z}$ 求解, 当 $\lambda \neq \lambda_1$, 得到 $y_1 = x^{\lambda_1}$, $y_2 = x^{\lambda_2}$, 当 $\lambda = \lambda_1$, 得到 $y_1 = x^{\lambda_1}$ 与 $y_2 = x^{\lambda_1} \ln x$

[注意: if we are looking for solutions in $x < 0$, then let $z = \ln(-x)$]

- 现在我们回到万方程, 其系数有高阶展开项, 不能用简单的 Euler 解, 所以我们在 Euler 解中乘上一个在 $x=0$ 附近析的函数 (高阶项的存在会修正系数)

deeper into Fourier Transformation

- (1) for $f(x)$, we have its Fourier expansion:

$$f(x) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \left(\int_{-\infty}^{\infty} f(s) e^{-iws} ds \right) e^{iwx} dw$$

where the inner integral is $\hat{f}(w)$, named Fourier Transformation.

define $F(f(x)) = \hat{f}(w)$, $F^{-1}(\hat{f}(w)) = f(x)$.

(2) for even $f(x)$: $\hat{f}(w) = \int_0^{\infty} f(x) \cos wx dx$ $f(x) = \frac{2}{\pi} \int_0^{\infty} \hat{f}(w) \cos wx dw$

for odd $f(x)$: $\hat{f}(w) = -\int_0^{\infty} f(x) \sin wx dx$ $f(x) = \frac{2}{\pi} \int_0^{\infty} \hat{f}(w) \sin wx dw$

- (3) the Fourier and inverse Fourier transforms are linear

$$F(af(x) + bg(x)) = a\hat{f}(w) + b\hat{g}(w)$$

$$F^{-1}(a\hat{f}(w) + b\hat{g}(w)) = af(x) + bg(x)$$

- $F(f(ax)) = \frac{1}{|a|} \hat{f}\left(\frac{w}{a}\right)$ if $a > 0$

- $F(f(-x)) = \hat{f}(-w)$

- $F(f(x-x_0)) = e^{-iwx_0} \hat{f}(w)$

- $F(e^{iwx_0} f(x)) = \hat{f}(w-w_0)$

- $F(\hat{f}(x)) = 2\pi f(-w)$

Proof: $f(x) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \hat{f}(w) e^{iwx} dw = \frac{1}{2\pi} \int_{-\infty}^{\infty} \hat{f}(s) e^{isx} ds$ let $w=s$

$$f(-w) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \hat{f}(s) e^{-iws} ds = \frac{1}{2\pi} \int_{-\infty}^{\infty} \hat{f}(x) e^{-iwx} dx$$
 let $s=x$

$$= \frac{1}{2\pi} F(\hat{f}(x))$$

- $F\left(\frac{d^n}{dx^n} f(x)\right) = (iw)^n \hat{f}(w)$

Proof: use induction.

- $F(xf(x)) = i\hat{f}'(w)$

此处常用

对于奇偶 $f(x)$, 计算 $\hat{f}(w)$
 时 $\int e^{-iwx} = \cos wx - i \sin wx$

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proof: $\int_{-\infty}^{\infty} x f(x) e^{-iwx} dx = \int_{-\infty}^{\infty} f(x) \frac{d}{dw} (e^{-iwx}) dx$
 $= i \frac{d}{dw} \hat{f}(w) = i \hat{f}'(w)$

• $F_c(f'(x)) = -f(0) + w \hat{f}_s(w)$

$F_s(f'(x)) = -w \hat{f}_c(w)$

$F_c(f''(x)) = -f'(0) - w^2 \hat{f}_c(w)$

$F_s(f''(x)) = w f'(0) - w^2 \hat{f}_s(w)$

$h(x) = \begin{cases} 1 & |x| \leq a \\ 0 & \text{otherwise} \end{cases}, \quad \hat{h}(w) = \frac{2}{w} \sin(aw)$

- if $f(x)$ is a complex-valued function and $[f(x)]^*$ is its complex conjugate

$F([f(x)]^*) = [\hat{f}(-w)]^*$

proof: $[\hat{f}(-w)]^* = \int_{-\infty}^{\infty} [f(x)]^* e^{-iwx} dx = F([f(x)]^*)$

- we define the convolution of two functions $f(x), g(x)$, define over $(-\infty, \infty)$ as $f(x) * g(x) = \int_{-\infty}^{\infty} f(x-u)g(u)du$
 an important result is convolution theorem: $F(f(x) * g(x)) = \hat{f}(w)\hat{g}(w)$

proof: $\int_{x=-\infty}^{\infty} \left(\int_{u=-\infty}^{\infty} f(x-u)g(u)du \right) e^{-iwx} dx$

$= \int_{u=-\infty}^{\infty} g(u) \left(\int_{x=-\infty}^{\infty} f(x-u) e^{-iwx} dx \right) du$

$= \int_{u=-\infty}^{\infty} g(u) \left(\int_{s=-\infty}^{\infty} f(s) e^{-iw(s+u)} ds \right) du$ let $s = x - u$

$= \left(\int_{u=-\infty}^{\infty} g(u) e^{-iwu} du \right) \left(\int_{s=-\infty}^{\infty} f(s) e^{-iws} ds \right) = \hat{g}(w)\hat{f}(w)$

- if $f(x)$ is a real-valued matrix, then $\frac{1}{2\pi} \int_{-\infty}^{\infty} |\hat{f}(w)|^2 dw = \int_{-\infty}^{\infty} |f(x)|^2 dx$

proof: $F(f(x) * f(-x)) = \hat{f}(w)\hat{f}(-w) = \hat{f}(w)[\hat{f}(w)]^* = |\hat{f}(w)|^2$

$f(x) * f(-x) = \int_{-\infty}^{\infty} f(u-x)f(u)du = \frac{1}{2\pi} \int_{-\infty}^{\infty} |\hat{f}(w)|^2 e^{iwx} dw$

let $X=0$, then done.

(4) the Dirac delta-function

- Consider the following step-function

$f_k(x) = \begin{cases} \frac{k}{2} & \text{if } |x| < \frac{1}{k} \\ 0 & \text{if } |x| > \frac{1}{k} \end{cases}, \quad \delta(x) = \lim_{k \rightarrow \infty} f_k(x)$

clearly $\int_{-\infty}^{\infty} \delta(x) dx = 1$, and by mean-value theorem,

$\int_{-\infty}^{\infty} g(x) \delta(x-a) dx = g(a)$

- notice that:

$F(\delta(x)) = \int_{-\infty}^{\infty} \delta(x) e^{-iwx} dx = e^{iwo} = 1, \quad F^{-1}(1) = \delta(x)$

事实上, using inverse formula, we have: $\delta(x) = \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{\pm iwx} dw$

此外, 常考

ODEs

(1) $G(x, f(x), \frac{df}{dx}, \dots, \frac{d^k f}{dx^k}) = 0$

order is the order of the highest derivative k .

degree is the power of the highest derivative (when fractional power is removed)

the ODE is linear if G is a linear function of $f(x)$ and its derivatives

[注意: $\frac{d^2y}{dx^2} = x^2y + 1$ 是 linear!]

(2) for first order ODEs

① $\frac{dy}{dx} = f\left(\frac{y}{x}\right)$ [可以将 $\frac{ax+by}{Ax+By}$ 形式化简]

let $u = \frac{y}{x}$, $y = ux$, $\frac{dy}{dx} = u + x \frac{du}{dx}$

so $u + x \frac{du}{dx} = f(u)$

② $\frac{dy}{dx} + p(x)y = q(x)y^n$

let $u = y^{1-n}$, $\frac{du}{dx} = (1-n)y^{-n} \frac{dy}{dx}$

so $(1-n)y^{-n} \frac{du}{dx} + (1-n)p(x)y^{1-n} = (1-n)q(x)$

$\frac{du}{dx} + (n-1)p(x)u = (n-1)q(x)$

(3) for second order ODEs:

① $\frac{d^2y}{dx^2} = f\left(x, \frac{dy}{dx}\right)$

let $u = \frac{dy}{dx}$, so $\frac{du}{dx} = f(x, u)$

② $\frac{d^2y}{dx^2} = f\left(y, \frac{dy}{dx}\right)$

let $u = \frac{dy}{dx}$, $\frac{du}{dx} = \frac{du}{dy} \frac{dy}{dx} = u \frac{du}{dy}$

so $u \frac{du}{dy} = f(y, u)$

(3) for linear ODEs:

① the general form $f(x) = a_k(x) \frac{d^k y}{dx^k} + \dots + a_1(x) \frac{dy}{dx} + a_0(x)y$

ODE is homogeneous if $f(x) = 0$

② we define the differential operator $D[f] = \frac{d}{dx}[f]$, it is linear

linear ODEs are associated with linear operator $[Ly] = \sum_{i=0}^k a_i(x) D^i[y]$

③ a set $\{f_i(x)\}_{i=1}^k$ is said to be linearly independent if $C_1 f_1 + C_2 f_2 + \dots + C_k f_k = 0$ if and only if $C_1 = C_2 = \dots = C_k = 0$

④ the solution of homogeneous linear ODE $Ly = 0$ form a vector space, so the general solution is $C_1 y_1 + C_2 y_2 + \dots + C_k y_k$ where $B = \{y_i(x)\}_{i=1}^k$ is a set of linearly independent solutions forming a basis for solution vector space

⑤ Wronskian matrix: $(W_{k \times k})$

$$W[\{y_i(x)\}_{i=1}^k] = \det W = \det \begin{pmatrix} y_1(x) & y_2(x) & \dots & y_k(x) \\ \frac{dy_1(x)}{dx} & \frac{dy_2(x)}{dx} & \dots & \frac{dy_k(x)}{dx} \\ \vdots & \vdots & \ddots & \vdots \\ \frac{d^{k-1}y_1(x)}{dx^{k-1}} & \frac{d^{k-1}y_2(x)}{dx^{k-1}} & \dots & \frac{d^{k-1}y_k(x)}{dx^{k-1}} \end{pmatrix}$$

the set $\{y_i(x)\}_{i=1}^k$ is linearly independent if $\det W \neq 0$

[注意: 给定一个 k^{th} degree homogeneous linear ODE 和其 k 个答案, 要证

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证明此 k 个根 linearly independent, 通解为 $C_1 y_1 + C_2 y_2 + \dots + C_k y_k$

② $\det W = 0 \Rightarrow \{y_i(x)\}_{i=1}^k$ is linear dependent / consider x' and $|x|_x$

(4) when solving non-homogeneous linear ODE, the solution is $y_{CF} + y_{PL}$ (100)
where $\mathcal{L}[y_{CF}] = 0, \mathcal{L}[y_{PL}] = f(x)$

① for $\mathcal{L}(y) = \beta_k x^k \frac{d^k y}{dx^k} + \beta_{k-1} x^{k-1} \frac{d^{k-1} y}{dx^{k-1}} + \dots + \beta_1 x \frac{dy}{dx} + \beta_0 y = f(x)$

let $x = e^z, \frac{dy}{dx} = \frac{1}{x} \frac{dy}{dz}, \frac{d^2 y}{dx^2} = \frac{d}{dz} \left(\frac{dy}{dx} \right) \cdot \frac{dz}{dx} = \frac{1}{x^2} \left[\frac{d^2 y}{dz^2} - \frac{dy}{dz} \right] \dots$

② we can also use Fourier transformation to solve questions

$$\frac{d^2 y}{dx^2} + xy = 0$$

$$F\left(\frac{d^2 y}{dx^2}\right) + F(xy) = F(0)$$

$$(i\omega)^2 \hat{y}(\omega) + i \hat{y}'(\omega) = 0$$

(5) system of ODEs:

我们首先将 higher order ODEs 转化成多个 first order ODEs

example:

$$G_1(t, x, y, \frac{dx}{dt}, \frac{d^2 x}{dt^2})$$

$$G_2(t, x, y, \frac{dy}{dt}, \frac{d^2 y}{dt^2})$$

$$\Rightarrow \text{let } u = \frac{dx}{dt}, v = \frac{dy}{dt} \Rightarrow G_1(t, x, y, v, \frac{dv}{dt}), G_2(t, x, y, u, \frac{du}{dt}), \frac{dx}{dt} = u, \frac{dy}{dt} = v$$

随后, 形式可转化成:

$$\frac{d}{dt} \vec{y} = A \vec{y} + \vec{g}(t)$$

解可用 $\vec{y}_{as} = \vec{y}_{CF} + \vec{y}_{PL}$ 求解

① 如果 A diagonalisable, 对于 $\frac{d}{dt} \vec{y} = A \vec{y}$, eigenvalues 与 eigenvectors

则 $\vec{y}_{CF} = (\vec{v}_1, \vec{v}_2, \dots, \vec{v}_n) \begin{pmatrix} C_1 e^{\lambda_1 t} \\ C_2 e^{\lambda_2 t} \\ \vdots \\ C_n e^{\lambda_n t} \end{pmatrix}$, where $\vec{v}_i$ and λ_i are corresponding eigenvector and eigenvalue

proof:

$$P^{-1} \frac{d}{dt} \vec{y} = P^{-1} A P P^{-1} \vec{y} \quad \text{where } A = P \Lambda P^{-1}$$

$$\frac{d}{dt} (P^{-1} \vec{y}) = \Lambda (P^{-1} \vec{y})$$

② 若 A 不可被 diagonalised, 使用 Jordan Normal form 求解 $\frac{d}{dt} \vec{y} = A \vec{y}$
则 $W^{-1} A W = J$, 求解 W 时, 将已知元素用 eigenvectors 表达, 其余元素用 $\alpha, \beta, \dots$ 等

求解 $\frac{d}{dt} (W^{-1} \vec{y}) = J (W^{-1} \vec{y})$, 从下往上求解

$$\text{如果 } J = \begin{pmatrix} \lambda_1 & 0 & \dots \\ 0 & \lambda_2 & \dots \\ \vdots & \vdots & \ddots \end{pmatrix}, \text{ then } \begin{cases} z_n = C_n e^{\lambda_n t} \\ z_{n-1} = C_{n-1} e^{\lambda_n t} + C_n t e^{\lambda_n t} \\ z_{n-2} = C_{n-2} e^{\lambda_n t} + C_{n-1} t e^{\lambda_n t} + C_n \frac{t^2}{2} e^{\lambda_n t} \end{cases}, \vec{z} = W^{-1} \vec{y} \text{ or } \vec{y} = W \vec{z}$$

③ 求解 y_{PL} 用 Ansatz

$$y_{CF} = z_1 \vec{v}_1 + z_2 \vec{w}_2 + \dots + z_n \vec{w}_n, \text{ where } W = (\vec{v}_1, \vec{w}_2, \dots, \vec{w}_n)$$

[注意: 有关于 Jordan matrix 的内容]

① 根据 character equation 找出 λ_i , 随后分解出 geometric multiplicity

个 $\vec{v}_i$: (geometric multiplicity = $\dim(\ker(A - \lambda_i I))$)

② 根据 algebraic multiplicity 与 geometric multiplicity 求 Jordan block 的大小: geometric multiplicity = block 的个数 algebraic multiplicity = blocks 的大小的和

$d_1 = \dim(\ker(A - \lambda_i I))$: 大小至少为 1 的 block 的个数

$d_2 = \dim(\ker(A - \lambda_i I)^2)$: 大小至少为 2 的 block 的个数

③ 判断 $\vec{v}_i$ 具体对应多大的 block:

$\vec{v}_1$ 只属于 $\text{im}(A - \lambda_i I)$: $\vec{v}_1$ 是 2×2 block 的链头

$\vec{v}_2$ 只属于 $\text{im}(A - \lambda_i I)^2$: $\vec{v}_2$ 是 3×3 block 的链头

...

• qualitative analysis of ODEs

(1) • asymptotic behaviour: when $t \rightarrow \infty$, ...

• fixed points: $\vec{y}^*$ is a fixed point or an equilibrium point of a system of first order ODEs, if once $\vec{y}(t_0) = \vec{y}^*$ at some t_0 , then for all future times $t > t_0$, $\vec{y} = \vec{y}^*$

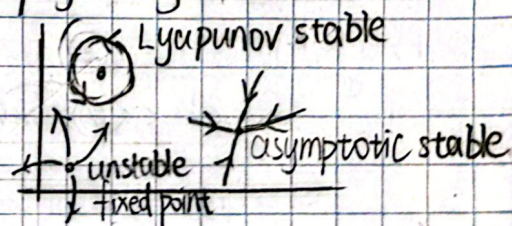
$$\frac{d\vec{y}}{dt} \Big|_{\vec{y}=\vec{y}^*} = \vec{0}$$

$$\text{if } \frac{d\vec{y}}{dt} = A\vec{y}, \text{ for fixed point: } \begin{cases} \vec{y}^* = \vec{0} & \text{if } \det A \neq 0 \\ \vec{y}^* \in \ker A & \text{if } \det A = 0 \end{cases}$$

• Lyapunov stability: a fixed point $\vec{y}^*$ is said to be Lyapunov stable, if for every $\epsilon > 0$, there exists a $\delta > 0$ such that, if $\|\vec{y}(0) - \vec{y}^*\| < \delta$, then for $\forall t \geq 0$, we have $\|\vec{y}(t) - \vec{y}^*\| < \epsilon$

• asymptotic stability: a fixed point $\vec{y}^*$ is said to be asymptotic stable, if it is Lyapunov stable and there exists $\delta > 0$ such that, if $\|\vec{y}(0) - \vec{y}^*\| < \delta$, then

$$\lim_{t \rightarrow \infty} \|\vec{y}(t) - \vec{y}^*\| = 0$$



(2) uniqueness of solutions of ODEs

solutions of ODEs are uniquely defined by

initial conditions except at some special

points (fixed points or singular points where trajectories start or end) in the phase plane. trajectories in the phase plane cannot cross

③ for linear system $\frac{d\vec{y}}{dt} = A\vec{y}$, eigenvectors of A represents invariant

lines, if A has eigenvectors $\vec{v}_1, \vec{v}_2, \dots$, then if $\vec{y}(0)$ is at the line of $\vec{v}_1$, then it will stay on the line forever. if $\lambda > 0$, $\vec{y}(t)$ grows along $\vec{v}_1$, if $\lambda < 0$, $\vec{y}(t) \rightarrow \vec{0}$.

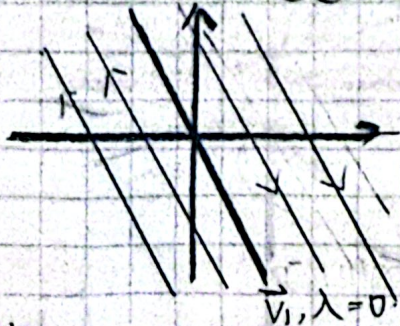
• Assume $\frac{d\vec{y}}{dt} = A\vec{y}$, $\vec{y} = C_1 e^{\lambda_1 t} \vec{v}_1 + C_2 e^{\lambda_2 t} \vec{v}_2$.

$\lambda_1 > 0 > \lambda_2$, then $\vec{y}$ parallel to $\vec{v}_1$ as $t \rightarrow \infty$

• 如果我们有 $\begin{cases} \frac{dx}{dt} = ax + by \\ \frac{dy}{dt} = cx + dy \end{cases}$, 可将其转换为 $A\vec{y} = \frac{d\vec{y}}{dt}$

通过计算 A 的 eigenvalue/eigenvector, 可大体图像

[注意 如果 $\Delta = 0$, $\tau = 0$. 通过 Jordan form 后可求得



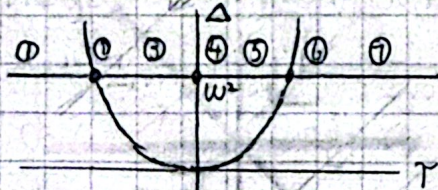
[rotational direction unknown]

• 对于 2D nonlinear system, 我们可通过 $\frac{d\vec{y}}{dt} = \vec{0}$ 寻找 fixed point.

• 对于 nD linear system of ODEs, we have stability if real part of all eigenvalues are negative.

• for a linear system, a bifurcation is defined as a change in stability of the system.

example ① $\frac{d}{dt} \begin{pmatrix} x \\ u \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ -\omega^2 & -2k \end{pmatrix} \begin{pmatrix} x \\ u \end{pmatrix}$, $\Delta = \omega^2$, $\tau = -2k$

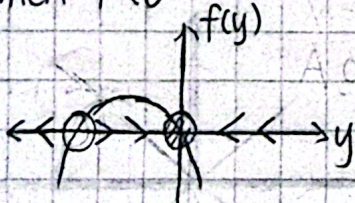


根据 ω, k 大小, 我们共有 7 种不同阶段
其中, when $k=0$ 时 stability 改变 (①)
so bifurcation points are at $k=0$

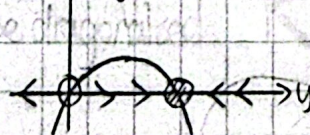
① $\frac{dy}{dt} = ry - y^2 = f(y)$

when $\frac{dy}{dt} = 0$, $y^* = 0$ or r

when $r < 0$



when $r > 0$



when $r = 0$

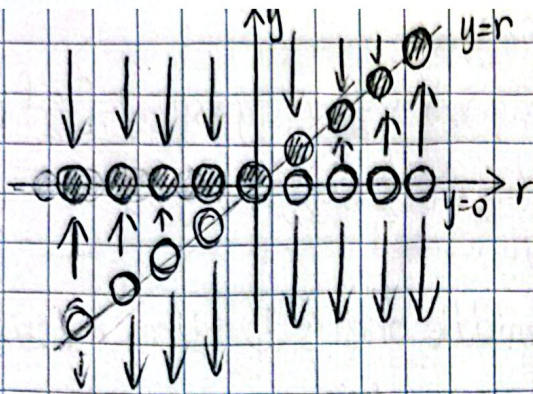


* 对于 1D non-linear system

* 先找出可能的 fixed point

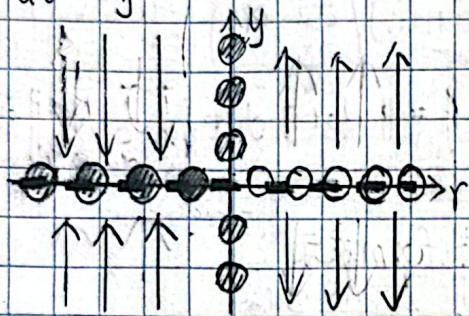
* 画出 $f(y)$ vs y 以判断各个情况下 fixed point 在哪里

根据结果画出 bifurcation diagram.



* 画 bifurcation diagram 先画出可能的 fixed point 表达式以简化
[注意: semi stable = saddle points]

② $\frac{dy}{dt} = \frac{r}{y}$



* 对于 singular point (即 undefined points) 用虚线标出
即便 singular point 与 fixed point 并不重合

fixed points dominate the dynamics of first order systems. phase points never reverses directions and the approach to equilibrium is always monotonic. so it's impossible to oscillate in one dimensional systems.

• multivariable calculus

- (1) level curve of a multivariable function $f(x,y)$ can be $\vec{x}_c = (x,y)_c$, where $f(\vec{x}_c) = C$
- (2) $\lim_{\vec{x} \rightarrow \vec{x}^*} f(\vec{x}) = C : \forall \epsilon > 0, \exists \delta > 0$, such that $\forall 0 < |\vec{x} - \vec{x}^*| < \delta, |f(\vec{x}) - C| < \epsilon$
- (3) f is continuous if and only if $\lim_{\vec{x} \rightarrow \vec{x}^*} f(\vec{x}) = f(\vec{x}^*)$

(4) $\frac{\partial f}{\partial x}(x,y) = \lim_{h \rightarrow 0} \frac{f(x+h,y) - f(x,y)}{h}$

$\frac{\partial}{\partial x}(\frac{\partial f}{\partial y}) = \frac{\partial}{\partial y}(\frac{\partial f}{\partial x})$, $\frac{\partial^2 f}{\partial x \partial y} = \frac{\partial^2 f}{\partial y \partial x}$
 $df = \sum_{i=1}^n (\frac{\partial f}{\partial x_i}) dx_i$, $\frac{df}{dt} = \sum_{i=1}^n (\frac{\partial f}{\partial x_i}) (\frac{dx_i}{dt})$

[注意: 有多个 variables, 每个 $(\frac{\partial f}{\partial x_i})$ 标号却标以区别表达式所包含的 variable]

(5) for an implicit function. $F(x,y,z) = 0$

since $df = \frac{\partial F}{\partial x} dx + \frac{\partial F}{\partial y} dy + \frac{\partial F}{\partial z} dz = 0$

$dz = -\frac{\partial F}{\partial x} dx - \frac{\partial F}{\partial y} dy$

$dz = -\frac{(\frac{\partial F}{\partial x})}{(\frac{\partial F}{\partial z})} dx - \frac{(\frac{\partial F}{\partial y})}{(\frac{\partial F}{\partial z})} dy$

} 得到的答案相同



BARCLAYS

Example 1:

$$u = f(x, y), y = g(x, t), \text{ find } \left(\frac{\partial u}{\partial x}\right)_t, \left(\frac{\partial u}{\partial t}\right)_x$$

$$\text{Sol: } du = \left(\frac{\partial u}{\partial x}\right)_y dx + \left(\frac{\partial u}{\partial y}\right)_x dy$$

$$dy = \left(\frac{\partial y}{\partial x}\right)_t dx + \left(\frac{\partial y}{\partial t}\right)_x dt$$

$$\therefore du = \underbrace{\left(\left(\frac{\partial u}{\partial x}\right)_y + \left(\frac{\partial u}{\partial y}\right)_x \left(\frac{\partial y}{\partial x}\right)_t\right)}_{\textcircled{1}} dx + \underbrace{\left(\frac{\partial u}{\partial y}\right)_x \left(\frac{\partial y}{\partial t}\right)_x}_{\textcircled{2}} dt$$

$$\therefore \left(\frac{\partial u}{\partial x}\right)_t = \textcircled{1}, \left(\frac{\partial u}{\partial t}\right)_x = \textcircled{2}$$

(6) Taylor expansion for multivariable functions.

$$f(x_0 + \Delta x, y_0 + \Delta y) = f(x_0, y_0) + \frac{\partial f}{\partial x}(x_0, y_0) \Delta x + \frac{\partial f}{\partial y}(x_0, y_0) \Delta y + \frac{1}{2!} \left[\frac{\partial^2 f}{\partial x^2}(x_0, y_0) (\Delta x)^2 + 2 \frac{\partial^2 f}{\partial x \partial y}(x_0, y_0) \Delta x \Delta y + \frac{\partial^2 f}{\partial y^2}(x_0, y_0) (\Delta y)^2 \right] + \dots$$

change of variables in multivariable functions

$$\begin{pmatrix} dx \\ dy \end{pmatrix} = \begin{pmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{pmatrix} \begin{pmatrix} du \\ dv \end{pmatrix} \text{ where the middle one is Jacobian matrix}$$

由此引出的应用:

$$\bullet (ds)^2 = (dx)^2 + (dy)^2 = (dx \ dy) \begin{pmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{pmatrix} \begin{pmatrix} du \\ dv \end{pmatrix} = (du \ dv) \begin{pmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{pmatrix} \begin{pmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{pmatrix} \begin{pmatrix} du \\ dv \end{pmatrix}$$

$$\bullet dA = dx dy = \|\vec{dx} \times \vec{dy}\| = \|\det \begin{pmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{pmatrix} \begin{pmatrix} \hat{i} & \hat{j} & \hat{k} \end{pmatrix}\| = \|\det J\| \det \begin{pmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{pmatrix} \begin{pmatrix} du \\ dv \end{pmatrix} = |\det J| du dv$$

[注意: 在计算多项式导数时有以下方法:

example 2: $u = u(x, y)$, 将 $\frac{\partial u}{\partial x}$ 用 polar form 表示

$$\bullet \frac{\partial u}{\partial x} = \frac{\partial u}{\partial r} \frac{\partial r}{\partial x} + \frac{\partial u}{\partial \theta} \frac{\partial \theta}{\partial x} = \cos \theta \frac{\partial u}{\partial r} - \frac{\sin \theta}{r} \frac{\partial u}{\partial \theta}$$

$$\frac{\partial}{\partial x} \left(\frac{\partial u}{\partial x} \right) = \frac{\partial}{\partial r} \left(\frac{\partial u}{\partial x} \right) \frac{\partial r}{\partial x} + \frac{\partial}{\partial \theta} \left(\frac{\partial u}{\partial x} \right) \frac{\partial \theta}{\partial x} =$$

$$= \cos \theta \frac{\partial}{\partial r} \left(\frac{\partial u}{\partial x} \right) - \frac{\sin \theta}{r} \frac{\partial}{\partial \theta} \left(\frac{\partial u}{\partial x} \right)$$

$$= \cos^2 \theta \frac{\partial^2 u}{\partial r^2} + \frac{\cos \theta \sin \theta}{r^2} \frac{\partial u}{\partial \theta} - \frac{\cos \theta \sin \theta}{r} \frac{\partial^2 u}{\partial r \partial \theta} + \frac{\sin^2 \theta}{r} \frac{\partial u}{\partial r}$$

$$- \frac{\cos \theta \sin \theta}{r} \frac{\partial^2 u}{\partial r \partial \theta} + \frac{\cos \theta \sin \theta}{r^2} \frac{\partial u}{\partial \theta} + \frac{\sin^2 \theta}{r^2} \frac{\partial^2 u}{\partial \theta^2}$$

如果计算中遇到 $\frac{\partial}{\partial r} \left(\frac{\partial r}{\partial x} \right)$, $\frac{\partial}{\partial \theta} \left(\frac{\partial r}{\partial x} \right)$, $\frac{\partial}{\partial r} \left(\frac{\partial \theta}{\partial x} \right)$, $\frac{\partial}{\partial \theta} \left(\frac{\partial \theta}{\partial x} \right)$ 时, 带入数字考虑大小

$$\bullet \frac{\partial}{\partial x} u = \left(\cos \theta \frac{\partial}{\partial r} - \sin \theta \frac{\partial}{\partial \theta} \right) u$$

$$\frac{\partial^2}{\partial x^2} u = \left(\cos \theta \frac{\partial}{\partial r} - \sin \theta \frac{\partial}{\partial \theta} \right) \left(\cos \theta \frac{\partial}{\partial r} - \sin \theta \frac{\partial}{\partial \theta} \right) u$$

将求偏导化作线性算子进行考虑

(5) 在求解 ODE 时, 如果 $\frac{dy}{dx} = -\frac{F(x, y)}{G(x, y)}$ 且 $\frac{\partial F}{\partial y} = \frac{\partial G}{\partial x}$, 则 $F(x, y) = \frac{\partial U}{\partial x}$, $G(x, y) = \frac{\partial U}{\partial y}$

这是因为 $du = \left(\frac{\partial u}{\partial x} \right) dx + \left(\frac{\partial u}{\partial y} \right) dy = 0$

已知 $\frac{\partial u}{\partial x}$, $\frac{\partial u}{\partial y}$ 后, 可先通过对 $\frac{\partial u}{\partial x}$ 积分得到 $H(x, y) + K(y)$, $K(y)$ 未知

再通过 $\frac{\partial}{\partial y} (H(x, y) + K(y)) = \frac{\partial u}{\partial y}(x, y)$ 求解 $K(y)$

如果 $\frac{dy}{dx} = -\frac{F(x, y)}{G(x, y)}$ 但 $\frac{\partial F}{\partial y} \neq \frac{\partial G}{\partial x}$, 则需添加 $\lambda(x)$ 或 $\lambda(y)$ 使得 $\frac{\partial(\lambda F)}{\partial y} = \frac{\partial(\lambda G)}{\partial x}$

此时 $\frac{\partial u}{\partial x} = \lambda F$, $\frac{\partial u}{\partial y} = \lambda G$

(a) sketch the graph.

① check continuity and singularity, and find their behaviour.

② obtain some level curves, for example $f(\vec{x}) = 0$.

③ find the stationary points

我们先寻找如下点: $\frac{\partial f}{\partial x} = \frac{\partial f}{\partial y} = 0$

随后我们可以用 Taylor expansion 判断性质.

$$f(\vec{x} + \Delta \vec{x}) - f(\vec{x}) = \frac{1}{2} \Delta \vec{x}^T \begin{pmatrix} \frac{\partial^2 f}{\partial x^2} & \frac{\partial^2 f}{\partial x \partial y} \\ \frac{\partial^2 f}{\partial y \partial x} & \frac{\partial^2 f}{\partial y^2} \end{pmatrix} \Delta \vec{x} \quad [-\text{阶导已为0}]$$

将 $\begin{pmatrix} \frac{\partial^2 f}{\partial x^2} & \frac{\partial^2 f}{\partial x \partial y} \\ \frac{\partial^2 f}{\partial y \partial x} & \frac{\partial^2 f}{\partial y^2} \end{pmatrix}$ 化为 $V(\vec{x}) \Lambda(\vec{x}) V^T(\vec{x})$ 的形式, 则:

- $\lambda_1, \lambda_2 > 0$: minimum point
- $\lambda_1, \lambda_2 < 0$: maximum point
- otherwise: saddle point

① 定理

- continuous on $[a, b]$, differentiable on (a, b)
 $(f(a) = f(b)) \Rightarrow (\exists c \in (a, b), f'(c) = 0)$
- continuous on $[a, b]$, differentiable on (a, b)
 $\exists c \in (a, b), f'(c) = \frac{f(b) - f(a)}{b - a}$
- continuous on $[a, b]$, for any f between $f(a)$ and $f(b)$, there exists $x \in [a, b]$ such that $f(x) = y$.
- continuous on $[a, b]$, $\exists c \in (a, b), f(c) = \frac{1}{b-a} \int_a^b f(x) dx$

(左斜率 = 右斜率) $\Rightarrow$ (导数存在)

the image of continuous function over a compact interval is compact

② 极限

- $\lim_{x \rightarrow \infty} x^{\frac{1}{p}} = 1$
- $\int_0^1 \frac{1}{x^p} dx$
 - convergent if $p < 1$
 - divergent if $p \geq 1$
- $\int_1^{\infty} \frac{1}{x^p} dx$
 - convergent if $p > 1$
 - divergent if $p \leq 1$
- $\sum \frac{1}{n^a}$
 - convergent if $a > 1$
 - divergent if $a \leq 1$

comparison 之前用 substitution/by parts/partial/ 拆分等进行预处理

③ integral technique

- $\sin x \cos y = \frac{1}{2} (\sin(x+y) + \sin(x-y))$
- $\sin x \sin y = \frac{1}{2} (\cos(x-y) - \cos(x+y))$
- $\cos x \cos y = \frac{1}{2} (\cos(x+y) + \cos(x-y))$
- $\sin x + \sin y = 2 \sin \frac{x+y}{2} \cos \frac{x-y}{2}$
- $\sin x - \sin y = 2 \cos \frac{x+y}{2} \sin \frac{x-y}{2}$
- $\cos x + \cos y = 2 \cos \frac{x+y}{2} \cos \frac{x-y}{2}$
- $\cos x - \cos y = -2 \sin \frac{x+y}{2} \sin \frac{x-y}{2}$

$$\sqrt{a^2 - x^2}, x = a \sin \theta$$

$$\sqrt{x^2 - a^2}, x = a \sec \theta$$

$$\sqrt{x^2 + a^2}, x = a \tan \theta$$

- 将 $f(x)$ 与 x 轴围成部分绕 y 轴旋转后的体积: $\int_a^b 2\pi x f(x) dx$
- 求 curve 的 moment: $M_x = \int_a^b x f(x) dx, M_y = \frac{1}{2} \int_a^b (f(x))^2 dx$

④ 如何证明 series divergent

- upper sum $\neq$ lower sum when $n \rightarrow \infty$
- 判断 $a_n \neq 0$

考虑重写 series 以应对 $(-1)^n / \cos$ 等 e.g.

使用求导/积分证明时注意上下限 $\int_a^b f(x) dx$

⑤ Taylor expansion

$$f(x) = \sum_{i=0}^n \frac{f^{(i)}(x_0)}{i!} (x - x_0)^i + R_n, R_n = \frac{(x - x_0)^{n+1}}{(n+1)!} f^{(n+1)}(c)$$

$$e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \dots$$

$$\ln(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \dots$$

$$\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \dots$$

$$\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \dots$$

$$= \int_0^x \frac{(x-t)^n}{n!} f^{(n+1)}(t) dt$$

$$= \frac{(x-x_0)^{n+1}}{(n+1)!} f^{(n+1)}(c)$$

$$(all x)$$

$$(-1 < x \leq 1)$$

$$(all x)$$

$$(all x)$$

套用 Taylor 展开时注意 $|x|$ 限定
通过 Taylor 展开判断 $f^{(n)}(x)$ 时一定注意公式!

巧用 binomial expansion 应对 $f(x)$,
证明展开成 ∞ 证明 $R \rightarrow 0$,
此时大概要分类讨论.

⑥ Fourier transformation

$$\text{period } 2\pi: f(x) = \frac{1}{2} a_0 + \sum_{k=1}^{\infty} a_k \cos kx + b_k \sin kx, a_k = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \cos kx dx$$

$$b_k = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \sin kx dx$$

$$f(x) = \sum_{-\infty}^{\infty} \gamma_k e^{ikx}$$

$$\gamma_k = \frac{1}{2\pi} \int_{-\pi}^{\pi} f(x) e^{-ikx} dx$$

$$\text{period } 2L: f(x) = \frac{1}{2} a_0 + \sum_{k=1}^{\infty} a_k \cos \frac{k\pi x}{L} + b_k \sin \frac{k\pi x}{L}, a_k = \frac{1}{L} \int_{-L}^L f(x) \cos \frac{k\pi x}{L} dx$$

$$b_k = \frac{1}{L} \int_{-L}^L f(x) \sin \frac{k\pi x}{L} dx$$

$$f(x) = \sum_{-\infty}^{\infty} \gamma_k e^{i \frac{k\pi x}{L}}$$

$$\gamma_k = \frac{1}{2L} \int_{-L}^L f(x) e^{-i \frac{k\pi x}{L}} dx$$

5

• period ∞ $\int_{-\infty}^{\infty} (f(x))' dx = \frac{1}{2} a_0 + \sum_{n=1}^{\infty} (a_n^2 + b_n^2)$
 $f(x) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} f(t) e^{-i\omega t} dt e^{i\omega x} d\omega$

① use power series solve DE Fourier Transformation

(1) 根据题型设 $y = \sum a_n x^n$ 或 $y = x^k (\sum a_n x^n)$

(2) 代入解是，将所有 a_n 用 $a_0, a_1, \dots$ 等表示

(3) 总结答案

(4) 判断 radius of convergence + v.

$\lim_{x \rightarrow \infty}$: 指数 $\gg$ 多项式 $\gg$ 对数

$\lim_{x \rightarrow 0}$ 多项式 $\gg$ 对数 $\gg$ 指数

$\int_1^{\infty} e^{-x^p} dx$ exists for all $p > 0$

$\int_1^{\infty} (\ln x)^p dx$ never exists for all p

$\int_0^1 (\ln x)^p dx$ exists for $p > -1$

type of bifurcation:

saddle-node: 平衡点: $0 \leftrightarrow 2$ (一稳一不稳)

transcritical: 平衡点: $2 \leftrightarrow 2$ (交换平衡性)

supercritical pitchfork: 平衡点: $1 \leftrightarrow 3$ (稳定对称分支)

极限: { 洛比达 [换元? 变化?]
 泰勒展开 [解析时打用]

积分 [先拆分] { bounded [洛比达/泰勒展开]

夹逼 [by part/substitution]

泰勒展开

by part/substitution

① Fourier Transformation

$$\begin{aligned} \hat{f}(w) &= \int_{-\infty}^{\infty} f(x) e^{-iwx} dx, & f(x) &= \frac{1}{2\pi} \int_{-\infty}^{\infty} \hat{f}(w) e^{iwx} dw \\ \hat{f}_c(w) &= \int_0^{\infty} f(x) \cos wx dx, & f(x) &= \frac{2}{\pi} \int_0^{\infty} \hat{f}_c(w) \cos wx dw \\ \hat{f}_s(w) &= \int_0^{\infty} f(x) \sin wx dx, & f(x) &= \frac{2}{\pi} \int_0^{\infty} \hat{f}_s(w) \sin wx dw \end{aligned}$$

[注意: $\hat{f}(w) = 2\hat{f}_c(w) - 2i\hat{f}_s(w)$ 配合 $f(x)$ 奇偶性使用]

$$F(af(x) + bg(x)) = a\hat{f}(w) + b\hat{g}(w)$$

$$F(f(ax)) = \frac{1}{|a|} \hat{f}\left(\frac{w}{a}\right) \quad \text{if } a > 0$$

$$F(f(-x)) = \hat{f}(-w)$$

$$F(f(x-x_0)) = e^{-iwx_0} \hat{f}(w)$$

$$F(e^{iwx_0} f(x)) = \hat{f}(w-w_0)$$

$$F(\hat{f}(x)) = 2\pi f(-w)$$

$$F\left(\frac{d^n}{dx^n} f(x)\right) = (iw)^n \hat{f}(w)$$

$$F(x f(x)) = i \hat{f}'(w)$$

$$F([f(x)]^*) = [\hat{f}(-w)]^*$$

$$F(f * g(x)) = \hat{f}(w) \hat{g}(w)$$

$$f * g(x) = \int_{-\infty}^{\infty} f(x-u) g(u) du$$

$$\bullet \text{ if } f(x) \text{ is real-valued matrix } \frac{1}{2\pi} \int_{-\infty}^{\infty} |\hat{f}(w)|^2 dw = \int_{-\infty}^{\infty} |f(x)|^2 dx$$

$$\bullet \int_{-\infty}^{\infty} \delta(x) dx = 1, \int_{-\infty}^{\infty} g(x) \delta(x-a) dx = g(a)$$

$$F(\delta(x)) = 1, \delta(x) = \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{\pm iwx} dw$$

* pre-result:

$$\bullet \text{ if } h(x) = 1 \text{ for } |x| < a, \hat{h}(w) = \frac{2}{w} \sin(aw)$$

$$\bullet \text{ if } f(x) = e^{-\alpha|x|}, \hat{f}(w) = \frac{2\alpha}{\alpha^2 + w^2}$$

② ODEs

• order: order of highest derivative

degree: power of highest derivative (fractional are removed)

linear: linear combination of $y, y', y'', \dots$ [没有 $x!$]

$$\bullet \frac{dy}{dx} = f\left(\frac{y}{x}\right)$$

$$\frac{dy}{dx} + p(x)y = q(x)y^n \quad \therefore \text{let } z = y^{1-n}$$

$$\frac{d^2y}{dx^2} = f\left(x, \frac{dy}{dx}\right)$$

$$\frac{d^2y}{dx^2} = f\left(y, \frac{dy}{dx}\right)$$

• Wronskian matrix:

$$\det W = \det \begin{pmatrix} y_1(x) & y_2(x) & \dots & y_k(x) \\ \frac{dy_1(x)}{dx} & \frac{dy_2(x)}{dx} & \dots & \frac{dy_k(x)}{dx} \\ \vdots & \vdots & \ddots & \vdots \\ \frac{d^{k-1}y_1(x)}{dx^{k-1}} & \frac{d^{k-1}y_2(x)}{dx^{k-1}} & \dots & \frac{d^{k-1}y_k(x)}{dx^{k-1}} \end{pmatrix}$$

[注意: $(\det W \neq 0) \rightarrow$ (linear independent)]

(linear independent) $\rightarrow (\det W \neq 0)$

给定 k^{th} order homogeneous linear ODE 的 k 个根, 证明 $\det W$ 不为 0

$$6. x^n \frac{d^n y}{dx^n} + a_{n-1} x^{n-1} \frac{d^{n-1} y}{dx^{n-1}} + \dots + a_1 x \frac{dy}{dx} + a_0 y = f(x) \quad \text{let } x = e^z$$

- 结合 Fourier Transformation:

$$F(xf(x)) = i\hat{f}'(\omega) \quad F(f^{(n)}(x)) = (i\omega)^n \hat{f}(\omega)$$

- 解 system of ODE:

化为 $\frac{d}{dt}\vec{y} = A\vec{y} + g(t)$. 则 $\vec{y} = \vec{y}_{CF} + \vec{y}_{PL}$

$\vec{y}_{CF}$: 如果 A diagonalisable, $\vec{y}_{CF} = (\vec{v}_1 \vec{v}_2 \dots \vec{v}_n) \begin{pmatrix} c_1 e^{\lambda_1 t} \\ c_2 e^{\lambda_2 t} \\ \vdots \\ c_n e^{\lambda_n t} \end{pmatrix}$

如果 A 不可 diagonalisable, 用 Jordan form, etc:

$J = \begin{pmatrix} \lambda_1 & 0 & 0 \\ 0 & \lambda_2 & 0 \\ 0 & 0 & \lambda_3 \end{pmatrix}$, $W = [\vec{v}_1 \vec{v}_2 \vec{u}_3]$, $A = WJW^{-1}$, 其中 $\vec{u}_3$ 设数时解

则 $z_0 = c_0 e^{\lambda_1 t}$, $z_2 = c_2 e^{\lambda_2 t} + c_3 t e^{\lambda_2 t}$, $z_1 = c_1 e^{\lambda_1 t}$

$\vec{y}_{CF} = z_1 \vec{v}_1 + z_2 \vec{v}_2 + z_3 \vec{u}_3$

③ qualitative analysis of ODEs

- fixed point: $\frac{d}{dt}\vec{y} = A\vec{y}$, $\vec{y}^* = \vec{0}$ if $\det A \neq 0$.

$\vec{y}^* \in \ker A$ if $\det A = 0$

Lyapunov stable: $\forall \varepsilon > 0, \exists \delta > 0$, if $\|\vec{y}(0) - \vec{y}^*\| < \delta$, $\|\vec{y}(t) - \vec{y}^*\| < \varepsilon$

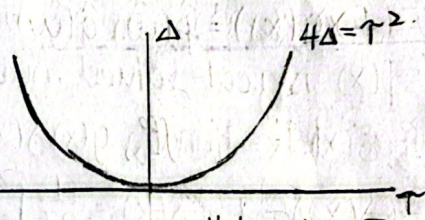
asymptotic stable: Lyapunov stable, $\exists \delta > 0$, if $\|\vec{y}(0) - \vec{y}^*\| < \delta$, $\lim_{t \rightarrow \infty} \|\vec{y}(t) - \vec{y}^*\| = 0$

$\begin{cases} \frac{dx}{dt} = f(x, y) \\ \frac{dy}{dt} = g(x, y) \end{cases}$

可求 $\frac{dy}{dx} = \frac{g(x, y)}{f(x, y)}$

可求 $\frac{d}{dt}\vec{y} = A\vec{y}$, 后画图

通过右图判断 bifurcation



- 1D non-linear system 求 bifurcation. $\frac{dy}{dt} = f(y)$

先求出可能的 fixed point 与 singular point

在 $y-f(y)$ 图象中判断各个情况中 fixed point 在哪里

在 $r-y$ 图像 bifurcation diagram 中先画出所有 fixed point 的线, 然后具体标点.

考虑附加边界 (如 $x > 0, y > 0$ 等)

~~考虑附加边界~~

④ multivariable calculus

- 有多个 variable 时, 对 $\frac{\partial f}{\partial x_i}$ 标号却标以区分表示所包含的 variables

$z = f(x, y)$, $\frac{dz}{dt} = \frac{\partial z}{\partial x} \frac{dx}{dt} + \frac{\partial z}{\partial y} \frac{dy}{dt}$, $\frac{\partial^2 z}{\partial x \partial y} = \frac{\partial^2 z}{\partial y \partial x}$

- Taylor expansion:

$f(x+\Delta x, y+\Delta y) = f(x, y) + \left[\frac{\partial f}{\partial x}(x, y)\Delta x + \frac{\partial f}{\partial y}(x, y)\Delta y \right] + \frac{1}{2!} \left[\frac{\partial^2 f}{\partial x^2}(x, y)(\Delta x)^2 + 2 \frac{\partial^2 f}{\partial x \partial y}(x, y)(\Delta x)(\Delta y) + \frac{\partial^2 f}{\partial y^2}(x, y)(\Delta y)^2 \right] + \dots$

- change of variables:

$\begin{pmatrix} dx \\ dy \end{pmatrix} = \begin{pmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{pmatrix} \begin{pmatrix} du \\ dv \end{pmatrix}$

$\frac{dy}{dx} = -\frac{F(x, y)}{G(x, y)}$: $\begin{cases} \frac{\partial F}{\partial y} = \frac{\partial G}{\partial x} \\ \frac{\partial \lambda F}{\partial y} = \frac{\partial \lambda G}{\partial x} \end{cases}$

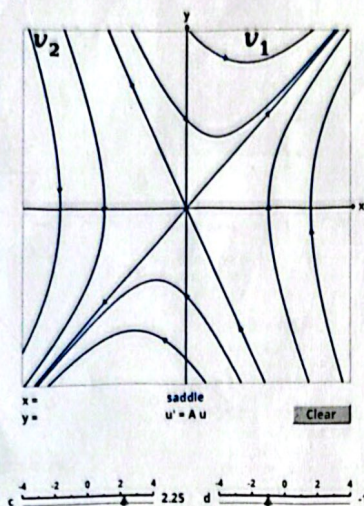
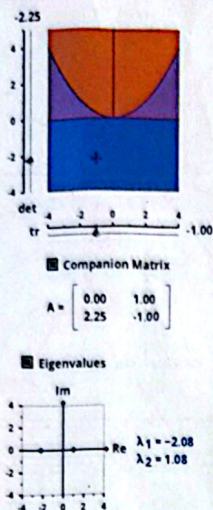
- stationary points:

先 $\frac{\partial z}{\partial x} = \frac{\partial z}{\partial y} = 0$, 找候选点

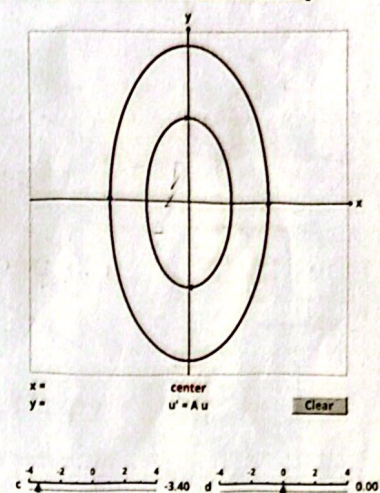
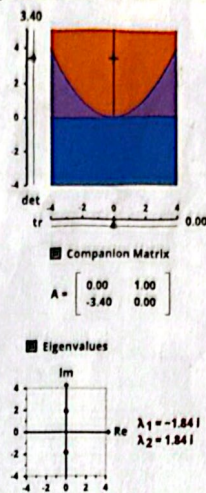
再考虑 $\begin{pmatrix} \frac{\partial^2 z}{\partial x^2} & \frac{\partial^2 z}{\partial x \partial y} \\ \frac{\partial^2 z}{\partial y \partial x} & \frac{\partial^2 z}{\partial y^2} \end{pmatrix}$. $\lambda_1, \lambda_2 > 0$, minimum
 $\lambda_1, \lambda_2 < 0$ maximum

- sketch the graph: check the continuity and singularity
obtain some level curve
find stationary points

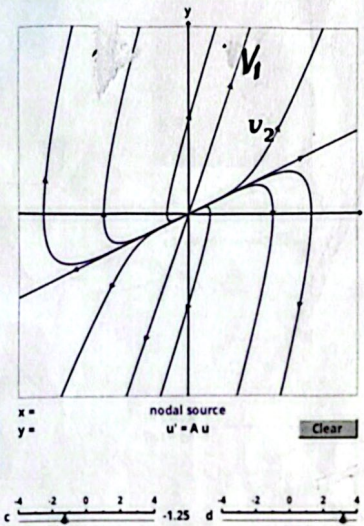
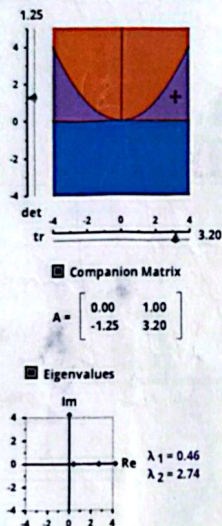
When $\lambda_1 > 0 > \lambda_2$:
[此时的原点为 unstable point]



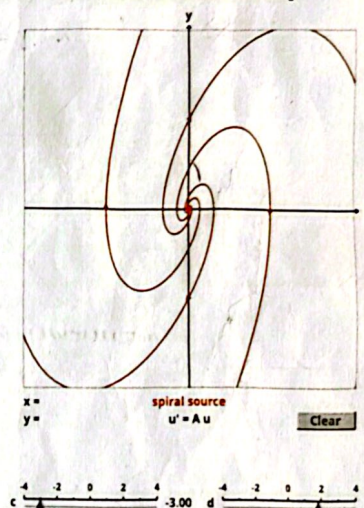
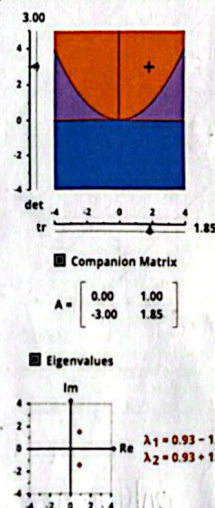
When $\lambda = \pm i\omega$:
[rotational direction unknown and should be calculated]



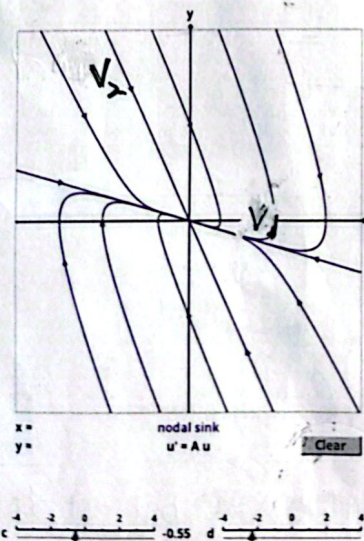
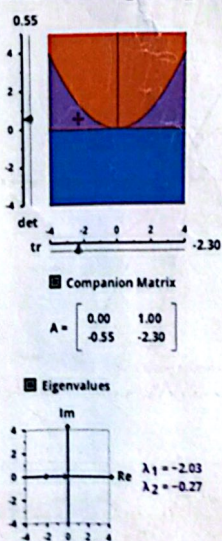
When $\lambda_1 > \lambda_2 > 0$:



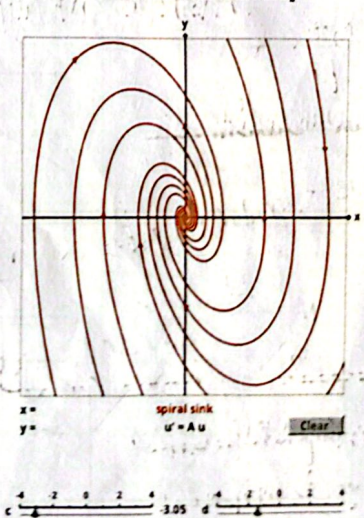
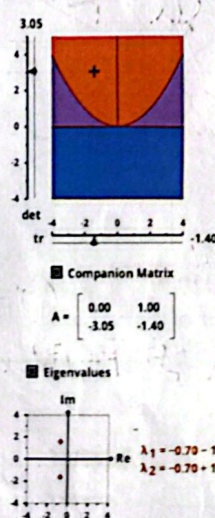
When $\lambda = \alpha \pm i\omega, \alpha > 0$:
[rotational direction unknown and should be calculated]



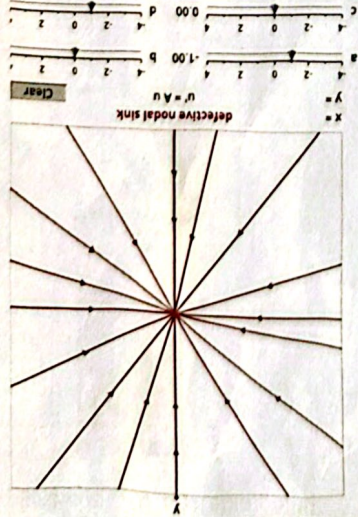
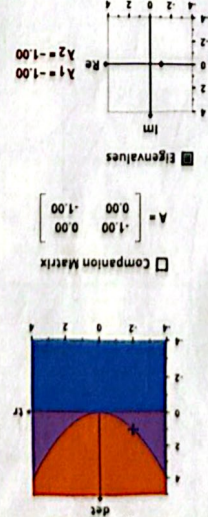
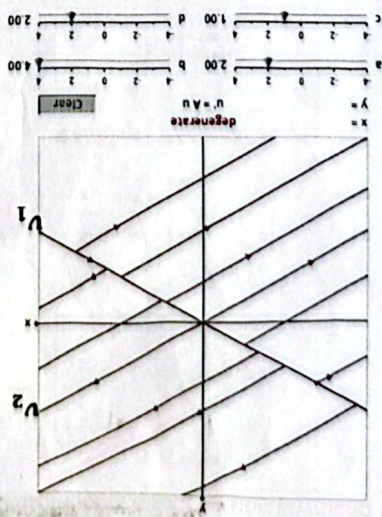
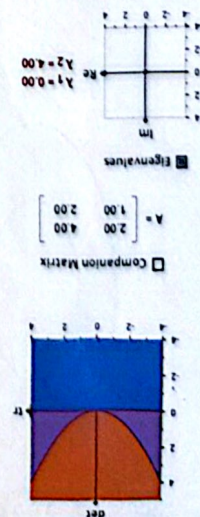
When $0 > \lambda_1 > \lambda_2$:



When $\lambda = \alpha \pm i\omega, \alpha < 0$:
[rotational direction unknown and should be calculated]

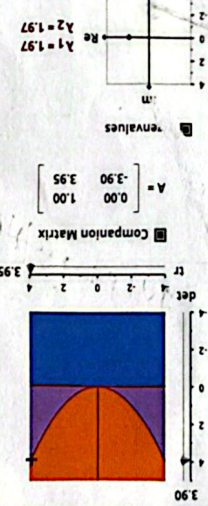
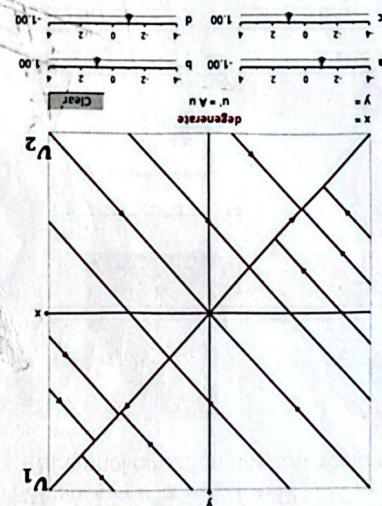
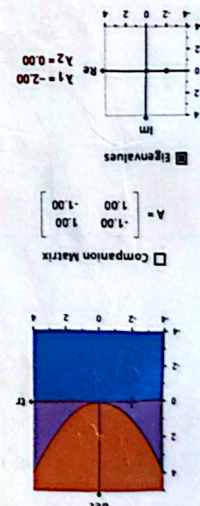


When $\lambda_1 = 0, \lambda_2 > 0$:



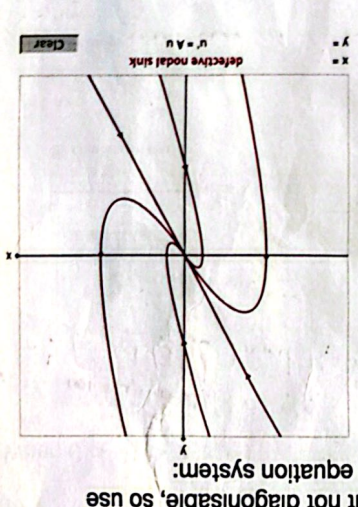
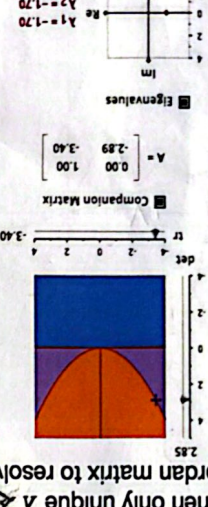
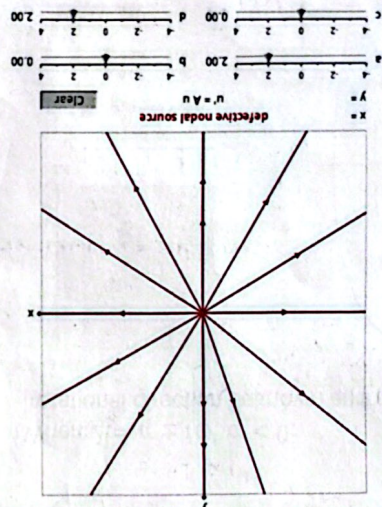
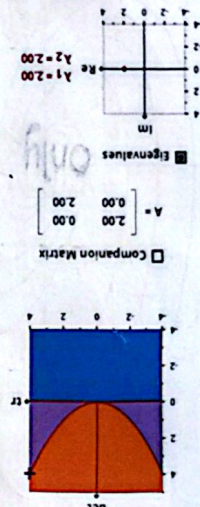
When only unique $\lambda < 0$, but not diagonalisable:

When $\lambda_1 = 0, \lambda_2 > 0$:



When only unique $\lambda > 0$, but not diagonalisable, so use Jordan matrix to resolve the equation system:

When only unique $\lambda > 0$, but diagonalisable:



When only unique $\lambda < 0$, but not diagonalisable, so use Jordan matrix to resolve the equation system:

only in form λI

only

When only unique $\lambda = 0$, but not diagonalisable so use Jordan matrix to resolve the equation system

